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A New Extension of Fan-KKM Theory and Equilibrium Theory on Hadamard Manifolds

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Abstract. In this paper, an extension of the Fan-KKM lemma to Hadamard manifolds is established. By using it, some existence results for equilibrium problems on Hadamard manifolds are provided. Finally, as an application of the main results, an existence result of a solution of the mixed variational inequality problem in the setting of Hadamard manifolds is stated.

Key Words and Phrases: equilibrium problem, Hadamard Manifold, KKM Map, mixed variational inequality.

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1. Introduction

It has been shown in [3, 24] that a vast number of the optimization problems such as variational inequality, convex minimization, fixed point and Nash equilibrium problems can be formulated as the equilibrium problem related to the Bi-function $F : K \times K \to \mathbb{R}$ in which K is a subset of Hilbert space H given as follows: Find $x \in K$ such that $F(x, y) \geq 0$ for every $y \in K$.

So far many methods and algorithms have been established to reach and approximate the equilibrium point (see [13, 17, 16, 7]). In a majority of equilibrium problems the subset K is assumed to be convex, as the result of the fact that convexity is an important tool to investigate the equilibrium problems. Corresponding to convexity in linear vector spaces, geodesic convexity is a well-known similar useful tool in non-linear metric spaces framework for optimization problems; see [30].

The main motivation to investigate equilibrium and optimization problems in the framework of Riemannian manifolds is the fact that by endowing the space with suitable Riemannian metric, some constrained optimization problems can be seen as unconstrained

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ones and optimization problems with un-convex objective function can be rewritten as convex ones in Riemannian manifold framework; (see [30, 26]). This attitude, in fact, poses a method through which many of the related optimization problems can be revisited in Riemannian manifolds framework. For instance, in [8] an example of a non-monotone vector field can be found which can be rewritten as a monotone vector field in the framework of Riemannian manifolds.

It should be noticed that Riemannian manifolds, especially Hadamard manifolds, which hold the topological structure, are the proper framework in these fields for developing concepts and methods from Euclidian spaces to non-linear structure. Therefore, the key idea in above mentioned problems is to transfer the concepts and methods from Euclidian space to Riemannian manifolds that has been shown in [1, 23, 31, 21]. The equilibrium problem was first investigated as a minimax inequality problem with a great generalization of KKM theory for Hausdorff topological vector spaces by Ky Fan [11]. In 1961, Fan [10] has proved the KKM lemma for Hausdorff topological vector spaces and later in 1984 stated and proved the developed KKM lemma by omitting compactness condition and presented the new version of Fan (1961) in [12] for Hausdorff topological vector spaces. In 2009, Zhou and Hang [33], proved a version of KKM's lemma for Hadamard manifolds under suitable conditions. In 2012 [6] Colao et al. developed and proved Fan's KKM [10] for Hadamard manifolds and later solved equilibrium problems and some related problems based on this extension.

Mixed variational inequality problem is one of the most important problems in optimization which has vast applications in economy, pure and applied mathematics. At first, this problem was brought up in the linear spaces framework and has been studied extensively. For instance, [14, 32] explain the concepts more extensively. Colao et al. in [6] brought up mixed variational inequality problem on Hadamard manifolds and proved the existence of a solution of this problem in Hadamard manifolds framework under suitable conditions.

In this paper, at first, we generalize KKM's lemma due to Fan in [12] on Hadamard manifolds with constant sectional curvature -1 and then we state and prove the existence theorem for the solution of equilibrium problem for Hadamard manifolds with sectional curvature -1. Then, as an application of this equilibrium problem, we develop and solve the mixed variational inequality problem on Hadamard manifolds with sectional curvature -1 under weaker conditions. The organization of this paper is as follows:

In Section 1 some concepts and basic results in Riemannian manifold which will be used throughout the paper are presented.

In section 2, at first, we obtain and prove the lemmas and theorems that are essential in this paper. Then inspired by them, we bring up and prove Fan's KKM lemma (1984) [12], under weaker condition for Hadamard manifolds with sectional curvature -1. After that, as a consequence of this KKM lemma, we develope and prove the existence theorem for the solution of equilibrium problem on Hadamard manifold with sectional curvature -1.

Finally, as an application of this equilibrium theorem, we generalize the mixed variational inequality under weaker conditions for Hadamard manifolds with sectional curvature -1.

2. Preliminaries

In this section, we recall some concepts and basic results which are needed in this paper (for more details on these concepts and results we refer the readers to [9, 4, 28, 18, 19, 20]). Recall that the definition of smooth function between two finite dimensional smooth manifolds is as follows:

Definition 1. *[18]. Let M and N be two smooth manifolds of dimensions m, n, respectively. A map $F : M \to N$ is called smooth at $p \in M$ if there exist charts (U, φ) , (V, ψ) with $p \in U \subset M$ and $F(p) \in V \subset N$ with $F(U) \subset V$ and the composition $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth. F is called smooth if it is smooth at every $p \in M$.*

Let M be a connected smooth manifold of dimension n. The tangent space of M at x is denoted by T_xM and $\mathcal{X}(M)$ is the space of vector fields over M, in other words, if $V \in \mathcal{X}(M)$, then V is a smooth mapping of M into TM which associate to each $x \in M$ a vector $V(x) \in T_xM$. Firstly, we recall some fundamental concepts about tangent space of manifold which are useful in this paper.

Proposition 1. *[9]. The tangent vector X_p to the curve $c : (-\varepsilon, \varepsilon) \to M$ at $t = 0$ can be considered as a map $c'(0) : C^\infty(c(0)) \to \mathbb{R}$ given by the formula*

$$X_p(f) = c'(0)(f) = \left.\frac{d(f \circ c)}{dt}\right|_{t=0}, \forall f \in C^\infty(p), c(0) := p,$$

where $C^\infty(p)$ is the set of functions defined on some open neighbourhood of p and smooth at p.

Remark 1. *[9]. Every tangent vector at p on a smooth manifold M is a linear function defined on $C^\infty(p)$ which satisfies the Leibniz property:*

$$X_p(fg) = X_p(f)g + fX_p(g); \forall f, g \in C^\infty(p)$$

Definition 2. *[9]. Let M be a smooth manifold. A Riemannian metric g on M is a smooth family of inner products on the tangent spaces of M. Namely, g associates to each $p \in M$ a positive definite symmetric bilinear form on*

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

and the smoothness condition on g refers to the fact that the function

$$p \to g_p(X(p), Y(p)) \in \mathbb{R}$$

must be smooth as in Definition 1 for every locally defined smooth vector fields X, Y in M.

Equivalently, a Riemannian metric is a symmetric, positive definite $(0,2)$ tensor field on M . A Riemannian manifold is a pair (M,g) where M is a differentiable manifold and g is a Riemannian metric on M.

As a consequence of this definition, it is deduced that for each $p \in M$ the Riemannian metric induces a norm on T_pM. The following theorem states the existence result which guarantees such a Riemannian metric for every smooth manifold.

Theorem 1. *[9]. Any smooth manifold can be given a Riemannian metric.*

So, as a consequence of the above theorem, we can deduce that every smooth manifold can be considered as Riemannian manifold.

