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On Some Properties of Convolution in Morrey Type Spaces

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Abstract. Morrey space $M^{p,\alpha}$ and its subspace $MC^{p,\alpha}$ where the continuous functions are dense are considered. Basic properties of convolution are extended to these spaces. It is proved that the convolution in $MC^{p,\alpha}$ can be approximated by finite linear combinations of shifts. Approximate identity in $MC^{p,\alpha}$ is also considered.

Key Words and Phrases: Morrey space, convolution, approximate identity.

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1. Introduction

The concept of Morrey space was introduced by C.Morrey in 1938 in the study of the smooth properties of solutions of elliptic equations with VMO (Vanishing Mean Oscillation) coefficients. Since then, various problems related to this space have been intensively studied. Playing an important role in the qualitative theory of elliptic differential equations (see, for example, [1, 2]), this space also provides a large class of examples of mild solutions to the Navier-Stokes system [3]. In the context of fluid dynamics, Morrey spaces have been used to model fluid flow when vorticity is a singular measure supported on certain sets in R^n [4]. There appeared lately a large number of research works which considered fundamental problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc in these spaces (see, for example, [5] and the references above). More details about Morrey spaces can be found in [6, 7, 18, 19].

In view of the aforesaid, there has recently been a growing interest in the study of various problems in Morrey-type spaces. For example, some problems of harmonic analysis and approximation theory have been considered in [6, 7, 8, 9,

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10, 11, 12, 13]. In [13, 15], the basicity of the classical exponential system, as well as its perturbations in subspaces of Morrey space of functions defined on $[\pi, \pi]$ was investigated with the method of boundary value problem for analytic functions on complex domain. Note that some questions arising in boundary value problems of analytic function theory have been considered and studied in [17, 18, 19]. In [16], an analogue of the classical Young inequality for the convolution of periodic functions belonging to global Morrey type spaces on R^n is obtained.

In the present paper Morrey space $M^{p,\alpha}$ and its subspace $MC^{p,\alpha}$ where the continuous functions are dense are considered. Basic properties of convolution are extended to these spaces. It is proved that the convolution in $MC^{p,\alpha}$ can be approximated by finite linear combinations of shifts. Approximate identity in $MC^{p,\alpha}$ is also considered. The validity of the classical facts concerning the approximate identities is proved in Morrey type spaces.

2. Needful information

We will need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$.

By the Morrey-Lebesgue space $M^{p,\alpha}(\Gamma)$, $0 \leq \alpha \leq 1$, $p \geq 1$, we mean a normed space of all functions$f(\xi)$ measurable on Γ equipped with a finite norm $\|\cdot\|_{M^{p,\alpha}(\Gamma)}$:

$$\|f\|_{M^{p,\alpha}(\Gamma)} = \sup_B \left(\left| B \bigcap \Gamma \right|_\Gamma^{\alpha-1} \int_{B \bigcap \Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty.$$

$M^{p,\alpha}(\Gamma)$ is a Banach space and $M^{p,1}(\Gamma) = L_p(\Gamma)$, $M^{p,0}(\Gamma) = L_\infty(\Gamma)$. The embedding $M^{p,\alpha_1}(\Gamma) \subset M^{p,\alpha_2}(\Gamma)$ is valid for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Thus, $M^{p,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall \alpha \in [0,1]$, $\forall p \geq 1$. The case of $\Gamma \equiv [-\pi, \pi]$ will be denoted by $M^{p,\alpha}(-\pi, \pi) \equiv M^{p,\alpha}$.

Denote by $\tilde{M}^{p,\alpha}$ the linear subspace of $M^{p,\alpha}$ consisting of functions whose shifts are continuous in $M^{p,\alpha}$, i.e. $\|f(\cdot + \delta) - f(\cdot)\|_{M^{p,\alpha}} \to 0$ as $\delta \to 0$. The closure of $\tilde{M}^{p,\alpha}$ in $M^{p,\alpha}$ will be denoted by $MC^{p,\alpha}$. It is easy to prove the following

Theorem 1 ([13]). *Infinitely differentiable functions on* $[0, 2\pi]$ *are dense in the space* $MC^{p,\alpha}$.

