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Factorization in the Space of Henstock-Kurzweil Integrable Functions

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Abstract. In this work, we extend the factorization theorem of Rudin and Cohen to $HK(\mathbb{R})$, the space of Henstock-Kurzweil integrable functions. This implies a factorization for the isometric spaces $\mathcal{A}_C$ and $\mathcal{B}_C$. We also study in this context the Banach algebra of functions $HK(\mathbb{R})\cap BV(\mathbb{R})$, which is also a dense subspace of $L^2(\mathbb{R})$. In some sense this subspace is analogous to $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. However, while $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ factorizes as $L^1(\mathbb{R})\cap L^2(\mathbb{R}) * L^1(\mathbb{R})$, via the convolution operation *, it is shown in the paper that $HK(\mathbb{R})\cap BV(\mathbb{R}) * L^1(\mathbb{R})$ is a Banach subalgebra of $HK(\mathbb{R})\cap BV(\mathbb{R})$.

Key Words and Phrases: factorization, Banach algebra, Henstock-Kurzweil integral, bounded variation function.

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1. Introduction

For appropriate functions $f, g : \mathbb{R} \to \mathbb{R}$, the convolution is defined as

$$f * g(t) = g * f(t) := \int_{\mathbb{R}} f(t-y)g(y)dy. \tag{1}$$

Thus, $f * g$ is a superposition of translates of f taken with the weights $g(y)dy$, and we can expect that $f * g$ inherits properties of f as well as of g, [9]. For example, the space of Lebesgue integrable functions $L^1(\mathbb{R})$, is a Banach algebra with the convolution (1). In fact, W. Rudin [18] proved that

$$L^1(G) * L^1(G) = L^1(G),$$

for $G = \mathbb{R}^n$ and any locally compact abelian group [19]. In [5] and [8], the results of Rudin were generalized to Banach algebras with a bounded approximate unit. Moreover,

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the result of Cohen [5] states that if A is a Banach algebra with product $\circ$ and bounded approximate unit, then $A \circ A = A$. This type of results are important because they give different representations of the algebra. The subject was later developed in a more general setting in [8].

On the other hand, integration theory has been developed. For example, we can consider the Banach space of Henstock-Kurzweil integrable functions on $\mathbb{R}$, $\widehat{HK(\mathbb{R})}$, which strictly contains $L^1(\mathbb{R})$. Recently this integral has been used in order to generalize the classical Fourier transform [12].

The convolution operator has been studied on several spaces, for example, this operator is well defined for ultradistributions. This subject has been developed in recent years, where surjectivity characterizations and existence of right inverses for convolution operations are given, among others. See [6] and the references therein.

In this paper, the convolution of functions is extended to obtain a factorization theorem for $HK(\mathbb{R})$ and therefore for the related spaces $\mathcal{A}_C$ and $\mathcal{B}_C$ defined in [16]. Also, we define and analyse some Banach algebras contained in $HK(\mathbb{R})$ and embedded continuously in $L^2(\mathbb{R})$. In particular, we show that the subspace $HK(\mathbb{R}) \cap BV(\mathbb{R}) * L^1(\mathbb{R})$ is a Banach subalgebra of $HK(\mathbb{R}) \cap BV(\mathbb{R})$, which is also a dense subspace of $L^2(\mathbb{R})$. We recall that this subspace of $L^2(\mathbb{R})$ is, in some sense, the analogue of the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for the classical Fourier transform, in the context of the HK-Fourier Transform. See [10], [11] and Definition 2.10 in [12].

2. Preliminaries

We follow the notation from [15] in order to introduce basic definitions of the Henstock-Kurzweil integral theory. The Henstock-Kurzweil integrable real valued functions on I will be denoted as $\mathcal{HK}(I)$, where $I := [a, b]$ is any interval in $\overline{\mathbb{R}} = [-\infty, \infty]$. The Alexiewicz seminorm on $\mathcal{HK}(I)$ is defined in [15] by

$$||f||_A := \sup_{x \in I} \left| \int_a^x f \right|.$$

It is equivalent to

$$||f||'_A := \sup_{s,t \in I, -\infty < s < t < \infty} \left| \int_s^t f \right|.$$

Furthermore

$$||f||_A \leq ||f||'_A \leq 2||f||_A. \tag{2}$$

We remark that $\mathcal{HK}(I)$ is not a Banach space, see [17].

The quotient space $\mathcal{HK}(I)/\mathcal{W}(I)$ will be denoted by $HK(I)$. Here $\mathcal{W}(I)$ is the subspace of $\mathcal{HK}(I)$ for which the Alexiewicz seminorm vanishes. The completion of the Henstock-Kurzweil space in the Alexiewicz norm is denoted by $\widehat{HK(\mathbb{R})}$, [11], [17].

