Azerbaijan Journal of Mathematics
V. 7, No 2, 2017, July
ISSN 2218-6816

Projectivity and Unification Problem in the Variety Generated by Monadic Perfect MV-algebras

A. Di Nola*, R. Grigolia, R. Liparteliani

Abstract. A description and characterization of free and projective monadic MV-algebras in the variety generated by perfect MV-algebras is given. It is proved that the variety generated by monadic perfect MV-algebras has unitary unification type.

Key Words and Phrases: Łukasiewicz logic, perfect MV-algebra, monadic MV-algebra, unification problem.

2010 Mathematics Subject Classifications: 03G20,06D35

---

1. Introduction

The finitely valued propositional calculi, which have been described by Łukasiewicz and Tarski in [16], are extended to the corresponding predicate calculi. The predicate Łukasiewicz (infinitely valued) logic QL is defined in the following standard way. The existential (universal) quantifier is interpreted as supremum (infimum) in a complete MV-algebra. Then the valid formulas of predicate calculus are defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by Rutledge in [19]. Scarpellini in [20] has proved that the set of valid formulas is not recursively enumerable.

Monadic MV-algebras were introduced and studied by Rutledge in [19] as an algebraic models for the predicate calculus QL of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. Rutledge followed P.R. Halmos' study of monadic Boolean algebras. In view of the incompleteness of the predicate calculus, the result of Rutledge in [19], showing the completeness of the monadic predicate calculus, has been of great interest.

Let L denote a first-order language based on $\cdot, +, \to, \neg, \exists$ and let L_m denote a propositional language based on $\cdot, +, \to, \neg, \exists$. Let $Form(L)$ and $Form(L_m)$ be the sets of all

*Corresponding author.

formulas of L and L_m, respectively. We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation $\Psi : Form(L_m) \to Form(L)$ by putting:

- $\Psi(p) = p^*(x)$ if p is propositional variable,
- $\Psi(\neg\alpha) = \neg\Psi(\alpha)$,
- $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \to$,
- $\Psi(\exists\alpha) = \exists x\Psi(\alpha)$.

Through this translation Ψ, we can identify the formulas of L_m with monadic formulas of L containing the variable x. Moreover, it is routine to check that $\Psi(MLPC) \subseteq QL$, where $MLPC$ is the monadic Lukasiewicz propositional calculus [7].

For a detailed consideration of Łukasiewicz predicate calculus we refer to [1, 3, 12, 15, 16, 21, 22].

2. Preliminaries on monadic MV-algebras

The characterization of monadic MV-algebras as pair of MV-algebras, where one of them is a special kind of subalgebra (m-relatively complete subalgebra), is given in [7, 5]. MV-algebras were introduced by Chang in [6] as an algebraic model for infinitely valued Łukasiewicz logic.

An MV-algebra is an algebra $A = (A, \oplus, \odot, ^*, 0, 1)$ where $(A, \oplus, 0)$ is an abelian monoid, and the following identities hold for all $x, y \in A$: $x \oplus 1 = 1$, $x^{**} = x$, $0^* = 1$, $x \oplus x^* = 1$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, $x \odot y = (x^* \oplus y^*)^*$.

Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \text{ iff } x^* \oplus y = 1.$$

$(A, \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra:

$$x \odot y \leq x \wedge y \leq x \vee y \leq x \oplus y.$$

The unit interval of real numbers $[0, 1]$ endowed with the following operations: $x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), x^* = 1 - x$, becomes an MV-algebra. It is well known that the MV-algebra $S = ([0, 1], \oplus, \odot, ^*, 0, 1)$ generates the variety **MV** of all MV-algebras, i. e. $\mathcal{V}(S) = \mathbf{MV}$.

Let Q denote the set of rational numbers. Then $[0, 1] \cap Q$ is another MV-algebra, which also generates the variety **MV**.

An algebra $A = (A, \oplus, \odot, ^*, \exists, 0, 1)$ is said to be a monadic MV-algebra (MMV-algebra for short) [19, 7] if $A = (A, \oplus, \odot, ^*, 0, 1)$ is an MV-algebra and in addition $\exists$ satisfies the following identities:

E1. $x \leq \exists x$,

E2. $\exists(x \vee y) = \exists x \vee \exists y$,

E3. $\exists(\exists x)^* = (\exists x)^*$,

E4. $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$,

E5. $\exists(x \odot x) = \exists x \odot \exists x$,

E6. $\exists(x \oplus x) = \exists x \oplus \exists x$.

