Azerbaijan Journal of Mathematics
V. 7, No 2, 2017, July
ISSN 2218-6816

# On Third Order Coupled Systems with Full Nonlinearities

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**Abstract.** This work studies the solvability of the nonlinear third order coupled system composed by the differential equations

$$\begin{cases} -u'''(t) = f(t, v(t), v'(t), v''(t)) \\ -v'''(t) = h(t, u(t), u'(t), u''(t)), \end{cases}$$

with $f, h : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ $L^1$-Carathéodory functions and the two-point boundary conditions

$$\begin{cases} u(0) = u'(0) = u'(1) = 0 \\ v(0) = v'(0) = v'(1) = 0. \end{cases}$$

An adequate truncature together with Nagumo-type conditions allow the dependence of the nonlinearities on the second derivatives. By lower and upper solutions method we obtain strips where the unknown functions and their derivatives must lie, which provides some qualitative data on the solutions.

**Key Words and Phrases**: coupled systems, Green functions, Nagumo-type condition, coupled lower and upper solutions.

**2010 Mathematics Subject Classifications**: 34B15, 34B27, 34L30

## 1. Introduction

This work gives sufficient conditions for the existence of solution, positive or not, of the nonlinear third order coupled system composed by the differential equations

$$\begin{cases} -u'''(t) = f(t, v(t), v'(t), v''(t)) \\ -v'''(t) = h(t, u(t), u'(t), u''(t)), \end{cases} \tag{1}$$

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where $f,h:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ are $L^1$-Carathéodory functions, and the two-point boundary conditions

$$\begin{cases} u\,(0)=u'\,(0)=u'\,(1)=0, \\ v\,(0)=v'\,(0)=v'\,(1)=0. \end{cases} \tag{2}$$

Moreover, by applying lower and upper solutions technique, the localization part of the result allows us to have some qualitative data about solutions sign, growth or variation, as suggested in [19].

Higher order nonlinear systems of differential equations have had an increasing interest in last years, mostly due to their applications in several fields such as populations dynamics, mechanics, optimal control, harvesting; see [1, 3, 6, 7, 8, 9, 10, 13, 14, 15, 16, 18] and the references therein.

In particular, third order equations model many phenomenons in physics, engineering and physiology, among others. As examples, we mention the flow of a thin film of viscous fluid over a solid surface (see [2, 21]), the deflection of a curved beam having a constant or varying cross section, the solitary waves solution of the Korteweg–de Vries equation ([17]), the thyroid-pituitary interaction ([4]) or vehicles nonlinear suspensions ([11]).

The methods used in the literature for third order coupled systems can not deal with the second derivatives of the unknown functions. See, for example, [20] where the author proves the existence of at least three positive solutions for the boundary-value problem

$$\begin{cases} u'''\,(t)+a(t)\ f(t,u(t),v(t))=0, \quad 0<t<1, \\ v'''\,(t)+b(t)\ h(t,u(t),v(t))=0, \quad 0<t<1, \\ u\,(0)=u'\,(0)=0, \quad u'\,(1)=\beta u'(\eta), \\ v\,(0)=v'\,(0)=0, \quad v'\,(1)=\beta v'(\eta), \end{cases}$$

where $f,h:[0,1]\times[0,\infty)^2\to[0,\infty)$ are continuous and $0<\eta<1,\ \ 1<\beta<1/\eta$, $a(t),b(t)\in C([0,1],[0,\infty))$ and are not identically zero on $[\eta/\beta,\eta]$, applying the Leggett-Williams fixed point theorem. And [12], where the authors study the third order differential equations

$$u_i'''\,(t)+f_i\left(t,u_1(t),...,u_n(t),u_1'(t),...,u_n'(t)\right)=0,\ 0<t<1,\ i=1,...,n,$$

where $f_i:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ are continuous functions, with multi-point integral boundary conditions, via the Guo-Krasnosels'kii fixed point theorem in a cone.

Motivated by the above papers and by those applications which include a second derivative dependence, we consider problem (1), (2). Note that standard cone theory can not be applied to our problem because the second derivative of the Green´s functions, associated to the linear form of (1), changes sign.