Let g be a real valued function over $\mathbb{R}$. It is said that g is a bounded variation function over $[a,b] \subset \mathbb{R}$ if and only if

$$\mathrm{Var}(g,[a,b]) := \sup_P \sum_{i=1}^{n} |g(t_i) - g(t_{t-i})| < \infty,$$

where the supreme is taken over all possible partitions on $[a,b]$. It is said that g is a bounded variation function over $\mathbb{R}$ if and only if

$$\mathrm{Var}(g,\mathbb{R}) := \lim_{t\to\infty, s\to-\infty} \mathrm{Var}(g,[s,t])$$

exists in $\mathbb{R}$. We will denote the set of bounded variation functions over an interval I as $BV(I)$.

Note that if $g \in BV(\mathbb{R})$, then $\lim_{x\to\pm\infty} g(x) \in \mathbb{R}$. Thus, we can extend g on $\overline{\mathbb{R}} := [-\infty, \infty]$ as

$$g(\pm\infty) := \lim_{x\to\pm\infty} g(x).$$

Moreover, $\mathrm{Var}(g,\mathbb{R}) = \mathrm{Var}(g,\overline{\mathbb{R}})$. Thereby, we can use the same symbol $BV(\mathbb{R})$ for the bounded variation functions over $\overline{\mathbb{R}}$.

$BV_0(\mathbb{R})$ will denote the bounded variation functions over $\mathbb{R}$ having limits equal to zero at $\pm\infty$. It is easy to prove that $HK(\mathbb{R}) \cap BV(\mathbb{R}) \subset BV_0(\mathbb{R})$, see [14]. We consider the real vector space

$$B := HK(\mathbb{R}) \cap BV(\mathbb{R}) = HK(\mathbb{R}) \cap BV_0(\mathbb{R}),$$

with the norm $||f||_B := ||f||_A + ||f||_{BV}$, where $||f||_{BV} := \mathrm{Var}(f,\mathbb{R})$.

For $1 \le p < \infty$, the Banach space $L^p(\mathbb{R})$ over the field $\mathbb{R}$ with norm $\|\cdot\|_p$ consists of equivalence classes of measurable real valued functions over $\mathbb{R}$ such that $|f(x)|^p$ is Lebesgue integrable:

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

For a measurable function $f : \mathbb{R} \to \mathbb{R}$ the uniform norm is defined to be

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} |f(x)|.$$

We mention that if a continuous function f on $\mathbb{R}$ has limits at $\pm\infty$, then f is uniformly continuous.

Theorem 1. $(B, ||\cdot||_B)$ *is a real Banach space. Moreover, B is continuously embedded in $L^2(\mathbb{R})$, $HK(\mathbb{R})$ and $BV(\mathbb{R})$.*

Proof. Let (f_n) be a Cauchy sequence in B. Then it is also a Cauchy sequence in the norm $||\cdot||_{BV}$. Given $x \in \mathbb{R}$, for the partition $P = \{a, x, b\}$ one gets that

$$\begin{aligned} |(f_n - f_m)(x) - (f_n - f_m)(a)| &+ |(f_n - f_m)(b) - (f_n - f_m)(x)| \\ &\leq \mathrm{Var}(f_n - f_m, [a, b]) \\ &\leq \mathrm{Var}(f_n - f_m, \mathbb{R}). \end{aligned} \tag{3}$$

Taking $a \to -\infty$ and $b \to \infty$ in the expression (3) we obtain

$$\limsup_{a\to-\infty, b\to\infty} |(f_n - f_m)(x) - (f_n - f_m)(a)| + |(f_n - f_m)(b) - (f_n - f_m)(x))| \leq \mathrm{Var}(f_n - f_m, \mathbb{R}) \leq \epsilon.$$

Thus, for n and m great enough,

$$2|(f_n - f_m)(x)| \leq \epsilon.$$

This means that $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$. We define

$$f(x) := \lim_{n\to\infty} f_n(x).$$

To prove that $f \in BV(\mathbb{R})$, let $P = \{[x_{i-1}, x_i]\}_{i=1}^m$ be a partition of an interval $[a, b]$. One obtains

$$\begin{aligned} \sum_{k=1}^{n} |(f_n - f)(x_k) - (f_n - f)(x_{k-1})| &= \sum_{k=1}^{n} \lim_{m\to\infty} |(f_n - f_m)(x_k) - (f_n - f_m)(x_{k-1})| \\ &= \lim_{m\to\infty} \sum_{k=1}^{n} |(f_n - f_m)(x_k) - (f_n - f_m)(x_{k-1})| \\ &\leq \limsup_{m\to\infty} \mathrm{Var}(f_n - f_m, [a, b]) \\ &\leq \limsup_{m\to\infty} \mathrm{Var}(f_n - f_m, \mathbb{R}) \\ &\leq \epsilon. \end{aligned}$$