Let $\delta : [a,b] \to M$ be a piecewise smooth curve segment. Then the length of δ is defined and denoted by

$$L[\delta] = \int_a^b \left(g_{\delta(t)}\left(\delta'(t), \delta'(t)\right)\right)^{\frac{1}{2}} dt.$$

The following proposition guarantees a piecewise smooth curve between each pair of points in a smooth manifold.

Proposition 2. *[18]. If M is a connected smooth manifold, any two points of M can be joined by piecewise smooth curve segment.*

The following example shows that the result of Proposition 2 may fail when "connected" is replaced by "locally connected".

Example 1. *The subspace $(0,1) \cup (2,3)$ of the real line $\mathbb{R}$ is a locally connected manifold but there exist many pairs of points for which there isn't any piece-wise smooth curve that joins them.*

The distance between two points $x, y \in M$ which is defined in [18] is as follows:

$$d(x,y) =: \quad \inf\{L[\delta] \mid \delta : [a,b] \to M \ \textit{ piecewise smooth curve}, \\ \delta(a) = x, \ \delta(b) = y\}.$$

It should be noted that minimal curve plays an important role in geometry which is defined as follows:

Definition 3. *The piecewise smooth curve $\delta : [a,b] \to M$, $\delta(a) = x$ and $\delta(b) = y$, joining two points x ,y in M is called minimal curve between x and y if*

$$L[\delta] = d(x,y).$$

Notice that the Riemannian metric at each $p \in M$ induces an inner product on tangent space T_pM of manifold and then using this inner product on T_pM the length of piecewise smooth curve segment is defined. Then the distance function d on M is defined.

The following proposition shows that M is a metric space with the distance function d and states the relationship between topology induced by the metric of distance function on M and the inherent topology of M.

Proposition 3. *[20]. If M is a Riemannian manifold with metric g, then M is a metric space with the distance function d defined above. The metric topology agrees with the manifold topology.*

The following remark recalls the statement related to finite intersection property in metric space which is used in this paper.

Remark 2. *[5]. A metric space X is compact if and only if every collection $\mathcal{F}$ of closed sets with the finite intersection property has nonempty intersection.*

An affine connection is a geometric object on a smooth manifold which connects nearby tangent spaces, and this is useful to define the concept of geodesics between two points in Riemannian manifold.

Definition 4. *[28]. An affine (linear) connection on a smooth manifold M is a mapping*

$$\begin{cases} \nabla : X(M) \times X(M) \to X(M) \\ \qquad (X,Y) \mapsto \nabla_Y X \end{cases}$$

satisfying the following conditions:

- *For each fixed $Y \in \mathcal{X}(M)$ the map $X \to \nabla_X Y$ is $C^\infty(M)-$ linear . That is $\nabla_{fX+gY} V = f\nabla_X V + g\nabla_Y V$ for all $f, g \in C^\infty(M)$ and $X, Y, V \in \mathcal{X}(M)$.*
- *For each $X \in \mathcal{X}(M)$ the map $Y \to \nabla_X Y$ is R-linear. That is $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$ for all $a, b \in R$ and $Y_1, Y_2 \in \mathcal{X}(M)$.*
- *$\nabla_X(fY) = X(f)Y + f\nabla_X Y$ for all $f \in C^\infty(M)$ and $X, Y \in \mathcal{X}(M)$, where $X(f) \in C^\infty(M)$. ($C^\infty(M) := \{smooth\ \ fuctions\ \ on\ \ M\}$)*

In Riemannian geometry, the Levi-Civita connection is a specific connection on tangent bundle and it is useful to define geodesics on Riemannian manifold M.

Definition 5. *[28]. An affine connection ∇ is called a Levi-Civita connection if*

- *It preserves the metric, i.e.,* $\nabla g = 0$*. that is :*

$$\nabla_X g\left(Y, Z\right) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right),$$

for all vector fields X, Y, Z *on* M*, where* $\nabla_X g\left(Y, Z\right)$ *denotes the derivative of the function* $g\left(Y, Z\right)$ *along the vector field* X*.*

- *It is torsion-free, that is, for any vector fields* X *and* Y *we have*
$\nabla_X Y - \nabla_Y X = [X, Y]$*, where* $[X, Y]$ *is the Lie bracket of the vector fields* X *and* Y*.*

Proposition 4. *[20]. For each Riemannian manifold* (M, g)*, there exists a unique Levi-Civita connection on* M*.*

Let ∇ be the Levi-Civita connection associated with (M, g) and let δ be a smooth curve in M. A vector field $X \in \mathcal{X}\left(M\right)$ is said to be parallel along δ if $\nabla_{\delta'} X = 0$.
By using this fact, the concept of geodesics is defined as follows:

Definition 6. *[9]. A geodesics on a smooth manifold M is defined as a curve* $\delta\left(t\right)$ *such that parallel transport along the curve preserves the tangent vector to the curve, so*

$$\nabla_{\delta'(t)} \delta'\left(t\right) = 0,$$

at each point along the curve, where δ' *is the derivative with respect to t.*

Note that if δ' itself is parallel along δ, we say that δ is a geodesic and in this case $||\delta'\left(t\right)||$ is constant, where the norm induced by Riemannian metric is defined as

$$\parallel \delta'\left(t\right) \parallel := \left(g_{\delta(t)}\left(\delta'\left(t\right), \delta'\left(t\right)\right)\right)^{\frac{1}{2}}.$$

It should be noted that when $||\delta'\left(t\right)|| = 1$, then δ is said to be normalized.

Remark 3. *A geodesics joining x and y in M is said to be minimal if the geodesy induced by smooth curve* δ *between two points* x *and* y *is minimal.*

Recall that complete Riemannian manifold is useful to define Hadamard manifold which is defined as follows:

Remark 4. *[18]. A Riemannian manifold is complete if for any* $x \in M$ *all geodesics emanating from* x *are defined for all* $t \in \mathbb{R}$*.*

The following example shows that there exists Riemannian manifold which is not complete.

Example 2. *A simple example of a non-complete manifold is given by the punctured plane* $M := \mathbb{R}^2 \setminus \{0\}$ *(with its induced metric). Geodesics passing to the origin cannot be defined on the entire real line.*

Exponential map makes relationships between manifolds and their tangent spaces at each point. Also, it can be deduced that M is locally diffeomorphic to the Euclidean space $\mathbb{R}^m$. Thus, the manifold M has the same topology and differential structure as $\mathbb{R}^m$.

Definition 7. *[28]. Assuming that M is complete, the exponential map $exp_p : T_pM \to M$ at p is defined by $exp_p v = \delta_v(1,p)$ for each $v \in T_pM$, where $\delta(.) = \delta_v(.,p)$ is the geodesics starting at p with velocity v (that is, $\delta(0) = p$ and $\delta'(0) = v$). Then $exp_p tv = \delta_v(t,p)$ for each real number t.*

Note that the differential structures are invariant under diffeomorphism mappings, but algebraic structures are not invariant under diffeomorphism mappings such as linear structures and structures of vector spaces.

Theorem 2. *(**Hopf Rinow**)[4]. Let (M, g) be a connected Riemannian manifold. Then the following statements are equivalent:*

(i) The closed and bounded subsets of M are compact;

(ii) M is a complete metric space; (with respect to the metrics induced by distance function on M)

(iii) M is complete; that is, for every p in M, the exponential map exp_p is defined on the entire tangent space TpM.