Our arguments apply an integral system defined with the Green´s functions as the kernel component, and some auxiliary compact operators, in which an adequate truncature plays a key role. Coupled lower and upper solutions provide a localization tool to establish not only the equivalence between auxiliary and initial problems, but also to give some qualitative properties of the solution. Moreover, a Nagumo-type condition allows an *a priori* control on second derivatives, as in [5].

The paper is organized as follows: Section 2 contains the expression of the Green's functions, coupled lower and upper solutions definitions and *a priori* estimations for the second derivatives. The main theorem, an existence and localization result, is in Section 3. In last section we present an example to show the applicability of our main result.

## 2. Definitions and auxiliary results

Let $E = C^2[0,\, 1]$ be the Banach space equipped with the norm $\|\cdot\|_{C^2}$, defined by

$$\|w\|_{C^2} := \max\left\{\|w\|, \|w'\|, \|w''\|\right\},$$

where

$$\|y\| := \max_{t\in[0,\,1]} |y(t)|$$

and $E^2 = \left(C^2[0,\, 1]\right)^2$ with the norm

$$\|(u,v)\|_{E^2} = \max\left\{\|u\|_{C^2}, \|v\|_{C^2}\right\}.$$

For the reader's convenience, we present the definition of $L^1$-Carathéodory function:

**Definition 1.** *A function $g : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ is a $L^1$-Carathéodory function, if it satisfies the following properties:*

1. *$g(t,\cdot,\cdot,\cdot)$ is continuous in $\mathbb{R}^3$ for a.e. $t\in[0,1]$ .*
2. *$g(\cdot,x,y,z)$ is measurable in $[0,1]$ for all $(x,y,z)\in\mathbb{R}^3$.*
3. *For every $L>0$ there exists $\psi_L \in L^1[0,1]$ such that, for a.e. $t\in[0,1]$ and all $(x,y,z)\in\mathbb{R}^3$ with $\|(x,y,z)\|\le L$,*

$$|g(t,x,y,z)| \le \psi_L(t).$$

**Lemma 1.** *The pair of functions $(u(t),v(t)) \in \left(C^3[0,\,1],\,\mathbb{R}\right)^2$ is a solution of problem (1)-(2) if and only if $(u(t),v(t))$ is a solution of the following system of integral equations:*

$$\begin{cases} u(t) = \int_0^1 G(t,s) f(s,\, v(s),\, v'(s), v''(s))ds, \\ \\ v(t) = \int_0^1 G(t,s) h(s,\, u(s),\, u'(s), u''(s))ds, \end{cases} \tag{3}$$

*where $G(t,s)$ is the Green's function associated to problem (1)-(2), defined by*

$$G(t,s)=\begin{cases} -\frac{t^2 s}{2}-\frac{s^2}{2}+ts & , \ 0\le s\le t\le 1, \\ \\ -\frac{t^2 s}{2}+\frac{t^2}{2} & , \ t\le s\le 1. \end{cases} \tag{4}$$

The proof follows by standard arguments and is omitted.

**Definition 2.** *The pair of functions $(\alpha_1,\alpha_2)\in\left(C^3[0,1]\right)^2$ is called coupled lower solution of (1)-(2) if*

$$\begin{cases} -\alpha_1'''(t)\le f(t,\alpha_1(t),\alpha_1'(t),\alpha_1''(t)), \\ -\alpha_2'''(t)\le h(t,\alpha_2(t),\alpha_2'(t),\alpha_2''(t)), \end{cases}$$

*with*

$$\alpha_1(0)\le 0, \alpha_1'(0)\le 0, \alpha_1'(1)\le 0 \tag{5}$$

*and*

$$\alpha_2(0)\le 0, \alpha_2'(0)\le 0, \alpha_2'(1)\le 0.$$

*The pair $(\beta_1,\beta_2)\in\left(C^3[0,1]\right)^2$ is said to be coupled upper solutions of (1)-(2) if $\beta_1$ and $\beta_2$ verify the reversed inequalities.*

To control the growth of the second derivatives we need Nagumo-type conditions:

**Definition 3.** *The $L^1$-Carathéodory functions $f,h:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ satisfy Nagumo-type conditions if there are positive continuos functions $\phi_1,\phi_2$ such that*

$$|f(t,v_0,v_1,v_2)|\le\phi_1(v_2) \tag{6}$$

*and*

$$|h(t,u_0,u_1,u_2)|\le\phi_2(u_2) \tag{7}$$

*with*

$$\int_0^{+\infty}\frac{s}{\phi_1(s)}ds=+\infty \quad and \quad \int_0^{+\infty}\frac{s}{\phi_2(s)}ds=+\infty. \tag{8}$$