This proves that $f_n - f \in BV(\mathbb{R})$, yielding $f \in BV(\mathbb{R})$. Now we show that $f \in HK(\mathbb{R})$. Since $||f_n - f_m||_A < \epsilon$, then there exits $\tilde{f}$ in the completion of $HK(\mathbb{R})$, $\widehat{HK(\mathbb{R})}$, such that

$$\int_s^t f_n \to \int_s^t \tilde{f},$$

for all $s,t \in \mathbb{R}$. Moreover, (f_n) converges pointwise to f, so that by the Lebesgue Dominated Convergence Theorem we have

$$\int_s^t f_n \to \int_s^t f.$$

Thus,

$$\int_s^t \tilde{f} := \lim_{n\to\infty} \int_s^t f_n = \int_s^t f,$$

for all $s,t \in \mathbb{R}$, see [17]. Therefore, $f \in HK(\mathbb{R})$. So, B is a Banach space.

We prove now that B is continuously embedded in $L^2(\mathbb{R})$. For $f \in B$ and by [2, Multiplier Theorem] we have

$$\begin{aligned} ||f||_2^2 &= \int_{-\infty}^{\infty} (f(x))^2 dx \\ &\leq |Ff(x)||_{-\infty}^{\infty} + \left| \int_{-\infty}^{\infty} F df \right| \\ &\leq ||f||_A ||f||_{BV} \\ &\leq \frac{1}{2}(||f||_A^2 + ||f||_{BV}^2) \\ &\leq \frac{1}{2}(||f||_A + ||f||_{BV})^2, \end{aligned}$$

where $F(x) = \int_{-\infty}^{x} f(t)dt$. The other two embeddings follow directly.◀

Remark 1. *B is not compactly embedded in $(L^2(\mathbb{R}), ||\cdot||_2)$, neither in $(HK(\mathbb{R}), ||\cdot||_A)$, nor in $(BV(\mathbb{R}), ||\cdot||_{BV})$. It is easy to see that the sequence $f_n(x) := \chi_{[n,n+1]}(x)$ for $n \in \mathbb{N}$ is in B, but does not have Cauchy subsequences in those spaces.*

3. B as a Banach $L^1(\mathbb{R})$-module

Definition 1. *A topological algebra over the field $\mathbb{K}$ of real or complex numbers is a topological vector space A over $\mathbb{K}$ provided with an associative multiplication continuous with respect to both variables (i.e. jointly continuous).*

Definition 2. *A Banach algebra is a topological algebra which, regarded as a topological vector space, is a Banach space.*

These definitions can be seen in [20]. Now we will show that B is a Banach algebra according to these definitions.

Theorem 2. *B is a commutative Banach algebra with respect to the usual product of functions.*

Proof. Since B is a Banach space, it is a topological vector space. We will prove that the usual product of functions of B is a function in B, and that the map

$$B \times B \ni (f, g) \to fg \in B$$

is continuous.

When $f, g \in B$, by [2, Multiplier Theorem] we obtain $||fg||_A \leq ||f||_B ||g||_B$. Let us prove that $fg \in BV(\mathbb{R})$. As $f, g \in B \subset BV(\mathbb{R})$, then f and g are bounded functions. Moreover, $||f||_\infty \leq ||f||_{BV}$, implying

$$\begin{aligned} \mathrm{Var}(fg, [a, b]) &\leq ||f||_\infty ||g||_{BV} + ||g||_\infty ||f||_{BV} \\ &\leq 2||f||_B ||g||_B. \end{aligned}$$

This results in

$$||fg||_B \leq C||f||_B ||g||_B,$$

for some positive constant C. However, by Corollary 2.5 in [20], $(B, \cdot, || \cdot ||_B)$ can be considered a Banach algebra in the usual sense [3].◀

We recall that $L^1(\mathbb{R})$ is a real Banach algebra with respect to the convolution operation (1), see [4].

Note that $f * g(x)$ given by (1) is well defined as a Lebesgue integral for $f \in B$ and $g \in L^1(\mathbb{R})$.

The concept of Banach $A - module$ is defined in [8, Definition 32.14].