As a consequence of Hopf-Rinow theorem, we deduce that if Riemannian manifold M is complete, then (M, d) is a complete metric space, and bounded closed subsets are compact. Moreover, we know that if M is complete then any pair of points in M can be joined by a minimal geodesics.
In the sequel, we recall some topological concepts which are needed in the next section and play an important role in this paper (for more details see [19, 22])

At first, we recall the concept of path-homotopy and then the concept of simply connected topological space.

Definition 8. *[22]. Suppose X and Y are topological spaces, and let I denote the interval $[0, 1] \subset \mathbb{R}$. If $\alpha, \beta : X \to Y$ are two continuous maps, a homotopy from α to β is a continuous map $H : X \times I \to Y$ such that for every $x \in X$ the following relations are true:*

$$H(x, 0) = \alpha(x),$$

$$H(x, 1) = \beta(x).$$

We need a kind of definition of path which is called a loop. A loop based on point x is a continuous function $\alpha : [a, b] \to X$ such that $\alpha(a) = \alpha(b) = x$. In other words, a loop is a path whose starting point is equal to the ending point.

According to the above statements, recall the definition of simply connected topological spaces as follows:

Definition 9. *[19]. A topological space X is said to be simply connected if it is path connected and every loop in X can be continuously shrunk to the constant loop while its endpoints are kept fixed (i.e., the loop $c : [a, b] \to X$ such that $c(s) = x$ for all $s \in [a, b]$).*

Definition 10. *[22] Let X be a topological space. A covering space of X is a topological space E together with a continuous surjective map $\mathrm{P} : \mathrm{E} \to \mathrm{X}$ such that for any $x \in X$, there exists an open nieghborhood U of x such that $P^{-1}(U)$ is a union of disjoint open sets in E.*

A special covering space is called a universal covering space defined as follows:

Definition 11. *[22]. A covering space is universal covering space if it is simply connected.*

The following proposition shows the connection between two simply connected universal covers of the same space.

Proposition 5. *[19]. Any two simply connected coverings of the same space are isomorphic.*

In the following statement we recall the definition of Hadamard manifold which this paper is based on.

Definition 12. *[4]. A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. (The sectional curvature is one of the ways to describe the curvature of Riemannian manifolds; for more details about curvature see [9].)*

So from now on M always is a finite dimensional Hadamard manifold.

Example 3. *The real line $\mathbb{R}$ with its usual metric is a Hadamard manifold with constant sectional curvature equal to 0.*

The following proposition emphasizes that in Hadamard manifold for each pair of points there exists a unique normalized geodesics which is joining them minimally.

Proposition 6. *[28]. Let M be a Hadamard manifold and $p \in M$. Then $exp_p : T_pM \to M$ is a diffeomorphism, and for any two points $x, y \in M$ there exists a unique normalized geodesics joining x to y, $\delta_{x,y}$, which is minimal.*

Let $p, q \in M$ be two fixed points. By construction, the unique minimal geodesics $\delta : [0, 1] \to M$ joining these points is given by $\delta(t) = exp_p\left(texp_p^{-1}(q)\right)$.
So from now on, when referring to the geodesics joining two points we mean the unique minimal normalized one.
Recall that a geodesic triangle $\Delta(x_1, x_3, x_3)$ of a Hadamard manifold M is a set consisting of three points x_1, x_2, x_3 and three minimal geodesics joining these points. As in Euclidean geometry, the concept of convexity plays an important role also in Hadamard manifold. It is defined in Hadamard manifold as follows:

Definition 13. *[4]. Let M be a Hadamard manifold. A subset $K \subseteq M$ is said to be convex if for any two points x and y in K, the geodesics joining x to y is contained in K; that is, if $\delta : [a, b] \to M$ is a geodesics such that $x = \delta(a)$ and $y = \delta(b)$, then $\delta((1-t)a + tb) \in K$ for all $t \in [0, 1]$.*

Definition 14. *[18]. A real-valued function f defined on M is said to be convex if for any geodesics δ of M, the composition function $f \circ \delta : \mathbb{R} \to \mathbb{R}$ is convex; that is,*

$$(f \circ \delta)(ta + (1-t)b) \leq t(f \circ \delta)(a) + (1-t)(f \circ \delta)(b)$$

for any $a, b \in \mathbb{R}$, and $0 \leq t \leq 1$.

The following proposition is useful in this paper.

Proposition 7. *[25]. Assume that M is a Hadamard manifold and K is a nonempty subset of M. Let $x \in K$ and $\mathrm{u} \in \mathrm{T_x M}$. Define the function $g : M \to \mathbb{R}$ by*

$$g(y) := \langle u, exp_x^{-1} y \rangle .$$

Then both g and $-g$ are affine, in other words, g and $-g$ are convex functions.

Lemma 1. *[4]. Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold M. Then there exist unique $x', y', z' \in \mathbb{R}^2$ such that*

1. $d(x, y) = ||x' - y'||$
2. $d(x, z) = ||x' - z'||$
3. $d(y, z) = ||y' - z'||$.

The triangle $\Delta(x', y', z')$ is called comparison triangle of the geodesic triangle $\Delta(x, y, z)$, which is unique up to isometry of M. The following Lemma shows the relation between geodesic triangle and its comparison triangle. This Lemma states the geometric idea behind manifold with nonposetive sectional curvature.

Lemma 2. *[4]. Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold M and $\Delta(x', y', z')$ be its comparison triangle.*
Given any point q belonging to the geodesics which joins x to y, its comparison point is the point q' in the interval $[x', y']$ such that $d(q, x) = \parallel q' - x' \parallel$ and $d(q, y) = \parallel q' - y' \parallel$. Then

$$d(z, q) \leq \parallel z' - q' \parallel .$$

We introduce the Hyperbolic geometry briefly which is useful for this paper as follows. To comprehend the Hyperbolic geometry and the following statements which are concentrated in Hyperbolic geometry in this paper, see [29, 15, 27].

Hyperbolic geometry is a result of trying to prove parallel postulate of Euclidean geometry. For about 2000 years, mathematicians try to prove parallel postulate by use of the other postulates of Euclidean geometry. So Hyperbolic geometry arose out. For Hyperbolic geometry, some models are suggested such as Klein model and Half-plane model which explain this geometry by tools of Euclidian geometry. It should be noted that all of these models are related to each other. Hence, we consider upper-half plane as follows:

From the geometrical point of view, the hyperbolic plane is the open upper half plane which is denoted by H^2, in other words $H^2 := \{x + yi| \, y > 0\}$. A geodesics in H^2 is either a semicircle meeting the real axis at right angles, or a vertical ray emanating from a point on the axis. Let p and q be two points in H^2, o and r denote the points where the geodesics between p and q meets the real axis, the distance between two points p and q be defined as follows:

$$distance\,(p, q) := \left| ln \frac{|o - q|\,|p - r|}{|o - p|\,|q - r|} \right| .$$

Here $|o - p|$ is the Euclidean distance from o to p.
In the following paragraph, we explain the Klein model:
In the Klein model, points are inside the unit disk. The geodesics consist of all diameters of disk and all arcs of Euclidean circles that are contained within the disk. Note that these arcs are orthogonal to the boundary of the disk. Additionally, according to the all above mentioned information in two-dimensional models, n-dimensional half-plane model, which is called Hyperbolic plane and denoted by H^n , and n-dimensional Klein model are obtained by replacing lines by Hyperplanes and circles by hyperspheres.
The following statement shows that all this models are isomorphic.

