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# One-dimensional and Multidimensional Hardy Operators in Grand Lebesgue Spaces

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**Abstract.** Grand Lebesgue spaces on sets of infinite measure are defined using an additional characteristic $a(\cdot)$ called a grandizer. Conditions on the grandizer $a(x)$ for the Hardy operators to be bounded in the grand Lebesgue spaces $L_a^{p)}(\mathbb{R}^n)$ are found, and the lower and upper estimates for a sharp constant in the one-dimensional and multidimensional Hardy inequalities are given in dependence on the grandizer. For some special choice of the grandizer it is proved that this sharp constant is equal to the sharp constant for the classical Lebesgue spaces.

**Key Words and Phrases**: grand Lebesgue spaces, Hardy operators, Hardy inequalities, sharp constants, spherical means.

**2010 Mathematics Subject Classifications**: 46E30, 47B38

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## 1. Introduction

We study the one-dimensional and multidimensional Hardy operators in the so-called grand Lebesgue spaces. The theory of grand spaces, in particular the grand Lebesgue spaces, was intensively developed during the last decades. Such grand spaces $L^{p)}(\Omega)$, $1 < p < \infty$, on a bounded set $\Omega \subset \mathbb{R}^n$ were introduced in 1992 by T. Iwaniec and C. Sbordone [9] in connection with some applications for differential equations.

Operators of harmonic analysis were intensively studied in such spaces and they continue to attract attention of researchers in connection with their various applications ([1, 3, 4, 5, 6, 7, 10, 12, 13, 20, 11]). In these studies the set $\Omega$ was assumed to be of finite measure only, since the idea of the construction of the grand space was based on enlargening of $L^p$ when $p$ decreases.

In [24, 25, 30], an approach was proposed which allowed to introduce grand Lebesgue spaces $L_a^{p)}(\Omega)$, $1 < p < \infty$, on sets $\Omega \subseteq \mathbb{R}^n$ of not necessarily finite measure. This approach is based on introducing $a(x)^\varepsilon$ of a weight $a(x)$ with small $\varepsilon > 0$, into the norm of the grand space, see (1). We call the function $a$, defining the grand space $L_a^{p)}(\Omega)$, the *grandizer*. Such grand spaces $L_a^{p)}(\Omega)$ and some operators of harmonic analysis in these spaces have been studied in [28, 31, 29, 32]. The approach suggested in these papers made it possible to consider in grand spaces on $\mathbb{R}^n$ such operators as the Riesz potential, Hardy operators and others. In [26], the conditions on the grandizer $a(x)$ were obtained ensuring the validity of the Sobolev theorem for the Riesz potential in the grand spaces $L_a^{p)}(\mathbb{R}^n)$. In that paper the relationship between the Riesz potential and hypersingular integrals in the grand spaces was also investigated (see [23] regarding the hypersingular integrals).

There is a vast literature on the Hardy operators in the classical Lebesgue spaces. We refer to the books [8, 2, 19, 18, 14] and, for example, to the papers [17, 21, 22], where some other references can be found.

Note that various results concerning grand spaces were presented in the recent two volume book [15], including also some results for sets with infinite measure.

We consider the multidimensional Hardy operators

$$H_n^\alpha f(x) := |x|^{\alpha-n} \int\limits_{|y|<|x|} \frac{f(y)}{|y|^\alpha} dy, \quad \mathcal{H}_n^\beta f(x) := |x|^\beta \int\limits_{|y|>|x|} \frac{f(y)}{|y|^{n+\beta}} dy, \quad x \in \mathbb{R}^n,$$

for $n \geq 2$, and their semi-axis versions for $n = 1$:

$$H^\alpha f(x) := x^{\alpha-1} \int\limits_0^x \frac{f(t)}{t^\alpha} dt, \quad \mathcal{H}^\beta f(x) := x^\beta \int\limits_x^\infty \frac{f(t)}{t^{1+\beta}} dt, \quad x \in \mathbb{R}_+,$$

in the grand spaces $L_a^{p)}(\mathbb{R}^n)$ and $L_a^{p)}(\mathbb{R}_+)$, respectively.

Hardy operators in the natural setting, i.e. over $\mathbb{R}_+$ or $\mathbb{R}^n$, were not studied in grand spaces. In the one-dimensional case, in [20] a criterion for weighted boundedness of the Hardy operator on $[0, 1]$ was found.

In this work, we obtain conditions on the grandizer $a(x)$ for the Hardy operators to be bounded in the grand spaces $L_a^{p)}(\mathbb{R}^n)$ and give the upper estimate for the sharp constant in the Hardy inequalities in dependence on the grandizer $a(x)$. In the case of power type grandizer we provide two-sided estimates for norms of Hardy operators in $L_a^{p)}(\mathbb{R}^n)$. Under the special choice of the grandizer $a(x) = |x|^{-n}$ it is shown that the sharp constant is equal to the sharp constant for the classical Lebesgue spaces.

The paper is organized as follows. In Section 2 we provide necessary preliminaries, mainly on grand Lebesgue spaces. Section 3 contains the main results. In Section 3.1 we prove some technical estimates. In Section 3.2 we prove the boundedness of one-dimensional Hardy operators. We use this result in Section 3.3 to prove estimates for multidimensional Hardy operators via spherical means of the function $f$. Such estimates are stronger result than the boundedness of Hardy operators in grand spaces. Such a result in terms of spherical means is obtained for radial grandizers. Finally, in Section 3.4 we prove a theorem on the boundedness of Hardy operators in the case of general grandizers, but not via spherical means. The main results are given in Theorems 1, 2, 3, 4, 5, 6 7.

Notation.

$S^{n-1}$ is the unit sphere in $\mathbb{R}^n$ centered at the origin, $|S^{n-1}|$ is its area;

$B(x_0, r)$ is the ball in $\mathbb{R}^n$ of radius $r$ centered at the point $x_0$;

$\varphi_*(t) = \sup_{x>0} \frac{\varphi(xt)}{\varphi(x)}$.

## 2. Preliminaries

### 2.1. Grand Lebesgue spaces

Denote $L^p(\Omega, w) := \left\{f : \ \|f\|_{L^p(\Omega,w)} < \infty\right\}$, where

$$\|f\|_{L^p(\Omega,w)} = \left(\int\limits_{\Omega} |f(x)|^p w(x) dx\right)^{\frac{1}{p}}.$$

In the case $w \equiv 1$ we write $L^p(\Omega, w) = L^p(\Omega)$.