Note that, in general, the set of all continuous functions is not dense in $M^{p,\alpha}$. Corresponding examples can be found in many works (see, e.g., [7]). We give the example below.

Example 1. *Consider the function*

$$f_0(x) = |x|^{-\frac{\alpha}{p}}, x \in [-\pi, \pi].$$

From the results of Samko N. [22] it immediately follows that $f_0 \in L^{p,\alpha}$. *Let us show that* $\exists\, \delta_0 > 0$

$$\|f_0 - g\|_{p,\alpha} \geq \delta_0 > 0, \forall g \in C[-\pi, \pi].$$

It is clear that

$$\|f_0 - g\|_{p,\alpha}^p \geq \sup_{h \in (0,h_1)} \frac{1}{h^{1-\alpha}} \int_0^h |f_0(t) - g(t)|^p \, dt,$$

where $h_1 \in (0, \pi)$ *is an arbitrary number. Assume*

$$M = \sup_{|t| \leq h_1} |g(t)|^p.$$

We have

$$\int_0^h |f_0(t) - g(t)|^p \, dt \geq 2^{-p} \int_0^h |f_0|^p \, dt - \int_0^h |g(t)|^p \, dt \geq$$

$$\geq 2^{-p} \int_0^h t^{-\alpha} dt - Mh = \frac{2^{-p}}{1-\alpha} h^{1-\alpha} - Mh =$$

$$= h^{1-\alpha} \left(\frac{2^{-p}}{1-\alpha} - Mh^{\alpha} \right) \geq h^{1-\alpha} \left(\frac{2^{-p}}{1-\alpha} - Mh_1^{\alpha} \right).$$

Let us take sufficiently small $h_1 > 0$ *so that*

$$\frac{2^{-p}}{1-\alpha} - Mh_1^{\alpha} \geq \frac{2^{-p}}{2(1-\alpha)},$$

is true.

Thus, for an arbitrary function $g \in C[-\pi, \pi]$ *we have*

$$\|f_0 - g\|_{p,\alpha} \geq \frac{2^{-p}}{2(1-\alpha)} > 0,$$

i.e.

$$\inf_{g \in C[-\pi,\pi]} \|f_0 - g\|_{p,\alpha} > 0.$$

Thus, the following statement is true.

Proposition 1. *The space of continuous functions* $C[-\pi,\pi]$ *is not dense in the Morrey space* $L^{p,\alpha}$ *for* $\forall p \in [1, +\infty)$, $\forall \alpha \in (0, 1)$.

The following Hölder's inequality is also valid.

Lemma 1. *Let* $f \in L^{p,\alpha}(I) \wedge g \in L^{q,\alpha}(I), \frac{1}{p}+\frac{1}{q}=1$, $p \in [1, +\infty)$. *Then the following Hölder's inequality holds:*

$$\|fg\|_{L_1} \le |I|^{1-\alpha}\|fg\|_{1,\alpha} \le |I|^{1-\alpha}\|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

3. Main results

Consider the Morrey-Lebesgue space $M^{p,\alpha}$, $1<p<+\infty$, $0<\alpha<1$. Let $\frac{1}{p}+\frac{1}{q}=1$. Take $f \in M^{p,\alpha}; g \in M^{q,\alpha}$. Consider the convolution

$$(f*g)(x)=\int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy, x \in [-\pi,\pi].$$