Theorem 3. *The space B is a Banach $L^1(\mathbb{R}) - module$.*

Proof. Let $(f, g) \in B \times L^1(\mathbb{R})$. For a partition $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ of a bounded interval $[a, b]$, we calculate

$$\begin{aligned} \sum_{i=1}^n \left| \int_{\mathbb{R}} \big(f(x_i - y) - f(x_{i-1} - y)\big) g(y) dy \right| &\leq \sum_{i=1}^n \int_{\mathbb{R}} |f(x_i - y) - f(x_{i-1} - y)||g(y)| dy \\ &\leq \int_{-\infty}^{\infty} \mathrm{Var}(f, \mathbb{R}) |g(y)| dy \\ &= ||f||_{BV} ||g||_1. \end{aligned}$$

Therefore

$$\|f * g\|_{BV} := \mathrm{Var}(f * g, \mathbb{R}) \leq ||f||_{BV} ||g||_1. \tag{4}$$

Let us prove that $f * g \in HK(\mathbb{R})$. For $s < t$ arbitrary real numbers

$$\int_s^t dx \int_{-\infty}^{\infty} f(x-y)g(y)dy \quad = \quad \lim_{n\to\infty} \int_s^t \int_{-n}^{n} f(x-y)g(y)dydx, \tag{5}$$

where we apply the Lebesgue Dominated Convergence Theorem, due to

$$\left| \int_{-n}^{n} f(x-y)g(y)dydx \right| \leq \|f\|_\infty ||g||_1 \leq \|f\|_{BV}\, \|g\|_1.$$

From equation (5) and Fubini's Theorem we obtain

$$\begin{aligned} \left| \int_s^t f * g(x)dx \right| &= \left| \lim_{n\to\infty} \int_{-n}^{n} \int_s^t f(x-y)g(y)dxdy \right| \\ &= \left| \lim_{n\to\infty} \int_{-n}^{n} [F(t-y) - F(s-y)]g(y)dy \right| \\ &\leq \lim_{n\to\infty} \int_{-n}^{n} |g(y)|dy 2||F||_\infty \\ &= 2||g||_1||f||_A, \end{aligned}$$

where $F(x) = \int_{-\infty}^{x} f$. Thus,

$$\|f * g\|'_A := \sup_{[s,t]\subset\mathbb{R}} \left| \int_s^t f * g(x)dx \right| \leq 2||g||_1||f||_A. \tag{6}$$

The inequalities (4) and (6) and the equivalence of the norms (2) imply that

$$||f * g||_B = ||f * g||_{BV} + ||f * g||_A \leq ||f||_{BV}||g||_1 + ||f * g||'_A \leq 2||f||_B||g||_1. \tag{7}$$

The properties *i)* $f * (g + h) = f * g + f * h$, *ii)* $(f + \tilde{f}) * g = f * g + \tilde{f} * g$, *iii)* $f * (\alpha g) = \alpha(f * g) = (\alpha f) * g$ and *iv)* $f * (g * h) = f * (h * g)$, for $g, h \in L^1(\mathbb{R})$, $f, \tilde{f} \in B$ are easy to prove. On the other hand, since B is a Banach space, B is a Banach $L^1(\mathbb{R}) - module$.◄

4. Factorization

Definition 3. *A bounded approximate unit in a Banach algebra* $(A, \|\cdot\|)$ *is a sequence or net* $\{e_\alpha\}$ *such that* $\|e_\alpha\| < C$ *for some positive constant* C *and*

$$\lim_\alpha \|a - a\, e_\alpha\| = \lim_\alpha \|a - e_\alpha\, a\| = 0,$$

for each $a \in A$.

W. Rudin, [18] used properties of the Fourier transform in order to show that

$$L^1(\mathbb{R}) * L^1(\mathbb{R}) = L^1(\mathbb{R}).$$

The Fourier transforms of functions in B share some properties of functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ see [10], [11], [12]. On the other hand, the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ can be factored as $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) * L^1(\mathbb{R})$, see [9, Theorem 1.3.2], whereas $B * L^1(\mathbb{R})$ is a proper subset of B as we show below.

We denote the space of real valued continuous functions over $\mathbb{R}$ vanishing at infinity by $C_0(\mathbb{R})$.

Let A be a set of real valued functions on $\mathbb{R}$ for which the convolution operation with elements of $L^1(\mathbb{R})$ is defined. Then $A * L^1(\mathbb{R})$ denotes the set

$$\{f * g : \; f \in A, \; g \in L^1(\mathbb{R})\}.$$

Note that B is a dense subset in $HK(\mathbb{R})$ with the Alexiewicz norm, because B contains the linear combinations of characteristic functions of intervals which are dense in $HK(\mathbb{R})$, see [11, Lemma 2.1], [15, Theorem 7].