Proposition 8. *[27]. All of models in Hyperbolic geometry are isomorphic.*

3. Main results

At first, we introduce the following concepts and lemmas which play a crucial role in this paper and then by inspiration taken from these lemmas, we prove that convex hull of every compact set in finite dimensional Hadamard manifold with constant sectional curvature -1 is compact.
Also, we bring up KKM lemma [12] which concentrates on Hausdorff topological vector spaces for Hadamard manifolds with constant sectional curvature -1 and then develop and prove the existence theorem of equilibrium problem for Hadamard manifolds with

sectional curvature −1. Finally, as a result of this equilibrium theorem, we develop and prove mixed variational inequality problem under weaker conditions.

Definition 15. *Let M be a Hadamard manifold. Define the function γ on $M\times M\times[0,1]$ as*

$$\begin{cases} \gamma : M\times M\times[0,1]\to M \\ \quad (x,y,t)\mapsto\gamma\,(x,y;t) \end{cases}$$

where $\gamma\,(x,y;t) := \delta\,(x,y;t)$ and $\delta\,(x,y;t)$ is a point of the unique minimal geodesics joining two points x,y.

Note that for all $t\in[0,1]$, $\delta\,(x,y;t)$ is the unique minimal geodesics joining two points x and y.
In the two following lemmas, at first we show that the function γ is continuous in each variable and then we prove the critical lemma that is useful to establish that γ is a continuous function on $M\times M\times[0,1]$.

Lemma 3. *The function $\gamma : M\times M\times[0,1]\to M$ is continuous function in each variable of $M\times M\times[0,1]$.*

Proof. Since δ is a geodesics by definition of γ it is easy to see that for two fixed points x and y in M, the function γ is continuous in third variable.
To prove that γ is continuous in first variable, we must show that

$$\lim_{x\to x_0}\gamma\,(x,y;t)=\gamma\,(x_0,y;t)\,.$$

Let $\Delta\,(x,y,x_0)$ be a geodesic triangle in Hadamard manifold M, and $\Delta\,(x',y',{x_0}')$ be its comparison triangle in $\mathbb{R}^2$, with the fixed point y and time t_0. We know that $\gamma\,(x,y;t_0)$ is a point on the geodesics joining x,y at $t=t_0$. The geodesics joining x ,y is denoted by $\gamma_{x,y}$. We also know that $\gamma\,(x_0,y;t_0)$ is a point on $\gamma_{x_0,y}$ in the $\Delta\,(x,y,x_0)$. Now we consider $\Delta\,(x_0,\gamma\,(x,y;t_0)\,,y)$ in M and its comparison geodesic triangle $\Delta\,({x_0}',\gamma\,(x',y';t_0)\,,y')$ in $\mathbb{R}^2$, where $\gamma\,(x',y';t_0)$ is a comparison point $\gamma\,(x,y;t_0)$ on geodesics (line segment) $\gamma_{x',y'}$.
According to this, $\gamma\,(x_0,y;t_0)$ is a point on $\gamma_{x_0,y}$ in $\Delta\,(x_0,\gamma\,(x,y;t_0)\,,y)$ and $\gamma\,({x_0}',y';t_0)$ is its comparison point on $\gamma_{{x_0}',y'}$ in $\Delta\,({x_0}',\gamma\,(x',y';t_0)\,,y')$.
Since $d\,(y,\gamma\,(x_0,y;t_0))=||y'-\gamma\,({x_0}',y';t_0)||$ and $d\,(x_0,y)=||{x_0}'-y'||$, we have

$$d\,(x_0,\gamma\,(x_0,y;t_0))=\left|\left|{x_0}'-\gamma\left({x_0}',y';t_0\right)\right|\right|.$$

So from Lemma 2, we have

$$\begin{aligned} d\,(\gamma\,(x,y,t_0)\,,\gamma\,(x_0,y;t_0)) &\leq \parallel\gamma\,(x',y',t_0)-\gamma\,({x_0}',y';t_0)\parallel \\ &\leq \parallel t_0x'+(1-t_0)y'-(t_0{x_0}'+(1-t_0)y')\parallel \\ &\leq \parallel t_0x'-t_0{x_0}'\parallel\leq t_0\parallel x'-{x_0}'\parallel, \end{aligned}$$

Since $d(x,x_0)=\| x'-x_0{}' \|$, as $x\to x_0$, we have

$$d(\gamma(x,y,t_0),\gamma(x_0,y;t_0))=0.$$

Similarly we can prove that

$$\lim_{y\to y_0}\gamma(x,y;t)=\gamma(x,y_0;t).$$

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Lemma 4. *Let M be a Hadamard manifold and x and y be arbitrary points in M. For every $u\in M$, the following statement is true:*

$$d(u,\gamma(x,y;t))\leq td(u,x)+(1-t)\,d(u,y) \tag{1}$$

Proof.

Firstly, the Lemma is true in the setting of an Euclidean space, since geodesics in Euclidean space are straight lines, since the geodesic between two points x and y is $tx+(1-t)\,y$, so $\gamma(x,y;t)=tx+(1-t)\,y$, and it is clear that the inequality (1) is satisfied.

By Lemma 1 for three points x,y and u in M, there exist unique points $x',y',u'\in\mathbb{R}^2$ such that $d(x,y)=||x'-y'||$, $d(x,u)=||x'-u'||$ and

$$d(y,u)=||y'-u'||.$$

We claim that there exists an isometry of a minimal geodesics between x and y which is denoted by $\sigma_{x,y}$ to line segment between x' and y', denoted by $[x',y']$.

In fact, since $\sigma_{x,y}$ is a minimal geodesics between x and y, there exists a continuous curve $c:[0,l]\to M$ such that $c(0)=x$, $c(l)=y$ and $d(x,y)=l$.

We define $\begin{cases}\varphi:\sigma_{x,y}\to[x',y']\\ \varphi(c(t))=\frac{l-t}{l}x'+\frac{t}{l}y'\end{cases}$. It is easy to see that φ is an isometry.

Since the geodesics c is an isometry, we get $d(c(t),c(t'))=|t-t'|$. For every fixed time $t=t_0$, choose the point $q=\gamma(x,y;t=t_0)$ of $\sigma_{x,y}$. Since φ is an isometry, there exists a corresponding point $q'=\varphi(q)=\gamma(x',y';t_0)$ on $[x',y']$. Since φ is an isometry, we have

$$d(q',x')=d(\varphi(q),\varphi(x))=d(q,x).$$

So it follows from Lemma 2 that $d(u,\gamma(x,y,t_0))\leq\| u'-q' \|$ and

$$d(u,\gamma(x,y;t_0))\leq\| u'-q' \|=\| u'-\gamma(x',y';t_0) \|.$$

Consequently

$$\begin{aligned}\| u'-\gamma(x',y';t_0) \| &\leq t_0\| u'-x' \|+(1-t_0)\| u'-y' \|\\ &= t_0d(u,x)+(1-t_0)\,d(u,y).\end{aligned}$$

So we have

$$d(u, \gamma(x, y; t)) \leq td(u, x) + (1-t)\, d(u, y).$$

◀

In the following lemma, by using the Lemmas 3 and 4 we prove that γ is continuous function on $M \times M \times [0,1]$.