As $M^{p,\alpha} \subset L_1$, the existence of the convolution follows from the classical facts. Applying Hölder's inequality with respect to Morrey space, we obtain

$$|(f*g)(x)| \le c_\alpha \|f(x-\cdot)\|_{p,\,\alpha}\|g\|_{q,\,\alpha}, \tag{1}$$

where $c_\alpha=(2\pi)^{1-\alpha}$. We have

$$\|f(x-\cdot)\|_{p,\alpha}=\sup_{I_\pi}\left(\frac{1}{|I_\pi|^{1-\alpha}}\int_{I_\pi}|f(x-y)|^p\,dy\right)^{1/p},$$

where $I_\pi = I \bigcap [-\pi,\pi]$, $I \subset R$ is an arbitrary interval. Put $I_\pi(x)=x-I_\pi$. It is clear that $|I_\pi(x)|=|I_\pi|$, $\forall x \in [-\pi,\pi]$. Making a change of variables, we obtain

$$\|f(x-\cdot)\|_{p,\alpha}=\sup_{I_\pi}\left(\frac{-1}{|-I_\pi|^{1-\alpha}}\int_{-I_\pi}|f(x+y)|^p\,dy\right)^{1/p}=$$

$$=\sup_{I_\pi}\left(\frac{-1}{|I_\pi(x)|^{1-\alpha}}\int_{I_\pi(x)}|f(t)|^p\,dt\right)^{1/p}\le\|f\|_{p,\alpha}.$$

Considering this relation in (1), we have

$$\|f*g\|_\infty \le c_\alpha\|f\|_{p,\alpha}\|g\|_{q,\alpha}.$$

Let T_δ be a shift operator, i.e. $(T_\delta f)(x)=f(x+\delta)$. If $f \in MC^{p,\alpha}$, then it is clear that

$$\|T_\delta f-f\|_{p,\alpha}\to 0, \delta\to 0.$$

We have

$$\|T_\delta (f*g) - f*g\|_\infty = \|T_\delta f*g - f*g\|_\infty =$$
$$= \|(T_\delta f - f)*g\|_\infty \le \|T_\delta f - f\|_{p,\alpha} \|g\|_{q,\alpha} \to 0, \delta \to 0.$$

Thus, the following theorem is true.

Theorem 2. *Let* $1<p<+\infty$, $0<\alpha<1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *If* $f\in M^{p,\alpha} \wedge g \in M^{q,\alpha}$, *then the convolution* $f*g$ *is defined everywhere on* $[-\pi,\pi]$ *and the following inequality is true:*

$$\|f*g\|_\infty \le c_\alpha \|f\|_{p,\alpha} \|g\|_{q,\alpha}.$$

Moreover, if $f\in MC^{p,\alpha}$ *or* $g\in MC^{q,\alpha}$, *then the convolution* $f*g$ *is continuous on* $[-\pi,\pi]$.

Note that this theorem in more general form was proved in [16]. We gave an independent and simple proof in order to keep the style of the work.

Let $f:[-\pi,\pi]\to R$ be a simple function, i.e. let $\bigcup_{k=1}^r E_k = [-\pi,\pi]$ be some partition of an interval $[-\pi,\pi]$ and $f(x)=c_k$, $\forall x\in E_k$, $k=\overline{1,r}$. We have

$$(f*g)(x) = \int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy = \int_{-\pi}^{\pi} f(y)\,g(x-y)\,dy =$$
$$= \sum_{k=1}^r c_k \int_{E_k} g(x-y)\,dy, \ \forall x\in[-\pi,\pi] \ .$$

Let us take an arbitrary interval $I\subset[-\pi,\pi]$. Consider

$$\left(\int_I |(f*g)(x)|^p dx\right)^{\frac{1}{p}} \le \sum_{k=1}^r |c_k| \left(\int_I \left|\int_{E_k} g(x-y)\,dy\right|^p dx\right)^{\frac{1}{p}} \le$$
$$\le \sum_{k=1}^r |c_k|\,|E_k|^{\frac{1}{q}} \left(\int_I \int_{E_k} |g(x-y)|^p dydx\right)^{\frac{1}{p}}.$$