Following [30], we define the grand Lebesgue spaces on sets $\Omega$ of finite or infinite measures:

$$L_a^{p)}(\Omega) := \left\{ f : \|f\|_{L_a^{p)}(\Omega)} := \sup_{0<\varepsilon<p-1} \left(\varepsilon \int\limits_{\Omega} |f(x)|^{p-\varepsilon} a(x)^{\frac{\varepsilon}{p}}\, dx\right)^{\frac{1}{p-\varepsilon}} < \infty \right\}, 1 < p < \infty, \quad (1)$$

where $a(x)$ is an arbitrary non-negative function on $\Omega$, called grandizer. The choice of the grandizer may be dictated by problems of study in grand spaces. In [24, 25, 30] it was assumed that $a \in L^1(\Omega)$, which ensures the embedding $L^p(\Omega) \hookrightarrow L_a^{p)}(\Omega)$, see Lemma 1. In this paper we deal with $\Omega = \mathbb{R}_+$ or $\Omega = \mathbb{R}^n$ and when studying Hardy operators, we find it convenient to admit grandizers not necessarily integrable at the origin or infinity, and we always assume it to be locally integrable outside the origin:

$$a(x) \in L^1(B_{\delta N}) \qquad 0 < \delta < N < \infty, \quad (2)$$

where $B_{\delta N} = \{x : \ \delta < |x| < N\}$; in the case $\Omega = \mathbb{R}_+$ we write

$$a(x) \in L^1(\delta, N). \quad (3)$$

So defined grand space in general depends on the grandizer, although different choice of grandizers may lead to the same grand space (see [27]).

**Lemma 1.** *([32]) Let $a \in L^1(\Omega)$. Then the norm* (1) *is equivalent to the norm*

$$\sup_{0<\varepsilon<\varepsilon_0} \left( \varepsilon \int_\Omega |f(x)|^{p-\varepsilon} a(x)^{\frac{\varepsilon}{p}} \, dx \right)^{\frac{1}{p-\varepsilon}}$$

*for any $\varepsilon_0 > 0$.*

When $a \in L^1(\Omega)$, the following embeddings hold:

$$L^p(\Omega) \hookrightarrow L_a^{p)}(\Omega) \hookrightarrow L^{p-\varepsilon_1}(\Omega, a^{\frac{\varepsilon_1}{p}}) \hookrightarrow L^{p-\varepsilon_2}(\Omega, a^{\frac{\varepsilon_2}{p}}), \ 0 < \varepsilon_1 < \varepsilon_2 < p-1. \tag{4}$$

**Remark 1.** *In the case of bounded set $\Omega$, when it is usual to take $a(x) \equiv 1$, there always holds the embedding $L^p(\Omega) \hookrightarrow L_a^{p)}(\Omega)$, i.e. the grand Lebesgue space in this sense is an enlargening of the classical Lebesgue space. According to* (4)*, similar enlargening on unbounded sets is guaranteed by the assumption $a \in L^1(\Omega)$. This condition was usually assumed to hold in the study of grand spaces on unbounded sets, see for instance [30], [26], though grand spaces on such sets may be studied without this condition.*

**Remark 2.** *The procedure of enlargening the Lebesgue space by means of different grandizers may generate the same grand space. Different grand spaces may be obtained by using, for example, grandizers with power and exponential decay at infinity. For instance, when $a(x)$ has a power type decay, the function $\frac{1}{(1+|x|)^\gamma}$ is in $L_a^{p)}(\mathbb{R}^n)$ if and only if $\gamma \geq \frac{n}{p}$, while it belongs to $L_a^{p)}(\mathbb{R}^n)$, $a(x) = e^{-|x|}$, for all $\gamma \geq 0$. Note that $\bigcup_{p\leq q\leq\infty} L^q(\mathbb{R}^n) \subset L_a^{p)}(\mathbb{R}^n)$, $a(x) = e^{-|x|}$. When considering Hardy operators, we admit in particular exponential grandizers, see for instance Corollary 2.*

## 2.2. Dilation function

Let $a(x)$ be a positive, almost everywhere finite function on $\mathbb{R}_+$. The function

$$a_*(t) = \sup_{x>0} \frac{a(xt)}{a(x)} \tag{5}$$

is called a dilation of $a$, see [16, . 53]. The following properties of dilations hold:

$1^0$ If $x^\gamma a(x)$, $\gamma \in \mathbb{R}$, is non-increasing on $\mathbb{R}_+$, then $a_*(t) \leq \frac{1}{t^\gamma}$ for $t > 1$.

$2^0$ If $x^\gamma a(x)$, $\gamma \in \mathbb{R}$, is non-decreasing on $\mathbb{R}_+$, then $a_*(t) \leq \frac{1}{t^\gamma}$ for $t < 1$.

Let $a(x)$ be a positive, almost everywhere finite function on $\mathbb{R}^n \setminus \{0\}$. Define a multidimensional analogue of dilation by the formula

$$a_*(x) := \sup_{y\in\mathbb{R}^n} \frac{a(x|y|)}{a(y)}.$$

**Lemma 2.** *If* $a(x) = b(\rho)g(\sigma)$, $\rho = |x|$, $\sigma = \frac{x}{|x|}$, $b(\rho) \geq 0$, $g(\sigma) > 0$, *then* $a_*(x) \leq b_*(\rho)\frac{g(\sigma)}{\inf\limits_{\sigma\in S^{n-1}} g(\sigma)}$.

The proof is direct.

## 3. Main results

### 3.1. Auxiliary lemmas

Denote $Q_pf(x) = |x|^{-\frac{2n}{p}} f\left(\frac{x}{|x|^2}\right)$, $x \in \mathbb{R}^n$, so that $Q_p^2 = I$. In the case $n = 1$ we interpret the operator $Q_p$ as an operator on functions $f(x)$, $x > 0$. It is an isometry on classical Lebesgue spaces $L^p(\mathbb{R}^n)$:

$$\|Q_pf\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}, \ 1 \leq p \leq \infty. \tag{6}$$

The following relationship

$$H_n^\alpha = Q_p\mathcal{H}_n^\beta Q_p, \quad \alpha+\beta = \frac{p-2}{p}n, \tag{7}$$

between the Hardy operators is known, see, e.g., [21]. In the case of grand spaces the isometry (6) in general does not hold, but some modification of it is valid as stated in the following lemma. In this lemma and everywhere in the sequel we use the notation

$$\tilde{a}(x) = \frac{1}{|x|^{2n}} a\left(\frac{x}{|x|^2}\right).$$

**Lemma 3.** *The following isometry holds:*

$$\|Q_pf\|_{L_a^{p)}(\mathbb{R}^n)} = \|f\|_{L_{\tilde{a}}^{p)}(\mathbb{R}^n)}, \quad 1 < p < \infty, \tag{8}$$

*and*

$$\|\mathcal{H}_n^\beta\|_{L_a^{p)}(\mathbb{R}^n)} = \|H_n^\alpha\|_{L_{\tilde{a}}^{p)}(\mathbb{R}^n)}, \quad \alpha+\beta = \frac{p-2}{p}n. \tag{9}$$

*Proof.* We have

$$\|Q_pf\|_{L_a^{p)}(\mathbb{R}^n)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon\int\limits_{\mathbb{R}^n}\left|\frac{f\left(\frac{x}{|x|^2}\right)}{|x|^{\frac{2n}{p}}}\right|^{p-\varepsilon} a(x)^{\frac{\varepsilon}{p}}dx\right)^{\frac{1}{p-\varepsilon}}.$$

Whence after the change of variables $\frac{x}{|x|^2} = y$, $dx = \frac{1}{|y|^{2n}}dy$, we get (8).