Definition 4. *The convolution in* $\widehat{HK(\mathbb{R})} \times L^1(\mathbb{R})$ *is defined by*

$$\widehat{HK(\mathbb{R})} \times L^1(\mathbb{R}) \ni (f, g) \mapsto f * g := \lim_{n\to\infty} f_n * g,$$

where $(f_n) \subset B$ *converges to* f *in* $\widehat{HK(\mathbb{R})}$.

Note that if $(f_n) \subset B$ is a Cauchy sequence, then $F(x) = \lim_{n\to\infty} \int_{-\infty}^{x} f_n$ exists and in particular it is a uniformly continuous function on $\mathbb{R}$, see [11], [17].

Proposition 4. $\widehat{HK(\mathbb{R})}$ *is a Banach* $L^1(\mathbb{R})-$*module.*

Proof. We show that the map

$$\widehat{HK(\mathbb{R})} \times L^1(\mathbb{R}) \ni (f, g) \mapsto f * g$$

has range in $\widehat{HK(\mathbb{R})}$ and is continuous. Let us see first that $(f_n * g)$ is a Cauchy sequence in $\widehat{HK(\mathbb{R})}$ for (f_n) a Cauchy sequence converging to f. Relation (6) yields

$$||f_n * g - f_m * g||_A \leq ||(f_n - f_m) * g||'_A \leq 2||g||_1||f_n - f_m||_A.$$

Therefore, $f * g := \lim_{n\to\infty} f_n * g \in \widehat{HK(\mathbb{R})}$. Furthermore,

$$||f * g||_A := \lim_{n\to\infty} ||f_n * g||_A \leq \limsup_{n\to\infty} 2||g||_1||f_n||_A = 2||g||_1||f||_A.$$

The other properties in order that $\widehat{HK}(\mathbb{R})$ is a Banach $L^1(\mathbb{R}) - module$ are easy to prove [8, Definition 32.14].◀

The support of a function f is defined as $supp(f) := \overline{\{x, f(x) \neq 0\}}$, the closure of the set of points in $\mathbb{R}$ where f is non-zero. The space of real valued smooth functions with compact support in $\mathbb{R}$ is denoted by $C_c^\infty(\mathbb{R})$.

The Banach Module Factorization Theorem in [8, Theorem 32.22] says: *"Let A be a Banach algebra with a bounded approximate identity. If X is a Banach $A -$ module (with respect to the operation $\circ$), then $A \circ X$ is a closed linear subspace of X"*. In particular, if $A \circ X$ is dense in X, then $A \circ X = X$. So, we get the following result.

Corollary 1. $\widehat{HK}(\mathbb{R}) = \widehat{HK}(\mathbb{R}) * L^1(\mathbb{R}) = L^1(\mathbb{R}) * \widehat{HK}(\mathbb{R})$.

Proof. It is easy to show that $\widehat{HK}(\mathbb{R}) * L^1(\mathbb{R})$ is dense in $\widehat{HK}(\mathbb{R})$, because

$$L^1(\mathbb{R}) = L^1(\mathbb{R}) * L^1(\mathbb{R}) \subset \widehat{HK}(\mathbb{R}) * L^1(\mathbb{R}) \subset \widehat{HK}(\mathbb{R}).$$

In fact, for an arbitrary element $f \in \widehat{HK}(\mathbb{R})$, and $\varepsilon > 0$, we take $h \in HK(\mathbb{R}) \cap BV(\mathbb{R})$ such that

$$\|h - f\|'_A < \varepsilon/2.$$

Let $j \in C_c^\infty(\mathbb{R})$ be a positive function with $supp(j) \subset (-1, 1)$ and $\int_{-1}^{1} j(t)dt = 1$. We define for each positive number δ, the approximate of the unit in $L^1(\mathbb{R})$

$$j_\delta(x) := \frac{1}{\delta} j(x/\delta). \tag{8}$$

Application of Fubini's Theorem gives

$$\begin{aligned} \|h - h * j_\delta\|'_A &= \sup_{s,t} \left| \int_{-1}^{1} \big(H(t) - H(s) - H(t - \delta y) + H(s - \delta y)\big) j(y) dy \right| \\ &\leq 2\varepsilon. \end{aligned} \tag{9}$$