On the other hand

$$\int_I \int_{E_k} |g(x-y)|^p dydx = \int_{E_k} \int_I |g(x-y)|^p dxdy \ ,$$

and, as a result, we have

$$\frac{1}{|I|^{1-\alpha}} \int_I \int_{E_k} |g(x-y)|^p dydx = \int_{E_k} \frac{1}{|I|^{1-\alpha}} \int_I |g(x-y)|^p dxdy =$$

$$= \int_{E_k} \frac{1}{|I|^{1-\alpha}} \int_{I_y} |g(t)|^p \, dt dy,$$

where $I_y = I - y$. It is obvious that $|I| = |I_y|$. Then the following inequality is true:

$$\frac{1}{|I|^{1-\alpha}} \int_{I_y} |g(t)|^p \, dt \leq \|g\|_{p,\alpha}^p .$$

Taking into account this inequality, we obtain

$$\left(\frac{1}{|I|^{1-\alpha}} \int_I |(f * g)(x)|^p \, dx \right)^{\frac{1}{p}} \leq \sum_{k=1}^{r} |c_k| \, |E_k|^{\frac{1}{q}} \left(\int_{E_k} \frac{1}{|I|^{1-\alpha}} \int_{I_y} |g|^p \, dt dx \right)^{\frac{1}{p}} \leq$$

$$\leq \sum_{k=1}^{r} |c_k| \, |E_k|^{\frac{1}{q}} \, |E_k|^{\frac{1}{p}} \, \|g\|_{p,\alpha} = \sum_{k=1}^{r} |c_k| \, |E_k| \, \|g\|_{p,\alpha} = \|f\|_{L_1} \|g\|_{p,\alpha} \; .$$

Hence

$$\|f * g\|_{p,\alpha} \leq \|f\|_{L_1} \|g\|_{p,\alpha} \, , \; \forall f \in S[-\pi, \pi] \, , \tag{2}$$

where $S[-\pi, \pi]$ is the set of all simple functions on $[-\pi, \pi]$.

Now, let $f \in L_1(-\pi, \pi)$ be an arbitrary function. Take $\forall \{f_n\}_{n \in N} \subset S[-\pi, \pi]$: $\|f_n - f\|_{L_1} \to 0$, $n \to \infty$. Since $S[-\pi, \pi]$ is dense in $L_1(-\pi, \pi)$, the choice of such a sequence is always possible. Then it follows directly from the inequality (2) that the sequence $\{f_n * g\}_{n \in N}$ is fundamental in $M^{p,\alpha}$. Assume

$$f * g = \lim_{n \to \infty} f_n * g.$$

By virtue of inequality (2), the definition of $f * g$ does not depend on the choice of the sequence $\forall \{f_n\}_{n \in N}$. So, the following theorem is true.

Theorem 3. *Let $f \in L_1 \wedge g \in M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$. Then $f * g \in M^{p,\alpha}$, and, moreover, the following inequality is true:*

$$\|f * g\|_{p,\alpha} \leq \|f\|_{L_1} \|g\|_{p,\alpha} .$$

Remark 1. *Denote by $\mathscr{M}$ the space of measures on $\mathscr{T} \equiv [-\pi, \pi]$, i.e. $\mathscr{M}$ contains a distribution $F \in D$ satisfying the inequality*

$$|F(u)| \leq c \|u\|_\infty , \; \forall u \in C^\infty,$$

where $\|\cdot\|_\infty$ is a sup*-norm, C^∞ are infinitely differentiable functions with compact support on $(-\pi, \pi)$. Such measures are called Radon measures. It is known that (Riesz-Markov- Kakutani theorem for the compact space T) each functional* *(distribution) can be represented as an integral with respect to the unique regular Borel measure m on T :*

$$F(u) = \int_T u(x)\, dm(x).$$

$\mathscr{M}$ is a Banach space with regard to the norm

$$\|\mu\|_1 = \sup\{|\mu(u)| : u \in C[-\pi, \pi],\ \|u\|_\infty < 1\}.$$

For more details on these facts we refer the reader to [14].