Lemma 5. *Let M be a Hadamard manifold. Then the geodesics*

$$\begin{cases} \gamma & : \quad M \times M \times [0,1] \to M \\ & \quad (x, y; t) \mapsto \gamma(x, y; t) \end{cases}$$

is a continuous mapping on $M \times M \times [0,1]$.

Proof. Without loss of generality, we can suppose that $t_0 \leq t$. To prove γ is continuous on $M \times M \times [0,1]$, we must show that

$$\lim_{(x,y;t)\to(x_0,y_0;t_0)} \gamma(x, y; t) = \gamma(x_0, y_0; t_0).$$

Equivalently

$$\forall \varepsilon > 0, \exists \delta > 0, \quad d_\infty((x, y; t), (x_0, y_0; t_0)) < \delta \to d(\gamma(x, y; t), \gamma(x_0, y_0; t_0)) < \varepsilon,$$

where $d_\infty((x, y; t), (x_0, y_0; t_0)) := \max\{d(x, x_0), d(y, y_0), d(t, t_0)\}$.
For each given $\varepsilon > 0$, it suffices that we take $\delta := \min\{\delta_1, \delta_2, \delta_3\}$, where $\delta_1, \delta_2, \delta_3$, respectively, correspond to continuity of γ in first, second and third variable (Note that by Lemma 3 , we know that γ is continuous in each variable). From inequality (1) of Lemma 4, we have

$$\begin{aligned} d(\gamma(x, y; t), \gamma(x_0, y_0; t_0)) \leq\; & tt_0 d(x, x_0) + t(1-t_0)\, d(x, y_0) + \\ & t_0(1-t)\, d(y, x_0) + (1-t)(1-t_0)\, d(y, y_0). \end{aligned} \tag{2}$$

By Lemma 3, we know that γ is continuous in each variable. From continuity of γ in second and third variable and according to inequality (1) of Lemma 4, we have two cases for $d(x, y_0)$:

a) $d(x, y_0) \leq \frac{\varepsilon}{t(1-t)} - \frac{\varepsilon}{t(1-t_0)+t_0(1-t)} - \frac{(1-t)^2}{t(1-t)}\delta$

b) $d(x, y_0) \geq \frac{\varepsilon}{t(1-t)} - \frac{\varepsilon}{t(1-t_0)+t_0(1-t)} - \frac{(1-t)^2}{t(1-t)}\delta$,

And from continuity of γ in first and third variable and according to inequality (1) of Lemma 4, there exist two cases for $d(x_0, y)$:

c) $d(x_0, y) \geq \frac{\varepsilon}{t(1-t)} - \frac{t^2\delta}{t(1-t)} - \frac{\varepsilon}{t(1-t_0)+t_0(1-t)}$

d) $d(x_0, y) \leq \frac{\varepsilon}{t(1-t)} - \frac{t^2\delta}{t(1-t)} - \frac{\varepsilon}{t(1-t_0)+t_0(1-t)}$.

By comparing these cases and according to the inequality (2), and due to the t tending to t_0, it is not hard to see that

$$d(\gamma(x, y; t), \gamma(x_0, y_0; t_0)) \leq \varepsilon.$$

◄

It should be noted that in vector spaces we have linear structure. So we know that if E is a vector space, then for every subset X of E, the convex hull of X which is denoted by $co(X)$, is the set of affine convex combinations of finite subsets of X, in other words, each element of $co(X)$ is an affine finite combination element in X. This method in vector spaces is very useful and has many results.

As in this framework (Hadamard manifold) we don't have linear structure, in order to incept this method at this framework we need to have the following definition. This definition is useful in the rest of this paper.

Let M be a finite dimensional Hadamard manifold and $\{x_1, x_2, \ldots, x_n\}$ be a finite subset of M.

Define $D(x_1, x_2; t_1, t_2)$ as a point of geodesics joining two points x_1, x_2 at (t_1, t_2), such that $t_1, t_2 \in [0, 1]$ and $t_1 + t_2 = 1$, that is

$$D(x_1, x_2; t_1, t_2) := \gamma(x_2, x_1; t); \quad \textit{with} \;\; t = t_2 \;\; \textit{and} \;\; (1-t) = t_1.$$

(Note that since M is a Hadamard manifold and we know that in Hadamard manifolds for each pair of points there exists unique geodesics between them which is minimal). So always for each pair of points x, y in M, there exist such $t_1, t_2 \in [0, 1]$.

Note that in the normed space the geodesics between two points is a line segment between them, i.e $\delta(x, y; t) = tx + (1-t)y$. Then

$$D(x_1, x_2; t_1, t_2) := \gamma(x_2, x_1; t) = tx_2 + (1-t)x_1 = t_2x_2 + t_1x_1.$$

By induction we can define:

$$D(x_1, x_2, x_3; t_1, t_2, t_3) := \gamma\left(x_3, D\left(x_1, x_2; \frac{t_1}{1-t_3}, \frac{t_2}{1-t_3}\right); t_3\right),$$

such that $t_1 + t_2 + t_3 = 1$, $t_3 \neq 1$. $(t_1, t_2, t_3 \in [0, 1])$
So, we have:

Definition 16. *Let $x_1, x_2, \ldots, x_n$ be a finite number of elements of a Hadamard manifold M and $t_1, t_2, \ldots, t_n \in [0,1]$ be such that $\sum\limits_{i=1}^{n} t_i = 1$. For each positive integer n, inductively, we can define*

$$D\left(x_1, \ldots, x_n; t_1, \ldots, t_n\right) := \begin{cases} \gamma\left(x_n, D\left(x_1, \ldots, x_{n-1}; \frac{t_1}{1-t_n}, \ldots, \frac{t_{n-1}}{1-t_n}\right); t_n\right) & t_n \neq 1 \\ x_n & t_n = 1 \end{cases}$$

Note that the definition of convex hull here is the definition of convex hull in [6], which is as follows:

Definition 17. *[6] Let $A \subset M$. The convex hull of A is defined by*

$$co\,(A) := \bigcap \{E \subseteq M : E \text{ is a convex subset containing } A\}.$$

By Definition 16, we can define the set of all $D\left(x_1, \ldots, x_n; t_1, \ldots, t_n\right)$ for every $t_1, \ldots, t_n \in [0,1]$ as follows:

Definition 18. *Let $x_1, x_2, \ldots, x_n$ be a finite number of elements of a Hadamard manifold M and for every $t_1, t_2, \ldots, t_n \in [0,1]$ such that $\sum\limits_{i=1}^{n} t_i = 1$, define*

$$D\left(x_1, \ldots, x_n\right) := \left\{ D\left(x_1, \ldots, x_n; t_1, \ldots, t_n\right); \forall t_i \in [0,1], \text{ such that } \sum_{i=1}^{n} t_i = 1 \right\}.$$

Recall some important statements that are useful to prove the Proposition 9:

Remark 5. *[19]. The simply connected universal cover of a finite dimensional Hadamard manifold M with sectional curvature -1 is H^n.*

Remark 6. *[2]. Every geodesic triangle in H^n lies in 2-dimensional Hyperbolic plane $H^2 \subset H^n$.*

In the following Proposition, inspired by the above remarks, we can state and prove that for a finite subset $\{x_1, \ldots, x_n\}$ in M, $D\left(x_1, \ldots, x_n\right)$ mentioned in Definition 18, is equal to $co(\{x_1, \ldots, x_n\})$.