Furthermore, taking the inequalities (8) and (7) into account, we obtain

$$\|\mathcal{H}_n^\beta\|_{L_a^{p)}(\mathbb{R}^n)} = \sup_f \frac{\|\mathcal{H}_n^\beta f\|_{L_a^{p)}(\mathbb{R}^n)}}{\|f\|_{L_a^{p)}(\mathbb{R}^n)}} = \sup_f \frac{\|H_n^\alpha Q_p f\|_{L_{\tilde{a}}^{p)}(\mathbb{R}^n)}}{\|Q_p f\|_{L_{\tilde{a}}^{p)}(\mathbb{R}^n)}} = \|H_n^\alpha\|_{L_{\tilde{a}}^{p)}(\mathbb{R}^n)}.$$

◄

The next lemma provides estimates for

$$I_\mu(a) := \sup_{0<\varepsilon<p-1} \int_1^\infty t^{-\mu-1} \left[t^n a_*(t)\right]^{\frac{\varepsilon}{p(p-\varepsilon)}} dt,$$

$$J_\nu(a) := \sup_{0<\varepsilon<p-1} \int_0^1 t^{\nu-1} \left[t^n a_*(t)\right]^{\frac{\varepsilon}{p(p-\varepsilon)}} dt,$$

where $a_*(t)$ is the dilation of $a(x)$.

**Lemma 4.** *Let $1 < p < \infty$, $\mu > 0$, $\nu > 0$, $n = 1, 2, ...$ and $A(r)$ be a non-negative function. If $r^\gamma A(r)$ is non-increasing on $\mathbb{R}_+$ for some $\gamma > n - \mu p'$, then*

$$I_\mu(A) \leq \frac{1}{\mu - \frac{\max(n-\gamma, 0)}{p'}}.$$

*If $r^\lambda A(r)$ is non-decreasing on $\mathbb{R}_+$ for some $\lambda < n + \nu p'$, then*

$$J_\nu(A) \leq \frac{1}{\nu + \frac{\min(n-\lambda, 0)}{p'}}.$$

*Proof.* Let us prove the estimate for $I_\mu(A)$. It is easy to see that $A_*(t) \leq t^{-\gamma}$ for $t \geq 1$ under the assumptions of the lemma. Therefore

$$I_\mu(A) \leq \max_{0 \leq \varepsilon \leq p-1} \int_1^\infty t^{-\mu-1+\frac{\varepsilon(n-\gamma)}{p(p-\varepsilon)}} dt,$$

and the proof of the estimate is obtained by calculation of the integral and finding the maximum.

The proof of the estimate for $J_\nu(A)$ is similar. ◄

For lower estimates for the norms of the Hardy operators we need the following lemma.

**Lemma 5.** *Let* $a(x) = |x|^{-\lambda}$ *for* $x \in B(0,1)$*, where* $\lambda \in \mathbb{R}$*. Then* $|x|^{-\frac{\nu}{p}} \in L_a^{p)}(B(0,1))$*, if*

$$\frac{\nu}{p} + \frac{\lambda}{p'} < n \tag{10}$$

*when* $\lambda > n$ *and* $\nu \leq n$ *when* $\lambda \leq n$*.*

*Proof.* We have to verify that

$$\sup_{0<\varepsilon<p-1} \varepsilon \int\limits_{|x|<1} |x|^{-\frac{\nu}{p}(p-\varepsilon)-\lambda\frac{\varepsilon}{p}} dx < \infty.$$

Under the choice $\nu < \lambda$ the integral should exist for $\varepsilon = p-1$, which yields the condition (10). Under the choice $\nu \geq \lambda$ we have to ensure the existence of the integral for $\varepsilon \to 0$, which leads to the condition $\nu \leq n$, and then the supremum in the case $\nu = n$ is finite due to the factor $\varepsilon$. It remains to note that the conditions $\nu < \lambda$ and $\nu \geq \lambda$ arising in this calculation are guaranteed by the assumptions $n < \lambda$ and $n \geq \lambda$, respectively. ◄

### 3.2. One-dimensional case

In relation to Remark 1, note the following. The nature of the Hardy operators is such that it has a sense to admit grandizers $a$ non-integrable at the origin and infinity in order to obtain some natural results for these operators in grand spaces. This in particular can be seen in the theorems below.

Denote

$$c_\alpha(a) := \sup_{0<\varepsilon<p-1} \int\limits_1^\infty t^{\alpha+\frac{1}{p-\varepsilon}-2} a_*(t)^{\frac{\varepsilon}{p(p-\varepsilon)}}\, dt, \tag{11}$$

$$d_\beta(a) := \sup_{0<\varepsilon<p-1} \int\limits_0^1 t^{\beta+\frac{1}{p-\varepsilon}-1} a_*(t)^{\frac{\varepsilon}{p(p-\varepsilon)}}\, dt, \tag{12}$$

where $a_*$ is the dilation of $a$ as defined in (5).

**Theorem 1.** *Let* $1 < p < \infty$ *and* $a$ *be a non-negative function on* $\mathbb{R}_+$*, satisfying the integrability condition* (3)*.*

*The conditions* $c_\alpha(a) < \infty$ *and* $d_\beta(a) < \infty$ *are sufficient for the boundedness of the operators* $H^\alpha$ *and* $\mathcal{H}^\beta$*, respectively, in the grand space* $L_a^{p)}(\mathbb{R}_+)$*, and*

$$\|H^\alpha\|_{L_a^{p)}(\mathbb{R}_+)} \leq c_\alpha(a), \quad \|\mathcal{H}^\beta\|_{L_a^{p)}(\mathbb{R}_+)} \leq d_\beta(a). \tag{13}$$

*If $\alpha \geq \frac{1}{p'}$, then the operator $H^\alpha$ is not bounded in any grand space $L^{p)}_a(\mathbb{R}_+)$ with grandizer $a$ integrable in a neighbourhood $(0,\tau)$ of the origin, $\tau > 0$. Similarly, if $a \in L^1(N,\infty)$ for some $N > 0$, then the condition $\beta > -\frac{1}{p}$ is necessary for the boundedness of the operator $\mathcal{H}^\beta$ in $L^{p)}_a(\mathbb{R}_+)$, and*

$$\frac{1}{\frac{1}{p'} - \alpha} \leq \|H^\alpha\|_{L^{p)}_a(\mathbb{R}_+)}, \quad \alpha < \frac{1}{p'}, \tag{14}$$

$$\frac{1}{\frac{1}{p} + \beta} \leq \|\mathcal{H}^\beta\|_{L^{p)}_a(\mathbb{R}_+)}, \quad \beta > -\frac{1}{p}. \tag{15}$$

*If $\alpha < \frac{1}{p'}$ and $x^\gamma a(x)$ is non-increasing in the case of the operator $H^\alpha$ and $\beta > -\frac{1}{p}$ and $x^\lambda a(x)$ is non-decreasing in the case of the operator $\mathcal{H}^\beta$ for some $\gamma > \alpha p'$ and $\lambda < (\beta+1)p'$, then*

$$\|H^\alpha\|_{L^{p)}_a(\mathbb{R}_+)} \leq \frac{1}{\frac{\min\{\gamma,1\}}{p'} - \alpha}, \tag{16}$$

$$\|\mathcal{H}^\beta\|_{L^{p)}_a(\mathbb{R}_+)} \leq \frac{1}{\beta + 1 - \frac{\max\{\lambda,1\}}{p'}}. \tag{17}$$