The question of the validity of the following statement arises:

Statement 1. Let $1 \le p < +\infty \ \wedge\ 0 < \alpha < 1$. If $\mu \in \mathscr{M} \wedge f \in M^{p,\alpha}$, then $\mu * f \in M^{p,\alpha}$, and the following inequality holds:

$$\|\mu * f\|_{p,\alpha} \le \|\mu\|_1 \|f\|_{p,\alpha}.$$

The following theorem is also true.

Theorem 4. *Let $f \in L_1$ and $g \in E$, where E denotes any one of the spaces $C[-\pi, \pi]$ or $MC^{p,\alpha}$, $1 \le p < +\infty$, $0 < \alpha < 1$. Then the convolution $f * g$ in E can be approximated by finite linear combinations of shifts g, i.e. $\forall \varepsilon > 0$, $\exists \{a_k\}_1^n \subset [-\pi, \pi] \wedge \{\lambda_k\}_1^n \subset R$:*

$$\left\| f * g - \sum_{k=1}^{n} \lambda_k T_{a_k} g \right\|_E < \varepsilon.$$

Proof. The case of $E = C[-\pi, \pi]$ is known (see, e.g., [14]). Consider the case of $E = MC^{p,\alpha}$. Following the classical scheme, as a subset S_0 , such that the finite linear combinations of elements from S_0 are dense in L_1, we take a set of functions f, each of which coincides on $[-\pi, \pi]$ with the characteristic function of some interval $M = [a, b]$, $-\pi < a < b < \pi$, and continues further on periodically.

Let $\varepsilon > 0$ be arbitrary. Let us divide M into a finite number of subintervals I_k of length $|I_k| < \delta$. Take $\forall a_k \in I_k$. Let $f(x) = \chi_M(x)$. We have

$$(f * g)(x) - \sum_k |I_k|\, g(x - a_k) = \int_{\bigcup_k I_k} g(x - y)\, dy -$$

$$- \sum_k \int_{I_k} g(x - a_k)\, dy = \sum_k \int_{I_k} [g(x - y) - g(x - a_k)]\, dy = \sum_k h_k(x),$$

where

$$h_k(x) = \int_k [g(x - y) - g(x - a_k)]\, dy.$$

Consequently

$$\left\|(f*g)(x)-\sum_k |I_k|\, g(x-a_k)\right\|_{p,\alpha}\le \sum_k \|h_k\|_{p,\alpha}.$$

Let $I\subset[-\pi,\pi]$ be an arbitrary interval. We have

$$\|h_k\|^p_{L_p(I)}=\int_I |h_k|^p\,dx=\int_I\left|\int_{I_k}[g(x-y)-g(x-a_k)]\,dy\right|^p dx\le$$

$$\le\int_I\left(|I_k|^{p/q}\int_{I_k}|g(x-y)-g(x-a_k)|^p\,dy\right)dx=$$

$$=|I_k|^{p/q}\int_{I_k}\left(\int_I|g(x-y)-g(x-a_k)|^p\,dx\right)dy.$$

As a result, we obtain

$$\frac{1}{|I|^{1-\alpha}}\int_I|h_k|^p\,dx\le|I_k|^{p/q}\int_{I_k}\left(\frac{1}{|I|^{1-\alpha}}\int_I|g(x-y)-g(x-a_k)|^p\,dx\right)dy\le$$

$$\le|I_k|^{p/q}\int_{I_k}\|T_yg-T_{a_k}g\|^p_{p,\alpha}\,dy\Rightarrow$$

$$\Rightarrow\|h_k\|^p_{p,\alpha}\le|I_k|^{p/q}\int_{I_k}\|T_yg-T_{a_k}g\|^p_{p,\alpha}\,dy. \tag{3}$$

By Lemma 1, we have $\exists\delta>0:$

$$\|T_yg-T_{a_k}g\|_{p,\alpha}<\varepsilon,\ \forall y\in I_k.$$

From (3) it follows that

$$\|h_k\|^p_{p,\alpha}\le|I_k|^{p/q}\,|I_k|\,\varepsilon^p=|I_k|^p\,\varepsilon^p\Rightarrow\|h_k\|_{p,\alpha}\le|I_k|\,\varepsilon.$$