Here

$$H(t) := \int_{-\infty}^{t} h(y)dy.$$

The inequality (9) follows by the uniform continuity of H on $\mathbb{R}$. So, given $\varepsilon > 0$, there exists $\delta > 0$ small enough such that $||h - h * j_\delta||'_A \leq 2\varepsilon$. $\widehat{HK}(\mathbb{R}) * L^1(\mathbb{R})$ being closed, previous relation means that it contains the dense subspace $HK(\mathbb{R}) \cap BV(\mathbb{R})$ of $\widehat{HK}(\mathbb{R})$, yielding the result.◀

Talvila showed some properties for the convolution in the context of the distributional Denjoy integral, for example: $\mathcal{A}_C * L^1(\mathbb{R}) \subset \mathcal{A}_C$, [Theorem 8a, [16]] where $\mathcal{A}_C = \{f \in$

$\mathcal{D}'|f = F', F \in \mathcal{B}_C\}$, $\mathcal{D}'$ denotes the continuous linear functionals on the space of test functions $\mathcal{D} = C_c^\infty(\mathbb{R})$, and

$$\mathcal{B}_C = \{F : \mathbb{R} \to \mathbb{R} | F \ \ is \ \ continuous \ \ on \ \ \mathbb{R}, \ \ \lim_{x\to-\infty} F(x) = 0, \lim_{x\to\infty} F(x) \in \mathbb{R}\}$$

is a Banach space under the supremum norm. It was proved in [17] that $\mathcal{A}_C$ (with Alexiewicz norm) is isomorphic to $\mathcal{B}_C$ and in [13] it was shown that $\mathcal{B}_C$ is isomorphic to $\widehat{HK(\mathbb{R})}$. When $f \in L^1(\mathbb{R})$, it follows that $F(x) := \int_{-\infty}^x f(t)dt \in \mathcal{B}_C$ and $f \in \mathcal{A}_C$. Therefore, we have the following result.

Corollary 2. *$\mathcal{B}_C * L^1(\mathbb{R}) = \mathcal{B}_C$ and $\mathcal{A}_C * L^1(\mathbb{R}) = \mathcal{A}_C$.*

Proof. We give the proof for $\mathcal{A}_C * L^1(\mathbb{R}) = \mathcal{A}_C$. The other equality is quite similar.

We consider Definition 5 from [16] about the convolution mapping $* : \mathcal{A}_C \times L^1 \to \mathcal{A}_C$. We use the same symbol $*$ for the mapping. This mapping satisfies the following properties: For all $f, \ \widetilde{f} \in \mathcal{A}_C, \ g, \ \widetilde{g} \in L^1$, and $\alpha \in \mathbb{R}$:

1. $\mathcal{A}_C * L^1 \subset \mathcal{A}_C$,
2. $(f * g) * \widetilde{g} = f * (g * \widetilde{g})$,
3. $f * (\alpha\, g) = (\alpha\, f) * g = \alpha(f * g)$,
4. $f * (g + \widetilde{g}) = f * g + f * \widetilde{g}$,
5. $(f + \widetilde{f}) * g = f * g + \widetilde{f} * g$,
6. $||f * g||_A \leq ||f||_A ||g||_1$,

Properties (1), (2) and (6) hold by [16, Theorem 8 (a)-(b)]. As the elements in $\mathcal{A}_C$ are linear functionals, it is easy to prove (3), (4) and (5). Therefore, $\mathcal{A}_C$ is a Banach $L^1(\mathbb{R}) - module$, see [8, Definition 32.14]. On the other hand,

$$L^1(\mathbb{R}) = L^1(\mathbb{R}) * L^1(\mathbb{R}) \subset \mathcal{A}_C * L^1 \subset \mathcal{A}_C.$$

$L^1(\mathbb{R})$ is dense in $\mathcal{A}_C$, with Alexiewicz norm, [16, Proposition 7]. Then [8, The Banach Module Factorization Theorem] implies the result.◄

Now we consider the same question about factorization for the Banach algebra B. Note that $L^1(\mathbb{R})$ does not have inclusion relations with B, see for instance [11, corollary 3.3] and [1, example 1, p. 129]. Then the argument in Corollary 1 can not be used for B. In fact, the answer to our question is negative.

Proposition 5. *The $L^1(\mathbb{R}) - module$ B can not be factored as $B * L^1(\mathbb{R})$.*

Proof. By Theorem 3 and [8, The Banach Module Factorization Theorem], $B * L^1(\mathbb{R})$ is a closed linear subspace of B. We will show that B can not be factored as $B * L^1(\mathbb{R})$, for which it is sufficient to prove that $B * L^1(\mathbb{R}) \subset C_0(\mathbb{R})$.