*Proof.*

We first prove

$$\|\mathcal{H}^\beta f\|_{L^{p)}_a(\mathbb{R}_+)} \leq d_\beta(a) \|f\|_{L^{p)}_a(\mathbb{R}_+)}. \tag{18}$$

Since $\mathcal{H}^\beta f(x) = \int\limits_1^\infty \frac{f(xt)}{t^{1+\beta}} dt$, by Minkowsky inequality for $L^{p-\varepsilon}$ we get

$$\begin{aligned}\left\{\int\limits_0^\infty |\mathcal{H}^\beta f(y)|^{p-\varepsilon} a(y)^{\frac{\varepsilon}{p}} dy\right\}^{\frac{1}{p-\varepsilon}} &\leq \int\limits_1^\infty t^{-\beta-1} \left\{\int\limits_0^\infty |f(ty)|^{p-\varepsilon} a(y)^{\frac{\varepsilon}{p}} dy\right\}^{\frac{1}{p-\varepsilon}} dt \\ &= \int\limits_1^\infty t^{-\beta-1-\frac{1}{p-\varepsilon}} \left\{\int\limits_0^\infty |f(x)|^{p-\varepsilon} a\left(\frac{x}{t}\right)^{\frac{\varepsilon}{p}} dx\right\}^{\frac{1}{p-\varepsilon}} dt.\end{aligned}$$

Taking into account that $a\left(\frac{x}{t}\right) \leq a(x) a_*\left(\frac{1}{t}\right)$, $x \in \mathbb{R}_+$, and making the inversion change of the variable in the integral in $t$, we obtain the inequality

$$\left\{\int\limits_0^\infty |\mathcal{H}^\beta f(y)|^{p-\varepsilon} a(y)^{\frac{\varepsilon}{p}} dy\right\}^{\frac{1}{p-\varepsilon}} \leq \int\limits_0^1 t^{\beta+\frac{1}{p-\varepsilon}-1} a_*(t)^{\frac{\varepsilon}{p(p-\varepsilon)}} dt \left\{\int\limits_0^\infty |f(x)|^{p-\varepsilon} a(x)^{\frac{\varepsilon}{p}} dx\right\}^{\frac{1}{p-\varepsilon}}. \tag{19}$$

Hence (18) follows.

The estimate $\|H^\alpha f\|_{L^{p)}_a(\mathbb{R}_+)} \leq c_\alpha(a)\|f\|_{L^{p)}_a(\mathbb{R}_+)}$ is obtained from (18) by means of the formula (9).

Let as show the necessity of the conditions on $\alpha$ and $\beta$ mentioned in the theorem and prove the lower estimates for the operator norms. In the case of the operator $\mathcal{H}^\beta$ we choose the function

$$f_\delta(x) = \begin{cases} 0, & 0 < x \leq N, \\ x^{-\frac{1}{p}-\delta}, & x > N, \end{cases} \quad \delta > 0.$$

By the Hölder inequality with the exponent $\frac{p}{\varepsilon}$, it is easy to check that $\|f_\delta\|_{L^{p)}_a(\mathbb{R}_+)} < \infty$ for all $\delta > 0$. If $\mathcal{H}^\beta$ is bounded in $L^{p)}_a(\mathbb{R}_+)$, it must be defined on the functions $f_\delta \in L^{p)}_a(\mathbb{R}_+)$ for all $\delta > 0$. For $x > N$ we have

$$\mathcal{H}^\beta f_\delta = x^\beta \int\limits_x^\infty \frac{dt}{t^{\frac{1}{p}+\delta+\beta+1}},$$

which is finite, if $\beta > -\frac{1}{p} - \delta$. Since $\delta$ is arbitrary, we arrive at the condition $\beta \geq -\frac{1}{p}$. Moreover

$$\|\mathcal{H}^\beta\|_{L^{p)}_a(\mathbb{R}_+)} \geq \frac{\|\mathcal{H}^\beta f_\delta\|_{L^{p)}_a(N,\infty)}}{\|f_\delta\|_{L^{p)}_a(\mathbb{R}_+)}} = \frac{1}{\beta + \frac{1}{p} + \delta}.$$

Passing to the limit as $\delta \to 0$, we get the inequality (15). Also, in the case $\beta = -\frac{1}{p}$ the boundedness of the operator $\mathcal{H}^\beta$ is obviously excluded.

The lower estimate for the operator $\|H^\alpha\|$ is similarly obtained by means of the function

$$f_\delta(x) = \begin{cases} x^{-\frac{1}{p}+\delta}, & 0 < x \leq \tau \\ 0, & x > \tau, \end{cases} \quad \delta > 0.$$

The estimates (16) and (17) are consequences of the estimates (13) in view of Lemma 4. ◀

**Corollary 1.** *Let $a(x) = x^{-\gamma}(1+x)^{-\mu}$, where $\gamma > 0$, $\mu \geq 0$ and $\gamma < 1$ in the case of the operator $H^\alpha$ and $\gamma + \mu > 1$ in the case of the operator $\mathcal{H}^\beta$. Then the conditions $\alpha < \frac{1}{p'}$ and $\beta > -\frac{1}{p}$ are necessary for the boundedness of the operators $H^\alpha$ and $\mathcal{H}^\beta$, respectively, in the grand space $L^{p)}_a(\mathbb{R}_+)$ and*

$$\frac{1}{\frac{1}{p'} - \alpha} \leq \|H^\alpha\|_{L^{p)}_a(\mathbb{R}_+)} \leq \frac{1}{\frac{\gamma}{p'} - \alpha}, \tag{20}$$

$$\frac{1}{\frac{1}{p}+\beta} \leq \|\mathcal{H}^{\beta}\|_{L_a^{p)}(\mathbb{R}_+)} \leq \frac{1}{\beta+1-\frac{\gamma+\mu}{p'}} \tag{21}$$

*in the cases $\alpha p' < \gamma < 1$ and $1 < \gamma + \mu < (1+\beta)p'$, respectively.*

*Proof.* It is easy to see that $a_*(x) = \begin{cases} \frac{1}{x^{\gamma+\mu}}, & 0 < x < 1, \\ \frac{1}{x^{\gamma}}, & 1 < x < \infty, \end{cases}$ and then the direct calculations yield $c_\alpha(a) = \begin{cases} \frac{1}{\frac{\gamma}{p'}-\alpha}, & \gamma \leq 1, \\ \frac{1}{\frac{1}{p'}-\alpha}, & \gamma > 1, \end{cases}$ which proves the right-hand side estimate in (20). To obtain the left-hand side estimate, in view of Theorem 1 it suffices to note that $a$ is integrable in a neighbourhood of the origin.