Proposition 9. *Let M be a finite dimensional Hadamard manifold with constant sectional curvature -1. Then for finite elements $x_1, x_2, \ldots, x_n \in M$*

$$D\left(x_1, \ldots, x_n\right) = co\left\{x_1, \ldots, x_n\right\} \tag{3}$$

Proof. We prove this statement by induction. From Remark 5, we know that the simply connected universal covering space M is H^n and it is easy to see that M is a universal cover of itself. Also, by definition of Hadamard manifold, we know that M is simply connected. Since both M and H^n are simply connected universal covers of M, so according to Proposition 5, we deduce that M and H^n are isomporphic. So it is adequate to prove statement (3) for H^n.

For three elements x_1, x_2, x_3 in H^n we prove that $D(x_1, x_2, x_3) = co\{x_1, x_2, x_3\}$. It suffices to show that $D(x_1, x_2, x_3)$ is convex. As a consequence of Remark 6, we know that the three points x_1, x_2, x_3 in H^n lie in a copy of H^2, which is totally geodesic in H^n.

So any construction involved in taking geodesics between these three points in this copy of H^2 stays in H^2. So, it suffices to prove that the structure $D(x_1, x_2, x_3)$ is convex in the case of H^2.

According to structure of geodesics in the half-plane model H^2, we know that geodesics in H^2 is either a semicircle meeting the real axis at right angles, or a vertical ray emanating from a point on the real axis, and also we know that in H^2 any two discrete geodesic meeting together at most one point, so it is not hard to see that $D(x_1, x_2, x_3)$ is convex in H^2.

To prove the statement for finite elements $x_1, x_2, \ldots, x_n$, we can use the Klein model, in which all geodesics are straight line segments and all totally geodesic subspaces are intersections of the unit ball with Euclidean planes. The convex hull of n points in H^n is contained in an isometric copy of H^{n-1}, and according to the statement that any two discrete geodesics meeting together at most one point, it is not hard to see that $D(x_1, \ldots, x_n)$ is convex in H^n, and this completes the proof.◄

As a consequence of Proposition 9, we deduce that $D(x_1, \ldots, x_n)$ is convex in Hadamard manifold M with constant sectional curvature -1 that it is very essential in the following lemma to show that for a nonempty subset X of M the convex hull $co(X)$ is the set of convex combinations of finite subsets of X.

Lemma 6. *Let M be a finite dimensional Hadamard manifold with constant sectional curvature -1 and X be a nonempty subset of M, The convex hull of X is the set of convex combinations of finite subsets of X. In other words, $co(X)$ is equal to the subset $S(X)$ of M defined by*

$$\begin{aligned} S(X) \;=\; & \{D(x_0, x_1, \ldots, x_n; t_0, t_1, \ldots, t_n) \\ & \textit{with}\;\; n \geq 0;\; x_i \in X \;\; \textit{such that}\;\; t_i \in [0,1], \sum_{i=0}^{n} t_i = 1, \textit{for}\;\; i = 0, 1, .., n\}. \end{aligned}$$

Proof. Consider $a \in S(X)$. Then there exists nonnegative integer n and $t_0, t_1, t_2, \ldots,$ $t_n \in [0,1]$, $t_n \neq 1$ and $x_0, x_1, \ldots, x_n \in X$, such that $\sum_{i=0}^{n} t_i = 1$ and

$a = D(x_1, \ldots, x_n; t_1, \ldots, t_n)$. By Definition 16, we know that

$$\begin{aligned} a &= D(x_0, \ldots, x_n; t_1, \ldots, t_n) \\ &= \gamma\left(x_n, D\left(x_0, \ldots, x_{n-1}; \tfrac{t_0}{1-t_n}, \ldots, \tfrac{t_{n-1}}{1-t_n}\right); t_n\right), \end{aligned}$$

Since $x_n \in co(X)$, $D\left(x_0, \ldots, x_{n-1}; \frac{t_0}{1-t_n}, \ldots, \frac{t_{n-1}}{1-t_n}\right) \in co(X)$, and since $co(X)$ is convex, the minimal geodesics between them belongs to $co(X)$, i.e
$\gamma\left(x_n, D\left(x_0, \ldots, x_{n-1}; \frac{t_0}{1-t_n}, \ldots, \frac{t_{n-1}}{1-t_n}\right); t_n\right) \in co(X)$, so $a \in co(X)$.

To complete the proof of Lemma, we must show that $co(X) \subseteq S(X)$, so it is suffices to show that $S(X)$ is convex. As a consequence of Proposition 9, it is not hard to see that $S(X)$ is convex.◄

In the following theorem, we show that in finite dimensional Hadamard manifold with sectional curvature -1, the convex hull of every compact set is compact.

Theorem 3. *Let M be a finite dimensional Hadamard manifold with constant sectional curvature -1, and let X be a compact subset of M. Then $co(X)$ is compact.*

Proof. Let $\{x_n\}_{n\geq 0}$ be a sequence in $co(X)$. Let's prove that this sequence has a convergent subsequence. According to Lemma 6, for each $n = 0, 1, 2, \ldots$, there exists nonnegative integer d_n denoted by d ($d := d_n$) such that

$$x_n = D(a_{n,0}, a_{n,1}, \ldots, a_{n,d}; t_{n,0}, \ldots, t_{n,d}).$$

with $a_{n,k} \in X$, $t_{n,k} \geq 0$, $\sum_{k=0}^{d} t_{n,k} = 1$, for every $k = 0, 1, .., d$.

The sequence $\{a_{n,0}\}_{n\geq 0}$ belongs to the set X. Since X is compact, $\{a_{n,0}\}_{n\geq 0}$ has a convergent subsequence. Let $\{a_{n_1,0}\}_{n_1\geq 0}$ be such a subsequence.

Next, consider $\{a_{n,1}\}_{n\geq 0}$ in X. It also has a convergent subsequence $\{a_{n_2,1}\}_{n_2\geq 0}$. Then consider the sequence $\{a_{n,2}\}_{n\geq 0}$ and so on. Repeating this reasoning $d+1$ times, we end up with a sequence $\{n_d\}_{n\geq 0}$ of nonnegetive integers such that the $d+1$ sequences $\{a_{n_d,0}\}, \{a_{n_d,1}\}, \ldots, \{a_{n_d,d}\}_{n\geq 0}$ are convergent.

We consider now the sequence of real numbers $\{t_{n_d,0}\}_{n\geq 0}$, belonging to the compact interval $[0, 1]$. It has convergent subsequence. By repeating the argument above $d+1$ more times, we end up with a sequences $\{n_{2d}\}_{n\geq 0}$ of nonnegative integers such that the $d+1$ sequences $\{a_{n_{2d},0}\}, \{a_{n_{2d},1}\}, \ldots, \{a_{n_{2d},d}\}_{n\geq 0}$ in X are convergent and the $d+1$ sequence $\{t_{n_{2d},0}\}, \{t_{n_{2d},1}\}, \ldots, \{t_{n_{2d},d}\}_{n\geq 0}$ in $[0, 1]$ are convergent.

Then by Definition 16, and Lemma 5, we deduce that the subsequence $\{x_{n_{2d}}\}_{n\geq 0}$ of $\{x_n\}_{n\geq 0}$ is convergent.This completes the proof.◄

3.1. An extention of Fan's KKM to Hadamard manifolds

In this section, we explore the extension of the Fan's 1984 KKM Theorem [12] on Hadamard manifold with sectional curvature -1, under suitable conditions.