Next lemma gives *a priori* estimations for $u''(t)$ and $v''(t)$ :

**Lemma 2.** *Let $f,h:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ be $L^1$-Carathéodory functions satisfying (6), (7) and (8), in $[0,1]\times\mathbb{R}^3$. Then there exist $R_1,R_2>0$ (not depending on $(u,v)$ ) such that for every solution of (1) verifying*

$$\begin{aligned} \alpha_1^{(i)}(t) &\le u^{(i)}(t)\le\beta_1^{(i)}(t), \\ \alpha_2^{(i)}(t) &\le v^{(i)}(t)\le\beta_2^{(i)}(t), \ for\ i=0,1, \ and\ t\in[0,1], \end{aligned}$$

*we have*

$$||u''||<R_1 \quad and \quad ||v''||<R_2. \tag{9}$$

*Proof.* Let $(u, v)$ be a solution of (1) such that

$$\alpha_1(t) \leq u(t) \leq \beta_1(t),\ \alpha_1'(t) \leq u'(t) \leq \beta_1'(t),\ \text{for } t \in [0,1]\,, \tag{10}$$

and

$$\alpha_2(t) \leq v(t) \leq \beta_2(t),\ \alpha_2'(t) \leq v'(t) \leq \beta_2'(t),\ \text{for } t \in [0,1]\,.$$

Define $r > 0$ such that

$$r := \max\left\{\beta_1'(0) - \alpha_1'(1), \beta_1'(1) - \alpha_1'(0), \beta_2'(0) - \alpha_2'(1), \beta_2'(1) - \alpha_2'(0)\right\} \tag{11}$$

and take $R_1, R_2 > 0$ such that

$$\int_r^{R_1} \frac{s}{\phi_1(s)} ds > \max_{t\in[0,1]} \beta_1'(t) - \min_{t\in[0,1]} \alpha_1'(t) \tag{12}$$

and

$$\int_r^{R_2} \frac{s}{\phi_2(s)} ds > \max_{t\in[0,1]} \beta_2'(t) - \min_{t\in[0,1]} \alpha_2'(t)\,.$$

Let us prove the *a priori* estimation for $u''(t)$. For $v''(t)$ the technique is identical.

If, by contradiction, $|u''(t)| > r, \forall t \in [0,1]\,,$ in the case $u''(t) > r$, for $t \in [0,1]\,,$ by (10) and (11), we have the contradiction

$$\beta_1'(1) - \alpha_1'(0) \geq u'(1) - u'(0) = \int_0^1 u''(t) dt > \int_0^1 r\ dt \geq \beta_1'(1) - \alpha_1'(0).$$

In the case where $u''(t) < -r$, for $t \in [0,1]\,,$ we arrive at a similar contradiction. Therefore there exists $t \in [0,1]$ such that $|u''(t)| < r$.

If $|u''(t)| < r$, $\forall t \in [0,1]$, the proof would be finished assuming $R_1 > r$.

Consider that there is $t_0 \in [0,1[$ such that $|u''(t_0)| > r$. If $u''(t_0) > r$, there is $t^* \in [0,1]$, with $t^* < t_0$, $u''(t^*) = r$ and $u''(t) > r, \forall t \in ]t^*, t_0]$.

By a change of variable,

$$\begin{aligned}
\int_{u''(t^*)}^{u''(t_0)} \frac{s}{\phi_1(s)} ds &= \int_{t^*}^{t_0} \frac{u''(s)}{\phi_1(u''(s))} u'''(s)\, ds \\
&= \int_{t^*}^{t_0} \frac{u''(s)}{\phi_1(u''(s))} f\left(s, v(s), v'(s), v''(s)\right) ds
\end{aligned}$$

$$\leq \int_{t^*}^{t_0} u''(s)\,ds = u'(t_0) - u'(t^*)$$

$$\leq \max_{t\in[0,1]} \beta_1'(t) - \min_{t\in[0,1]} \alpha_1'(t) < \int_r^{R_1} \frac{s}{\phi_1(s)} ds.$$

As $t_0$ is taken arbitrarily on the values where $u''(t_0) > r$, we have $u''(t) < R_1, \forall t \in [0,1]$.