The estimate (21) is proved similarly. ◀

**Corollary 2.** *Let $a(x) = e^{-x}$. The operator $H^\alpha$ is bounded in $L_a^{p)}(\mathbb{R}_+)$, if $\alpha < \frac{1}{p'}$.*

*Proof.* Note that $a_*(t) \leq 1$, $t \geq 1$, for $a(x) = e^{-x}$. According to the arguments in the proof of Theorem 1, in view of Lemma 1 we see that there holds the boundedness

$$\|H^\alpha f\|_{L_a^{p)}(\mathbb{R}_+)} \leq c(\varepsilon_0) \sup_{0<\varepsilon<\varepsilon_0} \int_1^\infty t^{\alpha+\frac{1}{p-\varepsilon}-2} dt \|f\|_{L_a^{p)}(\mathbb{R}_+)},$$

where $c(\varepsilon_0)$ is the constant arising from the equivalence of norms provided by Lemma 1. The integral on the right-hand side converges if $\varepsilon = 0$ and consequently for all $0 < \varepsilon \leq \varepsilon_0$ with any sufficiently small $\varepsilon_0$: $\varepsilon_0 < p - \frac{1}{1-\alpha}$. ◀

In Theorems 2 and 3 we provide, in case of power grandizers, more precise two-sided estimates for the Hardy operators, including the case which allows the explicit calculation of the norm of the operators. We omit the proofs of these theorems, since they are contained in the proofs of Theorems 5 and 6, under the change of $\mathbb{R}^n$ by $\mathbb{R}_+$.

**Theorem 2.** *Let $a(x) = x^{-\lambda}$, $\lambda \in \mathbb{R}$. The condition $\alpha < \frac{\min\{1,\lambda\}}{p'}$ is sufficient and the condition $\alpha < \frac{\max\{1,\lambda\}}{p'}$ is necessary for the boundedness of the operator $H^\alpha$ in the grand space $L_a^{p)}(\mathbb{R}_+)$ and under these conditions*

$$\frac{1}{\frac{\max\{1,\lambda\}}{p'}-\alpha} \leq \|H^\alpha\|_{L_a^{p)}(\mathbb{R}_+)} \leq \frac{1}{\frac{\min\{1,\lambda\}}{p'}-\alpha}.$$

*In particular, when $\lambda = 1$, the operator $H^\alpha$ is bounded in grand space $L_a^{p)}(\mathbb{R}_+)$ if and only if $\alpha < \frac{1}{p'}$, and $\|H^\alpha\|_{L_a^{p)}(\mathbb{R}_+)} = \frac{1}{\frac{1}{p'}-\alpha}$.*

**Theorem 3.** *Let $a(x) = x^{-\lambda}$, $\lambda \in \mathbb{R}$. The condition $\beta > -\frac{\min\{1,\lambda\}}{p}$ is sufficient and the condition $\beta > -\frac{\max\{1,\lambda\}}{p}$ is necessary for the boundedness of the operator $\mathcal{H}^\beta$ in the grand space $L^{p)}_a(\mathbb{R}_+)$, and under these conditions*

$$\frac{1}{1 - \frac{\min\{1,\lambda\}}{p'} + \beta} \le \|\mathcal{H}^\beta\|_{L^{p)}_a(\mathbb{R}_+)} \le \frac{1}{1 - \frac{\max\{1,\lambda\}}{p'} + \beta}.$$

*In particular, when $\lambda = 1$, the operator $\mathcal{H}^\beta$ is bounded in grand space $L^{p)}_a(\mathbb{R}_+)$ if and only if $\beta > -\frac{1}{p}$, and $\|\mathcal{H}^\beta\|_{L^{p)}_a(\mathbb{R}_+)} = \frac{1}{\frac{1}{p}+\beta}$.*

### 3.3. Multidimensional case: estimates via spherical means

We call a function $f(x)$, $x \in \mathbb{R}^n$, radial, if it depends only on $|x|$. Let

$$F(\rho) := \frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}} f(\rho\sigma)d\sigma$$

and $\mathcal{F}(x) = F(|x|)$. We write $\mathcal{A}(x) = A(|x|)$ in the case of radial grandizer.

Multidimensional Hardy operators $H^\alpha_n$ and $\mathcal{H}^\beta_n$ are radical to one-dimensional Hardy operators $H^\alpha$ and $\mathcal{H}^\beta$, respectively, by the formulas

$$H^\alpha_n f(x) = |S^{n-1}| H^{\alpha-n+1} F(\rho);$$

$$\mathcal{H}^\beta_n f(x) = |S^{n-1}| \mathcal{H}^\beta F(\rho), \ \rho = |x|. \tag{22}$$

The following inequalities

$$\int\limits_{S_{n-1}} a(\rho\sigma)^\lambda d\sigma \le |S^{n-1}| A(\rho)^\lambda, \quad 0 < \lambda < 1, \tag{23}$$

$$|F(\rho)|^q \le \frac{1}{|S^{n-1}|} \int\limits_{S_{n-1}} |f(\rho\sigma)|^q d\sigma, \quad q \ge 1, \tag{24}$$

are obvious.

The lemmas below are direct consequences of the inequalities (23) and (24).

**Lemma 6.** *For a radial function $f$ the following holds:*

$$\|f\|_{L^{p)}_a(\mathbb{R}^n)} \le \|f\|_{L^{p)}_{\mathcal{A}}(\mathbb{R}^n)}$$

*where $a(x)$ is an arbitrary grandizer on $\mathbb{R}^n$ such that $\frac{1}{|S^{n-1}|} \int\limits_{S_{n-1}} a(\rho\sigma)d\sigma = \mathcal{A}(x)$.*

**Lemma 7.** *In case of non necessarily radial function $f$ and radial grandizer $\mathcal{A}$ the following estimate for spherical means is valid:*

$$\|\mathcal{F}\|_{L^{p)}_{\mathcal{A}}(\mathbb{R}^n)} \leq \|f\|_{L^{p)}_{\mathcal{A}}(\mathbb{R}^n)}. \tag{25}$$

By means of Theorem 1 the estimates

$$\|H^{\alpha}_{n} f\|_{L^{p)}_{a}(\mathbb{R}^n)} \leq C\|\mathcal{F}\|_{L^{p)}_{\mathcal{A}}(\mathbb{R}^n)}, \tag{26}$$

$$\|\mathcal{H}^{\beta}_{n} f\|_{L^{p)}_{a}(\mathbb{R}^n)} \leq C\|\mathcal{F}\|_{L^{p)}_{\mathcal{A}}(\mathbb{R}^n)} \tag{27}$$

for Hardy operators via spherical means will be proved.

In view of Lemma 7, the estimates via $L^{p)}$-norms of spherical means of the function $f$ are a stronger result than the estimates via $L^{p)}(\mathbb{R}^n)$-norms of the function $f$ itself, since the left-hand side in (25) may be finite, but the right-hand side infinite.