Sometimes we shall denote a monadic MV-algebra $A = (A, \oplus, \odot, ^*, \exists, 0, 1)$ by $(A, \exists)$, for brevity. We can define a unary operation $\forall x = (\exists x^*)^*$ corresponding to the universal quantifier.

Let A_1 and A_2 be any MMV-algebras. A mapping $h : A_1 \to A_2$ is an MMV-homomorphism if h is an MV-homomorphism and for every $x \in A_1$ $h(\exists x) = \exists h(x)$. Denote by **MMV** the variety and the category of MMV-algebras and MMV-homomorphisms.

From the variety of monadic MV-algebras **MMV** [7] select the subvariety **MMV**(**C**) which is defined by the following equation [9]:

$$(Perf)\ \ 2(x^2) = (2x)^2,$$

that is $\mathbf{MMV}(\mathbf{C}) = \mathbf{MMV} + (Perf)$. The main object of our interest are the varieties **MMV**(**C**).

An ideal I (a filter F) of an algebra $(A, \exists) \in \mathbf{MMV}$ is called *monadic ideal (filter)* (see [19, 7]), if I (F) is an ideal (a filter) of MV-algebra A (i.e. $A \supset I \neq \emptyset$ ($A \supset F \neq \emptyset$) and for every $x, y \in I$ ($x, y \in F$) (a) $x \oplus y \in I$ ($x \odot y \in F$); (b) $x \geq y, x \in I \Rightarrow y \in I$ ($x \leq y, x \in F \Rightarrow y \in F$)) and for every $a \in A$ we have $a \in I \Rightarrow \exists a \in I$ ($a \in F \Rightarrow \forall a \in F$). Note that if I (F) is a monadic ideal (filter) of $(A, \exists)$, then the set $\{\neg x : x \in I\}$ ($\{\neg x : x \in F\}$) is a monadic filter (ideal).

For every monadic MV-algebra $(A, \exists)$, there exists a lattice isomorphism between the lattice of all monadic ideals (filters) and the lattice of all congruence relations of $(A, \exists)$ [7].

There are MV-algebras which are not semisimple, i.e. the intersection of their maximal ideals (the radical of A, notation $Rad(A)$) is different from $\{0\}$. Non-zero elements from the radical of A are called infinitesimals. It is worth to stress that due to the existence of infinitesimals in some MV-algebras there is a remarkable difference of behaviour between Boolean algebras and MV-algebras.

Perfect MV-algebras are those MV-algebras generated by their infinitesimal elements or, equivalently, generated by their radical [4]. They generate the smallest non locally finite subvariety of the variety **MV** of all MV-algebras.

The class of perfect MV-algebras does not form a variety and contains non-simple subdirectly irreducible MV-algebras. It is worth stressing that the variety generated by all perfect MV-algebras, denoted by $\mathbf{MV(C)}$, is also generated by a single MV-chain, actually the MV-algebra C, defined by Chang in [6]. We name $MV(C)$-*algebras* all the algebras from the variety generated by C. Let L_P be the logic corresponding to the variety generated by perfect algebras which coincides with the set of all Łukasiewicz formulas that are valid in all perfect MV-chains, or equivalently that are valid in the MV-algebra C. Actually, L_P is the logic obtained by adding to the axioms of Łukasiewicz sentential calculus the following axiom: $(x \veebar x)\&(x \veebar x) \leftrightarrow (x\&x) \veebar (x\&x)$ (where $\veebar$ is strong disjunction, & strong conjunction in Łukasiewicz sentential calculus), see [4]. Note that the Lindenbaum algebra of L_P is an $MV(C)$-algebra. The perfect algebra C has relevant properties. Indeed, C generates the smallest variety of MV-algebras containing non-Boolean non-semisimple algebras. It is also subalgebra of any non-Boolean perfect MV-algebra.

The importance of the class of $MV(C)$ algebras and the logic L_P can be perceived by looking further at the role that infinitesimals play in MV-algebras and Łukasiewicz logic. Indeed, the pure first order Łukasiewicz predicate logic is not complete with respect to the canonical set of truth values $[0,1]$, see [20], [3]. The Lindenbaum algebra of the first order Łukasiewicz logic is not semisimple and the valid but unprovable formulas are precisely the formulas whose negations determine the radical of the Lindenbaum algebra, that is the co-infinitesimals of such algebra. Hence, the valid but unprovable formulas generate the perfect skeleton of the Lindenbaum algebra. So, perfect MV-algebras, the variety generated by them and their logic are intimately related with a crucial phenomenon of the first order Łukasiewicz logic.