If we assume $u''(t_0) < -r$, the method is analogous. Therefore, $||u''|| < R_1$.

Applying a similar technique and (7) it can be shown that $\|v''\| < R_2$, for some $R_2 > 0$.

The existence tool will be the Schauder's fixed point theorem:

**Theorem 1.** *([22]) Let $Y$ be a nonempty, closed, bounded and convex subset of a Banach space $X$, and suppose that $P : Y \to Y$ is a compact operator. Then $P$ has at least one fixed point in $Y$.*

## 3. Existence and localization theorem

The main theorem will provide the existence and the localization of a solution for the problem (1)-(2).

**Theorem 2.** *Let $f, h : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ be $L^1$-Carathéodory functions satisfying the Nagumo type conditions (6), (7) and (8).*
*If there are coupled lower and upper solutions of (1)-(2), $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$, respectively, such that*

$$(\alpha_1', \alpha_2') \leq (\beta_1', \beta_2'),$$

*that is,*

$$\alpha_1'(t) \leq \beta_1'(t) \text{ and } \alpha_2'(t) \leq \beta_2'(t), \ \forall t \in [0,1],$$

*then there is at least a pair $(u(t), v(t)) \in \left(C^3[0,\,1],\,\mathbb{R}\right)^2$ solution of (1)-(2) and, moreover, for $i = 0, 1$,*

$$\alpha_1^{(i)}(t) \leq u^{(i)}(t) \leq \beta_1^{(i)}(t)$$

*and*

$$\alpha_2^{(i)}(t) \leq v^{(i)}(t) \leq \beta_2^{(i)}(t), \ \forall t \in [0,1].$$

**Remark 1.** *If $\alpha_1'(t) \leq u'(t) \leq \beta_1'(t)$ for $t \in [0,1]$, then by integration in $[0,t]$, and, by (5) and (2),*

$$\alpha_1(t) \leq \alpha_1(t) - \alpha_1(0) \leq u(t) \leq \beta_1(t) - \beta_1(0) \leq \beta_1(t), \ \text{for } t \in [0,1].$$

*Analogously, if* $\alpha_2'(t) \le v'(t) \le \beta_2'(t)$, $\forall t \in [0,1]$, *then*

$$\alpha_2(t) \le v(t) \le \beta_2(t) \text{ for } t \in [0,1].$$

*Proof.* Define the operators $T_1 : E^2 \to E$ , $T_2 : E^2 \to E$ and

$$T(u,v) = (T_1(u,v), T_2(u,v)) \tag{13}$$

with

$$\begin{aligned}
(T_1(u,v))(t) &= \int_0^1 G(t,s) f\left(s, v(s), v'(s), v''(s)\right) ds, \\
(T_2(u,v))(t) &= \int_0^1 G(t,s) h\left(s, u(s), u'(s), u''(s)\right) ds.
\end{aligned}$$

By Lemma 1, the fixed points of $T$ are the solutions of (1)-(2). In the sequel, we prove that $T$ has a fixed point.

Consider the auxiliary operators $T^* : E^2 \to E^2$, $T^*(u,v) = (T_1^*(u,v), T_2^*(u,v))$, where $T_1^* : E^2 \to E$ is given by