In the theorem below we use the notation

$$c_{n,\alpha}(A) := |S^{n-1}| \sup_{0<\varepsilon<p-1} \int_1^\infty t^{\alpha-\frac{n}{(p-\varepsilon)'}-1} A_*(t)^{\frac{\varepsilon}{p(p-\varepsilon)}}\, dt,$$

$$d_{n,\beta}(A) := |S^{n-1}| \sup_{0<\varepsilon<p-1} \int_0^1 t^{\beta+\frac{n}{p-\varepsilon}-1} A_*(t)^{\frac{\varepsilon}{p(p-\varepsilon)}}\, dt.$$

**Theorem 4.** *Let $1 < p < \infty$, $\alpha, \beta \in \mathbb{R}$, $A(r)$ be a non-negative function on $\mathbb{R}_+$ satisfying the integrability condition* (3) *and $a(x)$ be an arbitrary grandizer on $\mathbb{R}^n$, the spherical mean of which is equal to $A(r)$.*

*I. If $c_{n,\alpha}(A) < \infty$ and $d_{n,\beta}(A) < \infty$, then the estimates* (26) *and* (27) *hold with $C = c_{n,\alpha}(A)$ and $C = d_{n,\beta}(A)$, respectively.*

*II. If $\mathcal{A} \in L^1(B(0,\tau))$ for some $\tau > 0$, then the condition $\alpha < \frac{n}{p'}$ is necessary for the inequality* (26) *to hold, and the sharp constant $C^*$ in* (26) *satisfies the inequality $\dfrac{|S^{n-1}|}{\frac{n}{p'}-\alpha} \leq C^*$. If $\mathcal{A} \in L^1(\mathbb{R}^n \setminus B(0,N))$ for some $N > 0$, then the condition $\beta > -\frac{n}{p}$ is necessary for the inequality* (27)*to hold, and the sharp constant $C^*$ in* (27) *satisfies the inequality $\dfrac{|S^{n-1}|}{\frac{n}{p}+\beta} \leq C^*$.*

*III. If $\alpha < \frac{n}{p'}$ and the function $r^{\gamma}A(r)$ is non-increating on $\mathbb{R}_+$ for some $\gamma > \alpha p'$, then* (26) *holds with the sharp constant*

$$C^* \leq \frac{|S^{n-1}|}{\frac{\min\{\gamma,n\}}{p'} - \alpha}.$$

*If* $\beta > -\frac{n}{p}$ *and the function* $r^{\lambda}A(r)$ *is non-decreasing on* $\mathbb{R}_+$ *for some* $\lambda < (\beta+n)p'$, *then* (27) *holds with the sharp constant*

$$C^* \le \frac{|S^{n-1}|}{n - \frac{\max\{\lambda,n\}}{p'} + \beta}. \tag{28}$$

*Proof.* The proof will be given for the operator $\mathcal{H}_n^{\beta}$. The proof for the operator $H_n^{\alpha}$ may be either similarly conducted, or derived from the relations (7) and (9).

I. In view of the equality in (22) and the inequality (23), we get

$$\begin{aligned}
\|\mathcal{H}_n^{\beta} f\|_{L_a^{p)}(\mathbb{R}^n)} &= |S^{n-1}| \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^{\infty} \rho^{n-1} \left|\mathcal{H}^{\beta}F(\rho)\right|^{p-\varepsilon} \int_{S_{n-1}} a(\rho\sigma)^{\frac{\varepsilon}{p}} d\sigma d\rho\right)^{\frac{1}{p-\varepsilon}} \\
&\le |S^{n-1}| \sup_{0<\varepsilon<p-1} \left(\varepsilon |S^{n-1}| \int_0^{\infty} \rho^{n-1} \left|\mathcal{H}^{\beta}F(\rho)\right|^{p-\varepsilon} A(\rho)^{\frac{\varepsilon}{p}} d\rho\right)^{\frac{1}{p-\varepsilon}} \\
&= |S^{n-1}| \sup_{0<\varepsilon<p-1} \left(\varepsilon |S^{n-1}| \int_0^{\infty} \left|\rho^{\frac{n-1}{p}} \mathcal{H}^{\beta}F(\rho)\right|^{p-\varepsilon} \left[\rho^{n-1}A(\rho)\right]^{\frac{\varepsilon}{p}} d\rho\right)^{\frac{1}{p-\varepsilon}} \\
&= |S^{n-1}| \sup_{0<\varepsilon<p-1} \left(\varepsilon |S^{n-1}| \int_0^{\infty} \left|\mathcal{H}^{\beta+\frac{n-1}{p}} F_p(\rho)\right|^{p-\varepsilon} \left[\rho^{n-1}A(\rho)\right]^{\frac{\varepsilon}{p}} d\rho\right)^{\frac{1}{p-\varepsilon}},
\end{aligned}$$

where $F_p(\rho) = \rho^{\frac{n-1}{p}} F(\rho)$. Taking the inequality (19) into account, we obtain

$$\begin{aligned}
\|\mathcal{H}_n^{\beta} f\|_{L_a^{p)}(\mathbb{R}^n)} &\le |S^{n-1}| \sup_{0<\varepsilon<p-1} \int_0^1 r^{\beta+\frac{n-1}{p}+\frac{1}{p-\varepsilon}-1} \left[r^{n-1}A_*(r)\right]^{\frac{\varepsilon}{p(p-\varepsilon)}} dr \\
&\quad \times \left(\varepsilon \int_0^{\infty} |F_p(\rho)|^{p-\varepsilon} \left[\rho^{n-1}A(\rho)\right]^{\frac{\varepsilon}{p}} d\rho\right)^{\frac{1}{p-\varepsilon}} \\
&= |S^{n-1}| \sup_{0<\varepsilon<p-1} \int_0^1 r^{\beta+\frac{n}{p-\varepsilon}-1} A_*(r)^{\frac{\varepsilon}{p(p-\varepsilon)}} dr \\
&\quad \times \left(\varepsilon \int_0^{\infty} |F(\rho)|^{p-\varepsilon} A(\rho)^{\frac{\varepsilon}{p}} |S^{n-1}| \rho^{n-1} d\rho\right)^{\frac{1}{p-\varepsilon}},
\end{aligned}$$

which proves (27).

II. We choose the function

$$f_\delta(x) = \begin{cases} 0, & |x| \le N, \\ |x|^{-\frac{n}{p}-\delta}, & |x| > N, \end{cases} \quad \delta > 0.$$

Since the spherical means $F_\delta$ of a radial function $f_\delta$ coincide with the function itself, by applying Hölder inequality with the exponent $\frac{p}{\varepsilon}$ it can be easily checked that $\|f_\delta\|_{L^{p)}_{\mathcal{A}}(\mathbb{R}^n)} < \infty$ for all $\delta > 0$. If (27) holds for the operator $\mathcal{H}^\beta_n$, then this operator should be defined on the functions $f_\delta \in L^{p)}_{\mathcal{A}}(\mathbb{R}^n)$ for all $\delta > 0$. For the existence of the integral defining $\mathcal{H}^\beta_n f_\delta$ it is necessary that $\beta > -\frac{n}{p} - \delta$ and then for $|x| > N$ we have

$$\mathcal{H}^\beta_n f_\delta(x) = \frac{|S^{n-1}|}{\beta + \frac{n}{p} + \delta} f_\delta(x).$$

Due to the arbitrariness of $\delta$ we arrive at the condition $\beta \ge -\frac{n}{p}$. We have

$$\begin{aligned} \|\mathcal{H}^\beta_n\|_{L^{p)}_{\mathcal{A}}(\mathbb{R}^n) \to L^{p)}_a(\mathbb{R}^n)} &\ge \frac{\|\mathcal{H}^\beta_n f_\delta\|_{L^{p)}_a(\mathbb{R}^n \setminus B(0,N))}}{\|f_\delta\|_{L^{p)}_{\mathcal{A}}(\mathbb{R}^n)}} \\ &= \frac{|S^{n-1}|}{\beta + \frac{n}{p} + \delta} \frac{\|f_\delta\|_{L^{p)}_a(\mathbb{R}^n \setminus B(0,N))}}{\|f_\delta\|_{L^{p)}_{\mathcal{A}}(\mathbb{R}^n \setminus B(0,N))}} \\ &\ge \frac{|S^{n-1}|}{\beta + \frac{n}{p} + \delta}, \end{aligned}$$

where we used Lemma 6. Passing to the limit as $\delta \to 0$, we obtain the lower estimate for the sharp constant $C^*$. Also, in the case $\beta = -\frac{n}{p}$ the boundedness of the operator $\mathcal{H}^\beta_n$ is obviously excluded.