As a result, we have

$$\left\|(f*g)(x)-\sum_k|I_k|\,T_{a_k}g\right\|_{p,\alpha}\le\sum_k|I_k|\,\varepsilon=|M|\,\varepsilon\le2\pi\varepsilon.$$

Since $\sum_k|I_k|\,T_{a_k}g$ is a finite linear combination of shifts g, it is clear that $f*g\in\overline{V_g}$, where $\overline{V_g}$ is a closed linear subspace in E, generated by shifts T_ag of the function g.

If $f \in L_1$ is an arbitrary element, then for $\forall \varepsilon > 0$ there exists a partition of $[-\pi, \pi]$ into a finite number of intervals M_k, and a number λ_k such that the inequality

$$\left\| f(\cdot) - \sum_k \lambda_k \chi_{M_k}(\cdot) \right\|_{L_1} < \varepsilon, \tag{4}$$

holds. It follows directly from the previous result that $\tilde{f} * g \in \overline{V_g}$, where $\tilde{f}(\cdot) = \sum_k \lambda_k \chi_{M_k}(\cdot)$. Then from (4) we obtain that $f * g \in \overline{V_g}$. ◄

Let us consider the approximate identities for convolutions in the space $M^{p,\alpha}$. By the approximate identity (for convolution) we mean $\left\{K_n^{(\cdot)}\right\}_{n\in N} \subset L_1(-\pi, \pi)$, satisfying the following conditions:

α) $\sup\limits_n \|K_n\|_{L_1} < +\infty$;

β) $\lim\limits_n \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)\, dx = 1$;

γ) $\lim\limits_{n\to\infty} \int_{\delta \le |x| \le \pi} |K_n(x)\, dx| = 0\ ,\ \ \forall \delta \in (0, \pi)\,.$

The following theorem is true.

Theorem 5. *Let $\{K_n\}_{n\in N}$ be an approximate identity. Then the following properties are true:*

i) $\lim\limits_{n\to\infty} \|K_n * f - f\|_\infty = 0\,,\ \forall f \in C[-\pi, \pi]$;

ii) $\lim\limits_{n\to\infty} \left\| \frac{d^m}{dx^m}(K_n * f) - \frac{d^m}{dx^m} f \right\|_\infty = 0\,,\ \forall f \in C[-\pi, \pi]$;

iii) $\lim\limits_{n\to\infty} \|K_n * f - f\|_{p,\alpha} = 0\,,\ \forall f \in MC^{p,\alpha}$, $1 \le p < +\infty\,,\ 0 < \alpha < 1$.

Proof. The assertions i)-ii) are classic facts (see, e.g., [14]). Let us prove assertion iii). Take $\forall f \in MC^{p,\alpha}$. Let $\varepsilon > 0$ be an arbitrary number. It is clear that $\exists g \in C[-\pi, \pi]$:

$$\|f - g\|_{p,\alpha} < \varepsilon.$$

Then paying attention to Theorem 3, from the property α) of an approximate identity we obtain

$$\|K_n * f - K_n * g\|_{p,\alpha} \le \|K_n\|_{L_1} \|f - g\|_{p,\alpha} \le M\varepsilon,$$

where M is a constant independent of n. Then from i) it follows that

$$\exists n_0 = n_0(\varepsilon) \in N : \|K_n * g\|_\infty < \varepsilon\,,\ \ \forall n \ge n_0.$$

Hence, we have

$$\|K_n * g - g\|_{p,\alpha} \le c\varepsilon, \forall n \ge n_0,$$

where $c > 0$ is some constant. As a result, we have

$$\|K_n * f - f\|_{p,\alpha} \le \|K_n * f - K_n * g\|_{p,\alpha} + \|K_n * g - g\|_{p,\alpha} +$$

$$+ \|g - f\|_{p,\alpha} \le (M + c + 1)\,\varepsilon, \forall n \ge n_0.$$

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