$$T_1^*(u(t), v(t)) = \int_0^1 G(t,s) F(s, u(s), v(s))\, ds,$$

with $F(t,u(t),v(t)) := F$ defined as

$$F=\begin{cases} f\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t)\right)-\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|}-\frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} & \text{if } u'(t)>\beta_1'(t),\\ & v'(t)>\beta_2'(t)\\ f\left(t,u(t),u'(t),u''(t)\right)-\frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} & \text{if } \alpha_1'(t)\le u'(t)\le\beta_1'(t),\\ & v'(t)>\beta_2'(t)\\ f\left(t,\alpha_1(t),\alpha_1'(t),\alpha_1''(t)\right)-\frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|}+\frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} & \text{if } u'(t)<\alpha_1'(t),\\ & v'(t)>\beta_2'(t)\\ f\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t)\right)-\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|} & \text{if } u'(t)>\beta_1'(t),\\ & \alpha_2'(t)\le v'(t)\le\beta_2'(t)\\ f\left(t,v(t),v'(t),v''(t)\right) & \text{if } \alpha_1'(t)\le u'(t)\le\beta_1'(t),\ \alpha_2'(t)\le v'(t)\le\beta_2'(t)\\ f\left(t,\alpha_1(t),\alpha_1'(t),\alpha_1''(t)\right)-\frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} & \text{if } u'(t)<\alpha_1'(t),\alpha_2'(t)\le v'(t)\le\beta_2'(t)\\ f\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t)\right)-\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|}+\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} & \text{if } u'(t)>\beta_1'(t),\\ & v'(t)<\alpha_2'(t)\\ f\left(t,u(t),u'(t),u''(t)\right)-\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} & \text{if } \alpha_1'(t)\le u'(t)\le\beta_1'(t),v'(t)<\alpha_2'(t)\\ f\left(t,\alpha_1(t),\alpha_1'(t),\alpha_1''(t)\right)-\frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|}-\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} & \text{if } u'(t)<\alpha_1'(t),\\ & v'(t)<\alpha_2'(t), \end{cases}$$

and $T_2^*: E^2\to E$ is given by

$$T_2^*(u\left(t\right),v\left(t\right))=\int_0^1 G\left(t,s\right)H\left(s,u\left(s\right),v(s)\right)ds,$$

with $H(t,u(t),v(t)) := H$ defined as

$$H = \begin{cases} h\left(t,\beta_2(t),\beta_2'(t),\beta_2''(t)\right) - \frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|} - \frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} & \text{if } u'(t) > \beta_1'(t), \\ & v'(t) > \beta_2'(t) \\ h\left(t,\beta_2(t),\beta_2'(t),\beta_2''(t)\right) - \frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} & \text{if } \alpha_1'(t) \le u'(t) \le \beta_1'(t),\ v'(t) > \beta_2'(t) \\ h\left(t,\beta_2(t),\beta_2'(t),\beta_2''(t)\right) + \frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} - \frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} & \text{if } u'(t) < \alpha_1'(t), \\ & v'(t) > \beta_2'(t) \\ h\left(t,v(t),v'(t),v''(t)\right) - \frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|} & \text{if } u'(t) > \beta_1'(t),\ \alpha_2'(t) \le v'(t) \le \beta_2'(t) \\ h\left(t,u(t),u'(t),u''(t)\right) & \text{if } \alpha_1'(t) \le u'(t) \le \beta_1'(t),\ \alpha_2'(t) \le v'(t) \le \beta_2'(t) \\ h\left(t,v(t),v'(t),v''(t)\right) - \frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} & \text{if } u'(t) < \alpha_1'(t),\ \alpha_2'(t) \le v'(t) \le \beta_2'(t) \\ h\left(t,\alpha_2(t),\alpha_2'(t),\alpha_2''(t)\right) + \frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|} - \frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} & \text{if } u'(t) > \beta_1'(t), \\ & v'(t) < \alpha_2'(t) \\ h\left(t,\alpha_2(t),\alpha_2'(t),\alpha_2''(t)\right) - \frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} & \text{if } \alpha_1'(t) \le u'(t) \le \beta_1'(t),\ v'(t) < \alpha_2'(t) \\ h\left(t,\alpha_2(t),\alpha_2'(t),\alpha_2''(t)\right) - \frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} - \frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} & \text{if } u'(t) < \alpha_1'(t), \\ & v'(t) < \alpha_2'(t). \end{cases}$$

As $f$ and $h$ are $L^1$-Carathéodory functions, it follows $F$ and $H$ are $L^1$-Carathéodory functions, too. Define the compact subset of $E^2$

$$K = \left\{(u,v) \in E^2 : \|(u,v)\|_{E^2} \le L\right\},$$

with $L > 0$ given by

$$L > \max\left\{R_1, R_2, \left\|\alpha_i^{(j)}\right\|, \left\|\beta_i^{(j)}\right\|, i = 1,2,\ j = 0,1,2\right\}, \tag{14}$$

where $R_1, R_2$ are defined in (9). Therefore, by Definition 1, for $(u,v) \in K$, there are positive functions $\psi_{1L}, \psi_{2L} : [0,1] \to (0,+\infty)$ such that $\psi_{1L}, \psi_{2L} \in L^1[0,1]$ and, for $(u,v) \in K$,