Let $f \in B$ and $g \in L^1(\mathbb{R})$. Since f is continuous except perhaps at countably many points, we get

$$f(x + r_n - y)g(y) \to f(x - y)g(y),$$

as $r_n \to 0$. If $x - y$ is not a discontinuity point, it means almost everywhere. Since f is bounded and g is L^1-integrable, by the dominated convergence Theorem we get

$$f * g(x + r_n) \to f * g(x), \quad r_n \to 0.$$

This implies continuity of $f * g$, so it belongs to $B \cap C(\mathbb{R}) \subset C_0(\mathbb{R})$.◄

$B*L^1(\mathbb{R})$ being a subspace strictly contained in B, it naturally brings us to the problem of characterizing the subspace.

We take $C^1(\mathbb{R})$ as the Banach space of real valued functions f with continuous derivative such that $\|f\|_\infty + \|f'\|_\infty < \infty$.

We study the space $B' = B \cap C^1(\mathbb{R})$, with given norm defined as

$$||f||_{B'} := ||f||_B + ||f'||_\infty.$$

Proposition 6. $(B', || \cdot ||_{B'})$ *is a Banach algebra strictly contained in* B.

Proof. Due to equivalence of the norm $||f||_{B'}$ with the norm $||f||_\infty + ||f'||_\infty + ||f||_B$, then the result holds by standard arguments.◄

We have the following characterization of $B' * L^1(\mathbb{R})$. It follows in particular that $B' * L^1(\mathbb{R})$ is a proper subset of B'. The characterization is in terms of $C_0(\mathbb{R})$.

Theorem 7.

$$B' * L^1(\mathbb{R}) = \{f \in C^1(\mathbb{R}) \cap B \ : \ f' \in C_0(\mathbb{R}) \ \}.$$

Proof. Let $(0, 0) \neq (f, g) \in B' \times L^1(\mathbb{R})$, then by similar arguments as previously one shows that $f * g \in C^1(\mathbb{R}) \cap B$. It remains to prove that

$$\frac{d}{dx} f * g(x) = f' * g(x) \in C_0(\mathbb{R}).$$

Take a sequence $(g_n) \in C_c^\infty(\mathbb{R})$ converging to g in L^1-norm. Then,

$$\|f' * (g - g_n)\|_\infty \leq \|f'\|_\infty \, \|g - g_n\|_1 \xrightarrow{n\to\infty} 0. \tag{10}$$

Integration by parts gives

$$f' * g_n(x) \quad = \quad \int_{-\infty}^{\infty} f'(x - y)g_n(y)dy$$

$$\begin{aligned} &= -f(x-\cdot)g_n(\cdot)\,\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(x-y)g_n'(y)dy \qquad (11)\\ &= \int_{-\infty}^{\infty} f(x-y)g_n'(y)dy. \end{aligned}$$

Since $\lim_{|x|\to\infty} f(x) = 0$, we can take $M_{\varepsilon,n}$ great enough so that

$$|f(x)| < \frac{\varepsilon}{2\|g_n'\|_1} \quad (\forall\,|x| \geq M_{\varepsilon,n}).$$

Suppose that $supp(g_n) \subset [-A_n, A_n]$. It follows that $|f(x-y)| < \varepsilon$ whenever $|x| > M_{\varepsilon,n}+A_n$ and $y \in supp(g_n)$, and we obtain from (11) that

$$\left| \int f'(x-y)g_n(y)dy \right| \leq \varepsilon \quad (\forall |x| \geq M_{\varepsilon,n} + A_n). \qquad (12)$$

(10) and (12) yield that $f' * g$ is the limit of the sequence $(f' * g_n)$ in the Banach space $(C_0(\mathbb{R}), ||\cdot||_\infty)$, implying

$$B' * L^1(\mathbb{R}) \subset \{f \in C^1(\mathbb{R}) \cap B \ : \ f' \in C_0(\mathbb{R})\ \}.$$

For the reverse contention, suppose $f \in B'$ with $f' \in C_0(\mathbb{R})$. To show that it is approximated in $B'-$ norm by elements in $B' * L^1(\mathbb{R})$, let us take $j_\delta(x)$ as in (8). The following estimate is valid by [2, Theorem 7.5] and (9):

$$\begin{aligned} \|f - f * j_\delta\|_B &= \|f - f * j_\delta\|_{BV} + \|f - f * j_\delta\|_A \\ &= \|f' - f' * j_\delta\|_1 + \|f - f * j_\delta\|_A \xrightarrow{\delta\downarrow 0} 0. \qquad (13) \end{aligned}$$

Therefore, we need only to verify that

$$\|f' - f' * j_\delta\|_\infty \xrightarrow{\delta\downarrow 0} 0. \qquad (14)$$

This is obtained from uniform continuity of f' on $\mathbb{R}$. Note that B' is a Banach $L^1(\mathbb{R})-$ *module* and $||f * g||_{B'} \leq 2||f||_{B'}||g||_1$. Therefore, $B' * L^1(\mathbb{R})$ is closed by The Banach Module Factorization Theorem [8]. This proves the statement.◀

Given $A \subseteq B$, we denote by $\overline{A}^B$ the closure of A in the norm of B.