III. To prove the inequality (28), it suffices to apply Lemma 4 and note that

$$d_{n,\beta}(A) = J_{\frac{n}{p}+\beta}(A).$$

◀

**Corollary 3.** *Let $a(x) = A(|x|)$. The conditions $c_{n,\alpha}(A) < \infty$ and $d_{n,\beta}(A) < \infty$ are sufficient and the conditions $\alpha < \frac{n}{p'}$ and $\beta > -\frac{n}{p}$ are necessary for the boundedness of the operators $H^\alpha_n$ and $\mathcal{H}^\beta_n$, respectively, in $L^{p)}_a(\mathbb{R}^n)$, and*

$$\|H^\alpha_n\|_{L^{p)}_a(\mathbb{R}^n)} \ge \frac{|S^{n-1}|}{\frac{n}{p'} - \alpha}, \quad \|\mathcal{H}^\beta_n\|_{L^{p)}_a(\mathbb{R}^n)} \ge \frac{|S^{n-1}|}{\frac{n}{p} + \beta}.$$

*Proof.* It suffices to use Lemma 7. ◀

**Theorem 5.** *Let $a(x) = |x|^{-\lambda}$, $\lambda \in \mathbb{R}$. The condition $\alpha < \frac{\min\{n,\lambda\}}{p'}$ is sufficient and the condition $\alpha < \frac{\max\{n,\lambda\}}{p'}$ is necessary for the boundedness of the operator $H_n^\alpha$ in the grand space $L_a^{p)}(\mathbb{R}^n)$ and under these conditions*

$$\frac{|S^{n-1}|}{\frac{\max\{n,\lambda\}}{p'} - \alpha} \leq C^* \leq \frac{|S^{n-1}|}{\frac{\min\{n,\lambda\}}{p'} - \alpha}.$$

*In particular, when $\lambda = n$, the inequality (26) holds if and only if $\alpha < \frac{n}{p'}$ and then $C^* = \frac{|S^{n-1}|}{\frac{n}{p'} - \alpha}$.*

*Proof.* The sufficiency of the condition $\alpha < \frac{\min\{n,\lambda\}}{p'}$ and the upper bound for $C^*$ follow from the part III of Theorem 4. Let us prove the necessity and the lower bound. In view of the part II of Theorem 4, we have only to treat the case $\lambda \geq n$. To this end, choose the function

$$f_\delta(x) = \begin{cases} |x|^{-n+\frac{\lambda}{p'}+\delta}, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

By Lemma 5, $f_\delta \in L_a^{p)}(\mathbb{R}^n)$ for $\delta > 0$. For $|x| < 1$ we have $H_n^\alpha f_\delta(x) = \frac{|S^{n-1}|}{\frac{\lambda}{p'}+\delta-\alpha} f_\delta(x)$, and then $\|H_n^\alpha f_\delta\|_{L_a^{p)}(\mathbb{R}^n)} \geq \frac{|S^{n-1}|}{\frac{\lambda}{p'}+\delta-\alpha} \|f_\delta\|_{L_a^{p)}(\mathbb{R}^n)}$ for all $\delta > 0$. Hence, passing to the limit as $\delta \to 0$, we obtain the inequality $\frac{|S^{n-1}|}{\frac{\lambda}{p'}-\alpha} \leq C^*$. ◀

The following theorem is proved in the same way. It may be also derived directly from Theorem 5 via the relation (9).

**Theorem 6.** *Let $a(x) = |x|^{-\lambda}$, $\lambda \in \mathbb{R}$. The condition $\beta > \frac{\max\{n,\lambda\}}{p'} - n$ is sufficient and the condition $\beta > \frac{\min\{n,\lambda\}}{p'} - n$ is necessary for the boundedness of the operator $\mathcal{H}_n^\beta$ in the grand space $L_a^{p)}(\mathbb{R}^n)$ and under these conditions*

$$\frac{|S^{n-1}|}{n - \frac{\min\{\lambda,n\}}{p'} + \beta} \leq C^* \leq \frac{|S^{n-1}|}{n - \frac{\max\{\lambda,n\}}{p'} + \beta}.$$

*In particular, when $\lambda = n$, the inequality (27) holds if and only if $\beta > -\frac{n}{p}$ and then $C^* = \frac{|S^{n-1}|}{\frac{n}{p}+\beta}$.*

### 3.4. Multidimensional case: general grandizers

In Theorem 4, for multidimensional Hardy operators we obtained a result stronger than the boundedness in the grand spaces. Namely, the estimates were given via one-dimensional norms of spherical means of function $f$, not $n$-dimensional norms of the functions $f$ themselves. This result was obtained under the assumption that the grandizer is a radial function. In the theorem below we use another approach, which allows to obtain estimates via $n$-dimensional norms of $f$ for general, i.e. not necessarily radial, grandizers.

Denote

$$k_p := \frac{|S^{n-1}|}{\frac{n}{p'} - \alpha}, \quad \ell_p := \frac{|S^{n-1}|}{\frac{n}{p} + \beta},$$

$$\mathcal{C}_{p,\alpha}(a) := \sup_{0<\varepsilon<p-1} k_{p-\varepsilon}^{\frac{1}{(p-\varepsilon)'}} \left( \int\limits_{|x|>1} |x|^{\alpha - \frac{n}{(p-\varepsilon)'} - n} a_*(x)^{\frac{\varepsilon}{p}} \, dx \right)^{\frac{1}{p-\varepsilon}},$$

$$\mathcal{D}_{p,\beta}(a) := \sup_{0<\varepsilon<p-1} \ell_{p-\varepsilon}^{\frac{1}{(p-\varepsilon)'}} \left( \int\limits_{|x|<1} |x|^{\beta - \frac{n}{(p-\varepsilon)'}} a_*(x)^{\frac{\varepsilon}{p}} \, dx \right)^{\frac{1}{p-\varepsilon}},$$

where $a_*(x) := \sup\limits_{y\in\mathbb{R}^n} \frac{a(x|y|)}{a(y)}$.

**Theorem 7.** *Let $1 < p < \infty$, $a$ be a non-negative function on $\mathbb{R}^n$, satisfying the integrability condition* (2).