$$|F(t,u(t),v(t))| \le \psi_{1L}(t), \text{ for a.e. } t \in [0,1], \tag{15}$$

and

$$|H(t,u(t),v(t))| \le \psi_{2L}(t), \text{ for a.e. } t \in [0,1]. \tag{16}$$

The Green's function $G(t,s)$ is continuous in $[0,1] \times [0,1]$ and, by Remark 1, functions $F(t,u(t),v(t))$ and $H(t,u(t),v(t))$ are bounded. Then $T_1^*(u,v)$ and $T_2^*(u,v)$ are well defined and continuous in $E^2$, and so, the operator $T^*$ is well defined and continuous in $E^2$.

**Step 1**: *$T_1^*$ and $T_2^*$ are completely continuous in* $\left(C^2[0,1]\right)^2$.

The operator $T_1^*$ is continuous in $\left(C^2[0,1]\right)^2$ as $G(t,s)$ and $\frac{\partial G(t,s)}{\partial t}$ are continuous and $f$ is a $L^1$-Carathéodory function. Moreover, $\frac{\partial^2 G(t,s)}{\partial t^2}$ is bounded and therefore

$$\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2} F(s,u(s),v(s))\,ds \text{ is continuous.}$$

In the same way, $T_2^*$ is continuous in $\left(C^2[0,1]\right)^2$.

**Claim 1.1.** *$T_1^*$ and $T_2^*$ are uniformly bounded in* $\left(C^2[0,1]\right)^2$.

Define

$$M(s) := \max\left\{\max_{0\le t\le 1}|G(t,s)|, \max_{0\le t\le 1}\left|\frac{\partial G}{\partial t}(t,s)\right|, \sup_{0\le t\le 1}\left|\frac{\partial^2 G}{\partial t^2}(t,s)\right|\right\}.$$

.

Then, by Lemma (2) and (15),

$$|(T_1^*(u(t),v(t))| \le \int_0^1 |G(t,s)|\,|F(s,u(s),v(s))|\,ds \le \int_0^1 M(s)\ \psi_{1L}(s)\,ds < k_0.$$

Analogously, it can be proved that

$$|\left(T_1^*(u(t),v(t))\right)'| < k_1 \ \ and \ \ |\left(T_1^*(u(t),v(t))\right)''| < k_2,$$

for some $k_0, k_1, k_2 > 0$.

As, for $T_2^*(u,v)$, by (7),

$$|(T_2^*(u(t),v(t))| \le \int_0^1 |G(t,s)|\,|H(s,u(s),v(s))|\,ds \le \int_0^1 M(s)\ \psi_{2L}(s)\,ds < \eta_0,$$

for $\eta_0 > 0$, by similar arguments we have

$$|\left(T_2^*(u(t),v(t))\right)'| < \eta_1 \text{ and } |\left(T_2^*(u(t),v(t))\right)''| < \eta_2,$$

for some $\eta_1, \eta_2 > 0$.

Therefore $T^*$ is uniformly bounded in $\left(C^2[0,1]\right)^2$.

**Claim 1.2.** *$T_1^*$ and $T_2^*$ are equicontinuous in* $\left(C^2[0,1]\right)^2$.

For the first operator $T_1^*$, consider $t_1$, $t_2 \in [0,1]$ and, without loss of generality, suppose $t_1 \leq t_2$. So, by (15),

$$\left|T_1^*(u,v)(t_1) - T_1^*(u,v)(t_2)\right| \leq \int_0^1 \left|G(t_1,s) - G(t_2,s)\right| \ \psi_{1L}(s)\,ds \to 0 \text{ as } t_1 \to t_2.$$

By similar arguments,

$$\left|(T_1^*(u,v))'(t_1) - (T_1^*(u,v))'(t_2)\right| \leq \int_0^1 \left|\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right| \ \psi_{1L}(s)\,ds \to 0$$

as $t_1 \to t_2$, and

$$\begin{aligned}\left|(T_1^*(u,v))''(t_1) - (T_1^*(u,v))''(t_2)\right| &\leq \int_0^1 \left|\frac{\partial^2 G(t_1,s)}{\partial t^2} - \frac{\partial^2 G(t_2,s)}{\partial t^2}\right| \ \psi_{1L}(s)\,ds \\ &\leq \int_{t_1}^{t_2} \psi_{1L}(s)\,ds \to 0 \text{ as } t_1 \to t_2.\end{aligned}$$

The proof that $T_2^*$ is equicontinuous in $\left(C^2[0,1]\right)^2$ follows as above.