Definition 19. *Let K be a nonempty subset of M and $F : K \to 2^K$ be a set valued mapping. F is called KKM if for every $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in K$*

$$co\left(\{x_1, x_2, \ldots, x_n\}\right) \subseteq \bigcup\nolimits_{i=1}^{n} F\left(x_i\right).$$

In 2012 [6] Colao et al. extended the Fan's 1961 KKM Theorem [10] from topological vector spaces to Hadamard manifolds.

Lemma 7. *[6]. Let M be a Hadamard manifold and K be a closed and convex subset of M, given $G : K \to 2^K$ be a mapping such that for each x, $G(x)$ is closed. Suppose that*

i) There exists $x_0 \in K$ such that $G\left(x_0\right)$ is compact;

ii) $\forall x_1, x_2, \ldots, x_n \in K, co\left(\{x_1, x_2, \ldots, x_n\}\right) \subseteq \bigcup_{i=1}^{n} G\left(x_i\right)$.

Then

$$\bigcap\nolimits_{x \in K} G(x) \neq \phi.$$

Ky Fan, in 1984, obtained the well-known Fan's 1984 KKM theorem in order to relax the compactness condition on topological vector spaces. In this paper, we develop the Fan's 1984 KKM Theorem[12] on Hadamard manifolds with sectional curvature -1 under weaker conditions.

The following Lemma recalls the version of Fan's 1984 KKM Lemma [12] on Hausdorff topological vector spaces.

Lemma 8. *[12]. Let C be a nonempty subset of a Hausdorff topological vector space Y and $F : C \to 2^C$ be a map such that*

(i) F has closed values;

(ii) F is a KKM map;

(iii) There exists a nonempty compact convex subset B of C such that $\bigcap_{x \in B} F(x)$ is compact.

Then

$$\bigcap\nolimits_{x \in C} F\left(x\right) \neq \phi.$$

Assume that M is a finite dimensional Hadamard manifold with constant sectional curvature -1 and K is a submanifold of M. Then we have the following lemma.

Lemma 9. *Let B be a nonempty convex compact subset of K. Then for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K, the set $co(\{x_1, x_2, \ldots, x_n\} \cup B)$ is compact in K.*

Proof. Since M is Hadamard manifold of finite dimension, and K is a submanifold of M, $\{x_1, \ldots, x_n\} \cup B$ is compact. Then it follows from Theorem 3 that

$$co(\{x_1, \ldots, x_n\} \cup B)$$

is compact in K.◀

It should be noted that the authors in [6] extended KKM Lemma [10] from topological vector spaces to Hadamard manifolds under suitable conditions. In our next result, we develop the version of Fan's KKM Theorem [12] from topological vector spaces to Hadamard manifolds with sectional curvature -1 under the conditions weaker than those in Lemma 8.

Theorem 4. *Let M be a Hadamard manifold with sectional curvature -1 and K be a nonempty convex submanifold of M. Suppose that the set-valued mapping $F : K \to 2^K$ satisfies the following conditions:*

(i) F has closed values;

(ii) F is a KKM map;

(iii) There exists a nonempty compact convex subset B of K and there exists a bounded subset S of K such that $\bigcap_{x\in B} F(x) \subseteq S$.

Then

$$\bigcap_{x\in K} F(x) \neq \phi.$$

Proof. First we show that $\bigcap_{x\in B} F(x) \neq \phi$. We define the set-valued map $H : B \to 2^B$ by $H(x) = B \cap F(x)$. For every $x \in B$ we have $H(x) \neq \emptyset$ (since B is nonempty, there exists a $x_0 \in B$ and since $x_0 \in F(x_0)$, this implies that $H(x_0) \neq \phi$). Also, since $F(x)$ for every x in K is closed and B is a compact subset of K, $H(x)$ is compact for every $x \in B$. Moreover, inspired by B is convex subset of K and F is KKM map on K, we deduce that the convex hull of every finite subset of B is contained in $B \cap (\bigcup_{i=1}^{i=n} F(x_i))$. This shows that H is KKM map on B. For every x in B, $H(x)$ is compact and nonempty and all conditions of Lemma 7 is satisfied for set valued mapping H. So from Lemma 7 we have $\bigcap_{x\in B} H(x) \neq \phi$ and according to the definition of H above, we have $\bigcap_{x\in B} F(x) \neq \phi$.

By (iii), there exists a bounded subset S of K such that $\bigcap\limits_{x\in B} F(x) \subseteq S$. Since S is bounded and $\bigcap_{x\in B} F(x) \neq \phi$, $\bigcap\limits_{x\in B} F(x)$ is bounded (as subset of metric spaces). Also, by (i) we know that $F(x)$ is closed for every x in K, so $\bigcap\limits_{x\in B} F(x)$ is closed and bounded subset of K. And, as K is a submanifold of M, it is also a Hadamard manifold. So by Hopf-Rinow theorem 2, we conclude that $\bigcap\limits_{x\in B} F(x)$ is compact.

Let $N = \bigcap\limits_{x\in B} F(x)$. Then N is nonempty and compact. Now, we are going to prove that the intersection $\bigcap_{x\in K} F(x)$ is nonempty. To see this, we consider an arbitrary finite subset $\{x_1, x_2, \ldots, x_n\}$ of K and $K' := B \cup \{x_1, x_2, \ldots, x_n\}$ and $L := co(K')$.
From Lemma 9, L is compact subset of K. Given $y \in L$, define $T(y) = L \cap F(y)$. It is easy to see that for every $y \in L$, $T(y) \neq \emptyset$. Since L is compact and $F(y)$ is closed for every $y \in L$, $T(y)$ is compact for every $y \in L$. Moreover, since L is a subset of K and F is KKM map on K, the convex hull of every finite subset of L is contained in $L \cap (\bigcup_{i=1}^{i=n} F(y_i))$. This implies that T is a KKM map on L. Thus all conditions of Lemma 7 are satisfied for T, so from Lemma 7 we derive that $\bigcap\limits_{y\in L} H(y) \neq \phi$. Therefore, definition of H and N imply that

$$N \cap (\bigcap_{i=1}^{i=n} F(x_i)) \supseteq L \cap (\bigcap\nolimits_{x\in B} F(x)) \cap (\bigcap\nolimits_{i=1}^{i=n} F(x_i)) \supseteq \bigcap\nolimits_{y\in L} T(y).$$

Thus for every finite elements $x_1, x_2, \ldots, x_n \in K$, we get
$\bigcap_{i=1}^{n} (N \cap F(x_i)) \neq \phi$, which implies that the collection $\{N \cap F(x) : x \in K\}$ has finite intersection property.
Thus by Remark 2 we conclude that $\bigcap_{x\in K} (N \cap F(x)) \neq \phi$ and consequently

$$\bigcap\nolimits_{x\in K} F(x) \neq \phi.$$

◄

Remark 7. *It should be noted that since every nonempty and open subset of smooth manifold is a submanifold, if we assume in the above theorem that K is nonempty, open and convex subset of finite dimensional Hadamard manifold M with sectional curvature -1, then Theorem 4 is also true.*

This is the open version Fan-KKM lemma, that is when the values of the set valued mapping are open.