*The conditions $\mathcal{C}_{p,\alpha}(a) < \infty$ and $\mathcal{D}_{p,\beta}(a) < \infty$ are sufficient for the boundedness of operators $H_n^\alpha$ and $\mathcal{H}_n^\beta$, respectively, in the grand space $L_a^{p)}(\mathbb{R}^n)$ and*

$$\|H_n^\alpha\|_{L_a^{p)}(\mathbb{R}^n)} \le \mathcal{C}_{p,\alpha}(a), \quad \|\mathcal{H}_n^\beta\|_{L_a^{p)}(\mathbb{R}^n)} \le \mathcal{D}_{p,\beta}(a).$$

*If $a \in L^1(B(0,\tau))$ for some $\tau > 0$, then the condition $\alpha < \frac{n}{p'}$ is necessary for the inequality $\|H_n^\alpha f\|_{L_a^{p)}(\mathbb{R}^n)} \le C\|f\|_{L_a^{p)}(\mathbb{R}^n)}$ to hold and $\frac{|S^{n-1}|}{\frac{n}{p'}-\alpha} \le \|H_n^\alpha\|_{L_a^{p)}(\mathbb{R}^n)}$ under this condition.*

*If $a \in L^1(\mathbb{R}^n \setminus B(0,N))$ for some $N > 0$, then the condition $\beta > -\frac{n}{p}$ is necessary for the inequality $\|\mathcal{H}_n^\beta f\|_{L_a^{p)}(\mathbb{R}^n)} \le C\|f\|_{L_a^{p)}(\mathbb{R}^n)}$ to hold and $\frac{|S^{n-1}|}{\frac{n}{p'}-\alpha} \le \|\mathcal{H}_n^\beta\|_{L_a^{p)}(\mathbb{R}^n)}$ under this condition.*

*Proof.*

For any $q>1$ by the Hölder inequality we obtain

$$|\mathcal{H}_n^\beta f(x)| \leq \int\limits_{|y|>|x|} \left(|x|^\beta |y|^{-\beta-n}\right)^{\frac{1}{q'}} |y|^{-\frac{n}{qq'}} \left(|x|^\beta |y|^{-\beta-n}\right)^{\frac{1}{q}} |y|^{\frac{n}{qq'}} |f(y)| dy$$

$$\leq \left(\int\limits_{|y|>|x|} |x|^\beta |y|^{-\beta-n} |y|^{-\frac{n}{q}} dy\right)^{\frac{1}{q'}} \left(\int\limits_{|y|>|x|} |x|^\beta |y|^{-\beta-n} |y|^{\frac{n}{q'}} |f(y)|^q dy\right)^{\frac{1}{q}}$$

$$= \ell_q^{\frac{1}{q'}} |x|^{-\frac{n}{qq'}} \left(\int\limits_{|y|>|x|} |x|^\beta |y|^{-\beta-\frac{n}{q}} |f(y)|^q dy\right)^{\frac{1}{q}}.$$

Furthermore, by Fubini theorem we get

$$\|\mathcal{H}_n^\beta f(x)\|_{L_a^{p)}(\mathbb{R}^n)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int\limits_{\mathbb{R}^n} |\mathcal{H}_n^\beta f(x)|^{p-\varepsilon} a(x)^{\frac{\varepsilon}{p}} dx\right)^{\frac{1}{p-\varepsilon}}$$

$$\leq \sup_{0<\varepsilon<p-1} \ell_{p-\varepsilon}^{\frac{1}{(p-\varepsilon)'}} \left(\varepsilon \int\limits_{\mathbb{R}^n} |x|^{\beta-\frac{n}{(p-\varepsilon)'}} \int\limits_{|y|>|x|} \frac{|f(y)|^{p-\varepsilon}}{|y|^{\beta+\frac{n}{p-\varepsilon}}} dy \; a(x)^{\frac{\varepsilon}{p}} dx\right)^{\frac{1}{p-\varepsilon}}$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon \ell_{p-\varepsilon}^{\frac{1}{(p-\varepsilon)'}} \left(\int\limits_{\mathbb{R}^n} |f(y)|^{p-\varepsilon} dy \int\limits_{|x|<1} |x|^{\beta-\frac{n}{(p-\varepsilon)'}} a(x|y|)^{\frac{\varepsilon}{p}} dx\right)^{\frac{1}{p-\varepsilon}}$$

$$\leq \sup_{0<\varepsilon<p-1} \varepsilon \ell_{p-\varepsilon}^{\frac{1}{(p-\varepsilon)'}} \left(\int\limits_{|x|<1} |x|^{\beta-\frac{n}{(p-\varepsilon)'}} a_*(x)^{\frac{\varepsilon}{p}} dx\right)^{\frac{1}{p-\varepsilon}}$$

$$\times \left(\int\limits_{\mathbb{R}^n} |f(y)|^{p-\varepsilon} a(y)^{\frac{\varepsilon}{p}} dy\right)^{\frac{1}{p-\varepsilon}}$$

$$= \mathcal{D}_{p,\beta}(a) \|f\|_{L_a^{p)}(\mathbb{R}^n)}.$$

The necessity of the condition $\beta > -\frac{n}{p}$ and the lower bound for the norm of the operator $\mathcal{H}_n^\beta$ are proved with the help of the same function $f_\delta(x)$ as in the proof of Theorem 4.

The proof of the bounedness of the operator $H_n^\alpha$ is similar. ◄

In the remark below for certain class of non-radial grandizers we provide some easy-to-check estimates for the constants $\mathcal{C}_{p,\alpha}(a)$ and $\mathcal{D}_{p,\beta}(a)$.

**Remark 3.** *Let* $a(x) = b(r)g(\sigma)$*, where* $r = |x|$*,* $\sigma = \frac{x}{|x|}$*,* $b$ *and* $g$ *be non-negative functions on* $\mathbb{R}_+$ *and* $S^{n-1}$*, respectively,* $\inf\limits_{\sigma\in S^{n-1}} g(\sigma) > 0$ *and* $g \in L^1(S^{n-1})$*. Then the constants* $\mathcal{C}_{p,\alpha}(a)$ *and* $\mathcal{D}_{p,\beta}(a)$ *have the following estimates:* $\mathcal{C}_{p,\alpha}(a) \leq |S^{n-1}|c_{p,\alpha}(b)c(g)^{\frac{1}{p'}}$*,* $\mathcal{D}_{p,\beta}(a) \leq |S^{n-1}|d_{p,\alpha}(b)c(g)^{\frac{1}{p'}}$*, where the constants* $c_{p,\alpha}$ *and* $d_{p,\beta}$*, corresponding to the one-dimensional case, were defined in* (11) *and* (12) *and*

$$c(g) = \frac{\frac{1}{|S^{n-1}|}\int\limits_{S^{n-1}} g(\sigma)d\sigma}{\inf\limits_{\sigma\in S^{n-1}} g(\sigma)}.$$

*To verify these estimates, it suffices to note that* $a_*(x) \leq b_*(r)\frac{g(\sigma)}{\inf\limits_{\sigma\in S^{n-1}} g(\sigma)}$*, see Lemma 2.*

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