By the Arzèla-Ascoli theorem, the operator $T^*(u,v)$ is completely continuous.

**Step 3**: *$T^* : E^2 \to E^2$ has a fixed point.*

In order to apply Theorem 1 for operator $T^*(u,v)$ it remains to prove that $T^*D \subset D$, for some closed, bounded and convex $D \subset E^2$.

Consider $D \subset E^2$ given by $D := \left\{(u,v) \in E^2 : \|(u,v)\|_{E^2} \leq \rho\right\}$, with $\rho > 0$ such that

$$\rho > \max\{R_1, R_2, L, k_i, \eta_i, i = 0,1,2\},$$

where $R_1, R_2$ are given by (9), $L$ by (14), and $k_i, \eta_i$, $i = 0,1,2$, are as in Claim 1.1.

By Claim 1.1, $\left\|(T_1^*(u,v)^{(i)}\right\| \leq k_i$, $i = 0,1,2$, and $\left\|(T_2^*(u,v)^{(i)}\right\| \leq \eta_i$, $i = 0,1,2$. Therefore $\|(T_1^*(u,v)\|_E < \rho$ and $\|(T_2^*(u,v)\|_E < \rho$, that is,

$$\|T^*(u,v)\|_{E^2} < \rho.$$

So, $T^*D \subset D$, and, by Theorem 1, $T^*$ has a fixed point $(u,v) \in D \subset E^2$.

**Step 4**: *This fixed point $(u,v)$ of $T^*$ is also a fixed point of $T$, given by (13).*

As $(u,v)$ is a fixed point of $T^*(u,v)$, it means that $(u,v)$ is a fixed point of $T_1^*(u,v)$ and of $T_2^*(u,v)$.

By standard arguments it can be shown that

$$-u'''(t) = F(t, u(t), v(t))$$

and

$$-v'''(t) = H(t, u(t), v(t)).$$

So, to prove this step it will be enough to show that

$$\alpha_1'(t) \le u'(t) \le \beta_1'(t) \text{ and } \alpha_2'(t) \le v'(t) \le \beta_2'(t), \ \forall t \in [0,1].$$

For the first inequality suppose, by contradiction, that there is $t \in [0,1]$ such that $\alpha_1'(t) > u'(t)$. Define

$$\max_{0 \le t < 1} \left(\alpha_1'(t) - u'(t)\right) := \alpha_1'(t_0) - u'(t_0) > 0.$$

By (2) and (5), $t_0 \neq 0$ because $\alpha'(0) - u'(0) = \alpha'(0) \le 0$. Analogously, $t_0 \neq 1$. So $t_0 \in ]0,1[$ and

$$\alpha_1''(t_0) = u_1''(t_0), \ \alpha_1'''(t_0) - u'''(t_0) \le 0.$$

As $(\alpha_1'(t) - u'(t)) \in C[0,1]$, there is $I \subset [0,1]$ such that $t_0 \in I$ and

$$\begin{array}{rcl} \alpha_1'(t) - u'(t) & > & 0, \\ \alpha_1'''(t) - u'''(t) & \le & 0, \ \forall t \in I. \end{array}$$

For all possible values of $v'(t_0)$, we obtain the following contradictions by the truncature $F$ and Definition 2:

If $v'(t_0) < \alpha_2'(t_0)$, and as $v'(t) - \alpha_2'(t) \in C[0,1]$, then there is $J_0 \subset [0,1]$ such that $t_0 \in J_0$ and $v'(t) - \alpha_2'(t) < 0$, $\forall t \in J_0$.

As $t_0 \in I \cap J_0$, we have $I \cap J_0 \neq \varnothing$ and

$$\begin{aligned} 0 &\ge \int_{I\cap J_0} \left(\alpha_1'''(t) - u'''(t)\right) dt \\ &= \int_{I\cap J_0} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right) - \frac{u'(t) - \alpha_1'(t)}{1 + |u'(t) - \alpha_1'(t)|}\right. \\ &\quad \left. - \frac{v'(t) - \alpha_2'(t)}{1 + |u'(t) - \alpha_2'(t)|}\right) dt \\ &> \int_{I\cap J_0} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right)\right) dt \ge 0. \end{aligned}$$

If $\alpha_2'(t_0) \leq v'(t_0) \leq \beta_2'(t_0)$, and as $v', \alpha_2', \beta_2' \in C[0,1]$, then there exists $J_1 \subset [0,1]$ such that $t_0 \in J_1$ and $\alpha_2'(t) \leq v'(t) \leq \beta_2'(t)$, $\forall t \in J_1$.