The following corollary is a special case of the previous theorem that is useful in this paper.

Corollary 1. *Let M be a Hadamard manifold with sectional curvature -1 and K be a convex submanifold of M. Suppose that the set-valued mapping $F : K \to 2^K$ satisfies the following conditions:*

(i) F has closed values;

(ii) F is a KKM map;

(iii) There exists a nonempty compact convex subset B of K such that $\bigcap_{x\in B} F(x)$ is compact.

Then

$$\bigcap_{x\in K} F(x) \neq \phi.$$

Proof. By Hopf-Rinow Theorem, if $\bigcap_{x\in B} F(x)$ is compact then $\bigcap_{x\in B} F(x)$ is bounded and closed, so by the previous theorem it is straightforward to deduce that

$$\bigcap_{x\in K} F(x) \neq \phi.$$

◄

We are now able to establish our existence results for the solutions of the equilibrium problem in the setting of Hadamard manifold.

Let M be a finite dimensional Hadamard manifold with sectional curvature -1 and K be a convex submanifold of M. Then we have the following theorem.

Theorem 5. *Let $F : K \times K \to \mathbb{R}$ be a bifunction such that*

(i) For any $x \in K$, $F(x,x) \geq 0$.

(ii) For every $x \in K$, the set $\{y \in K; F(x,y) < 0\}$ is convex.

(iii) For every $y \in K$, $\left\{ \begin{array}{c} K \to \mathbb{R} \\ x \mapsto F(x,y) \end{array} \right.$ is upper semicontinuous.

(iv) There exists a nonempty compact and convex subset B of M such that for each $x \in K\backslash B$ there exists $y \in B \cap K$ such that $F(x,y) < 0$.

Then there exists a point $x_0 \in B \cap K$, such that for every $y \in K$, the following inequality holds:

$$F(x_0, y) \geq 0.$$

Proof. Define the mapping $G : K \to 2^K$ by

$$G(y) := \{x \in K : F(x,y) \geq 0\}, \forall y \in K.$$

Since $F(.,y)$ is upper semicontinuous, $G(y)$ is closed for all $y \in K$. It follows from (iv) that the set $B \cap K$ is nonempty, convex and compact where $\bigcap\limits_{y\in B\cap K} G(y) \subseteq B$. Then

$\bigcap_{y\in B\cap K} G(y)$ is compact. In order to apply Theorem 4, we have to prove that G is a KKM mapping on K. In order to verify this, let $\{y_1, y_2, \dots, y_m\}$ be a finite subset of K then

$$co\left(\{y_1, y_2, \dots, y_m\}\right) \subseteq \bigcup_{i=1}^{m} G\left(y_i\right).$$

On the contrary, if G is not KKM mapping, then there exists a point $\hat{x}$, such that $\hat{x} \in co\left(\{y_1, y_2, \dots, y_m\}\right)$ but $\hat{x} \notin \bigcup_{i=1}^{m} G\left(y_i\right)$, that is for every $i \in \{1, 2, \dots, m\}$;

$$F\left(\hat{x}, y_i\right) < 0.$$

Hence $y_i \in \{y \in K; F\left(\hat{x}, y\right) < 0\}, \forall i \in \{1, \dots, m\}$. It follows from (ii) that the set $\{y \in K; F\left(\hat{x}, y\right) < 0\}$ is convex. Thus

$$\hat{x} \in co\left(\{y_1, y_2, \dots, y_m\}\right) \subseteq \{y \in K; F\left(\hat{x}, y\right) < 0\},$$

which contradicts (i). Then all of conditions of Theorem 4 are satisfied for the set valued mapping G. Hence by Theorem 4 there exists a point $x_0 \in K$ such that

$$x_0 \in \bigcap_{y\in K} G(y).$$

Then it follows

$$\bigcap_{y\in K} G(y) \subseteq \bigcap_{y\in B\cap K} G(y) \subseteq B.$$

Therefore $x_0 \in B \cap K$. Thus by the definition of $G(y)$, for every $y \in K$, we have

$$F\left(x_0, y\right) \geq 0.$$

◄

As an application of our main theorem we consider mixed variational inequality problem. That is one of the most important issues in optimization problems. This problem has been vastly studied in linear framework, see for instance [32, 14].

We develop and establish the mixed variational inequality problem under suitable conditions in Hadamard manifold framework with sectional curvature -1, by use of the equilibrium Theorem 5.

Let M be a finite dimensional Hadamard manifold with sectional curvature -1 and K be a convex submanifold of M. Assume that $T : K \to TM$ is a single valued vector field and $f : K \to \mathbb{R}$ is a real valued function. Consider the mixed variational inequality problem (MVIP) in the Hadamard manifold framework which is to find a point $x_0 \in K$ such that

$$\left\langle Tx_0, exp_{x_0}^{-1} y\right\rangle + f\left(y\right) - f\left(x_0\right) \geq 0, \forall y \in K.$$

The mixed variational inequality problem associated to T and f is denoted by MVIP(T,f).

Theorem 6. *Let $T : K \to TM$ be a continuous vector field and $f : K \to \mathbb{R}$ be a convex lower semicontinuous function. Assume that:*

(H) there exists a nonemty, convex and compact subset $B \subseteq M$ such that for each $x \in K \backslash B$ there exists $y \in B \cap K$, such that

$$\left\langle Tx, exp_x^{-1} y \right\rangle + f(y) - f(x) < 0. \tag{4}$$

Then MVIP(T,f) has a solution in $B \cap K$.

Proof. Define $F_{T,f}(x, y) : K \times K \to \mathbb{R}$ by

$$F_{T,f}(x, y) := \left\langle Tx, exp_x^{-1} y \right\rangle + f(y) - f(x), \forall x, y \in K.$$

For any $x \in K$, it is easy to see that $F_{T,f}(x, x) \geq 0$, so the function $F_{T,f}$, satisfies (i) of Theorem 5. It can be deduced from the continuity of T and lower semicontinuity of f that for every $y \in K$ the function $\begin{cases} F_{T,f}(., y) : K \to \mathbb{R} \\ x \mapsto F_{T,f}(x, y) \end{cases}$ is upper semicontinuous. This implies the validity of the hypothesis (iii) of Theorem 5.

The condition (H) is equivalent to the existence of a nonempty, convex and compact subset $B \subseteq M$ such that for each $x \in K \backslash B$ there exists $y \in B \cap K$, with

$$F_{T,f}(x, y) < 0,$$

which implies the condition (iv) of Theorem 5 is satisfied. We assert that the condition (ii) of Theorem 5 is valid for $F_{T,f}$. Indeed, define the map $g : K \to \mathbb{R}$, for any $x \in K$ as follows:

$$g(y) := \left\langle Tx, exp_x^{-1} y \right\rangle.$$

By Proposition 7, the map g is convex. And so for each $x \in K$, the function $F_{T,f}(x, .) : K \to \mathbb{R}$ is convex.

Hence the set $\{y \in K; F_{T,f}(x, y) < 0\}$ is convex. This completes the proof of the assertion. Since all conditions of Theorem 5 are satisfied for the function $F_{T,f}$, there exists a point $x_0 \in B \cap K$ such that

$$F_{T,f}(x_0, y) \geq 0, \forall y \in K.$$

Then $x_0 \in B \cap K$ is a solution of the MVIP(T,f).◄

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