Theorem 8.

$$B' \subset B * L^1 = \overline{B'}^B.$$

Proof. $B * L^1(\mathbb{R})$ is closed in B by Proposition 5. If $f \in B'$, then relation (13) is valid. So that $B' \subset B * L^1(\mathbb{R})$.

For $(f, g) \in B \times L^1(\mathbb{R})$, we will show that there is a sequence $h_n \in C^1(\mathbb{R}) \cap B$ converging in B to $f * g$. Choose $g_n \in C_c^\infty(\mathbb{R})$ that converges to g in $L^1(\mathbb{R})$. It follows from (7) that

$$\|f * g - f * g_n\|_B = \|f * (g - g_n)\|_B \leq 2\|f\|_B \|g - g_n\|_1 \xrightarrow{n\to\infty} 0.$$

We define $h_n := f * g_n$. Thus, $h_n \in B'$. This follows directly from Theorem 3 and $(g_n) \subset C_c^\infty(\mathbb{R})$, because of

$$\frac{d}{dx} f * g_n(x) = f * g_n'(x)$$

and

$$\|f * g_n\|_{B'} \leq 2\|f\|_B \ (\|g_n\|_1 + \|g_n'\|_1),$$

where we apply (7).◀

The following two theorems give a characterization of $B * L^1(\mathbb{R})$. The set of absolutely continuous functions over each compact interval in $\mathbb{R}$ is denoted by AC_{loc}.

Theorem 9. *$B_1 := AC_{loc} \cap B$ is a closed subalgebra of B.*

Proof. Given (f_n) a Cauchy sequence in B_1, there exists $f \in B$ such that

$$\|f_n - f\|_\infty \leq \|f_n - f\|_{BV} \xrightarrow{n\to\infty} 0,$$
$$||f_n - f||_A \xrightarrow{n\to\infty} 0.$$

It follows that $f \in C(\mathbb{R}) \cap B$. By [7, Theorem 4.14]

$$f_n(x) - f_n(s) = \int_s^x f_n'(t),$$

yielding

$$\lim_{s\to-\infty}[f_n(x) - f_n(s)] = f_n(x) = \int_{-\infty}^x f_n'(t).$$

[2, Hake's Theorem] implies that $f_n' \in HK(\mathbb{R})$ for each $n \in \mathbb{N}$. From [2, Theorem 7.5] one gets

$$\|f_n - f_m\|_{BV} = \int_{-\infty}^\infty |f_n' - f_m'| \xrightarrow{n,\ m\to\infty} 0,$$

yielding existence of some $g \in L^1(\mathbb{R})$ such that $f_n' \to g$, in L^1-norm. Furthermore,

$$f_n(x) - f(x) - f_n(s) + f(s) = \lim_{m\to\infty} \int_s^x f_n'(t) - f_m'(t) = \int_s^x f_n'(t) - g(t).$$

Thus $f \in B_1$, proving that B_1 is closed in B.

Since $u \cdot v \in B_1$, in the usual product $\cdot$ of functions and $||u \cdot v||_B \leq 2||u||_B||v||_B$ whenever $u, v \in B_1$, then B_1 is a Banach subalgebra of B.◀

Theorem 10. *The following equality holds:*

$$B * L^1(\mathbb{R}) = B_1.$$

Proof. [2, Theorem 7.5] implies that $B' \subset B_1$. Application of Theorem 8 and Theorem 9 gives that

$$B * L^1(\mathbb{R}) \subset B_1.$$

The reverse inclusion is implied by the following inequalities. With the approximation of the unit defined in (8), application of (9) and [2, Theorem 7.5] one gets

$$\|f * j_\delta - f\|_A \leq \|f * j_\delta - f\|'_A \xrightarrow{\delta \downarrow 0} 0,$$

$$\|f * j_\delta - f\|_{BV} = \|f' * j_\delta - f'\|_1 \xrightarrow{\delta \downarrow 0} 0,$$

where $f \in B_1$. As $B * L^1(\mathbb{R})$ is closed, the theorem follows. ◄

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