As $I \cap J_1 \neq \varnothing$, we have

$$\begin{aligned} 0 &\geq \int_{I\cap J_1} \left(\alpha_1'''(t) - u'''(t)\right) dt \\ &= \int_{I\cap J_1} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right) - \frac{u'(t) - \alpha_1'(t)}{1 + |u'(t) - \alpha_1'(t)|}\right) dt \\ &> \int_{I\cap J_1} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right)\right) dt \geq 0. \end{aligned}$$

If $v'(t_0) > \beta_2'(t_0)$, and as $v'(t) - \beta_2'(t) \in C[0,1]$, then there is $J_2 \subset [0,1]$ such that $t_0 \in J_2$ and $v'(t) - \beta_2'(t) > 0$, $\forall t \in J_2$.

As $t_0 \in I \cap J_2$, we have $I \cap J_2 \neq \varnothing$ and

$$\begin{aligned} 0 &\geq \int_{I\cap J_2} \left(\alpha_1'''(t) - u'''(t)\right) dt \\ &= \int_{I\cap J_2} \left(\alpha_1'''(t) - f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right) - \frac{u'(t) - \alpha_1'(t)}{1 + |u'(t) - \alpha_1'(t)|}\right. \\ &\quad \left. + \frac{v'(t) - \beta_2'(t)}{1 + |v'(t) - \beta_2'(t)|}\right) dt \\ &> \int_{I\cap J_2} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right)\right) dt \geq 0. \end{aligned}$$

Therefore, $\alpha_1'(t) \leq u'(t), \forall t \in [0,1]$. By similar arguments it can be proved that $u'(t) \leq \beta_1'(t), \forall t \in [0,1]$ , and so,

$$\alpha_1'(t) \leq u'(t) \leq \beta_1'(t), \ \forall t \in [0,1]. \tag{17}$$

Applying the same technique with the truncature $H$, it can be achieved that

$$\alpha_2'(t) \leq v'(t) \leq \beta_2'(t), \ \forall t \in [0,1]. \tag{18}$$

So, the fixed point $(u, v)$ of $T^*$ is also a fixed point of $T$, given by (13), and by Lemma 1, $(u(t), v(t))$ is a solution of the problem (1)-(2).

## 4. Example

Consider the system of nonlinear and nonautonomous differential equations

$$\begin{cases} -u'''(t) = & v^3(t) + e^{v'(t)} - 6\sqrt[3]{\left(v''(t)\right)^2}, \\ \\ -v'''(t) = & \frac{t}{4} - \arctan\left(u(t)\right) + \left(u'(t)\right)^3 + 2\left(u''(t)\right)^2, \end{cases} \tag{19}$$

with the boundary conditions (2).

In fact, (19) is a particular case of (1) with

$$f(t,x,y,z) = x^3 + e^y - 6\sqrt[3]{z^2}$$

and

$$h(t,x,y,z) = \frac{t}{4} - \arctan x + y^3 + 2z^2,$$

where $f$ and $h$ are $L^1$-Carathéodory functions.

By easy calculations, it can be seen that the functions

$$\begin{aligned} \alpha_1(t) &= -1, \quad \beta_1(t) = t^2, \\ \alpha_2(t) &= -t^2, \quad \beta_2(t) = 1, \end{aligned}$$

are the coupled lower and upper solutions of (19), (1).

By Theorem 2, there is a solution $(u,v)$ of (19), (2) such that

$$\begin{aligned} -1 &\leq u(t) \leq t^2, \quad -t^2 \leq v(t) \leq 1, \\ 0 &\leq u'(t) \leq 2t, \quad -2t \leq v'(t) \leq 0, \text{ for } t \in [0,1]. \end{aligned}$$

From the localization part, $u(t)$ is a nondecreasing function and $v(t)$ is a nonincreasing one.

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Received 20 April 2016
Accepted 07 March 2017