Let us introduce some notations: let $C_0 = \Gamma(Z,1)$, $C_1 = C \cong \Gamma(Z \times_{lex} Z, (1,0))$ with generator $(0,1) = c_1(= c)$, $C_m = \Gamma(Z \times_{lex} \cdots \times_{lex} Z, (1,0,...,0))$ with generators $c_1(= (0,0,...,1)),...,c_m(= (0,1,...,0))$, where the number of factors Z is equal to $m+1$ and $\times_{lex}$ is the lexicographic product and Γ is a well-known Mundici's functor translating a lattice ordered group with strong unit into MV-algebra. Let us denote $Rad(A) \cup \neg Rad(A)$ through $R^*(A)$, where $\neg Rad(A) = \{x^* : x \in Rad(A)\}$.

Let $(A, \oplus, \odot, ^*, \exists, 0, 1)$ be a monadic MV-algebra. Let $\exists A = \{x \in A : x = \exists x\}$. By [7], $(\exists A, \oplus, \odot, ^*, 0, 1)$ is an MV-subalgebra of the MV-algebra $(A, \oplus, \odot, ^*, 0, 1)$.

A subalgebra A_0 of an MV-algebra A is said to be relatively complete if for every $a \in A$ the set $\{b \in A_0 : a \leq b\}$ has a least element.

Let $(A, \oplus, \odot, ^*, \exists, 0, 1)$ be a monadic MV-algebra. By [19], the MV-algebra $\exists A$ is a relatively complete subalgebra of the MV-algebra $(A, \oplus, \odot, ^*, 0, 1)$, and $\exists a = inf\{b \in \exists A : a \leq b\}$.

A subalgebra A_0 of an MV-algebra A is said to be *m-relatively complete* [7], if A_0 is relatively complete and two additional conditions hold:

(#) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \odot a \Rightarrow v \geq a \& v \odot v \leq x)$,

(##) $(\forall a \in A)(\forall x \in A_0)(\exists v \in A_0)(x \geq a \oplus a \Rightarrow v \geq a \& v \oplus v \leq x)$.

By [7], there exists a one-to-one correspondence between
1) the monadic MV-algebras $(A, \exists)$;
2) the pairs (A, A_0), where A_0 is m-relatively complete subalgebra of A.

3. One-generated free monadic $MMV(C)$-algebras

According to the definition of monadic MV-algebras, m-relatively complete subalgebra of C coincides with C but not its two-element Boolean subalgebra. In other words, $(C, \exists)$ is monadic $MMV(C)$-algebra if $\exists x = x$. Let C^n be some non-negative integer. Then $(C^n, \exists)$ will be $MMV(C)$-algebra, where $\exists(a_1, ..., a_n) = max\{a_1, ..., a_n\}$ and $\forall(a_1, ..., a_n) = min\{a_1, ..., a_n\}$. In this case $\exists(C^n) = \{(x, ..., x) \in C^n : x \in C\}$. Note that $(C^n, \exists)$ is subdirectly irreducible [7]. For perfect MV-algebra $Rad^*(C^2)$ we also have $\exists(C^n) = \{(x, ..., x) \in C^n : x \in C\} \subset Rad^*(C^2)$.

Now we shall give examples of one-generated $MMV(C)$-algebras and show that there are infinitely many one-generated subdirectly irreducible $MMV(C)$-algebras unlike the one-generated subdirectly irreducible $MV(C)$-algebras. There is only one (up to isomorphism) subdirectly irreducible $MV(C)$-algebra C.

Lemma 1. *The following algebras are one-generated subdirectly irreducible $MMV(C)$-algebras:*

1) $(\mathbf{2}, \exists)$ with generator either 1 or 0, where $\mathbf{2}$ is two-element Boolean algebra,

2) $(\mathbf{2}^2, \exists)$ with generator either $(0,1)$ or $(1,0)$, where $\mathbf{2}^2$ is four-element Boolean algebra,

3) $(C, \exists)$ with generator either c or $\neg c$,

4) $(C^2, \exists)$ with generator either $(1, c), (\neg c, 0)$ or $(c, \neg c)$,

5) $(Rad^*(C^2), \exists)$ with generator either $(c, 0)$ or $(\neg c, 1)$,

6) $(C_2^2, \exists)$ with generator either $(c_1, \neg c_2)$ or $(\neg c_1, c_2)$,

7) $(Rad^*(C_2^2), \exists)$ generated by (c_1, c_2) or $(\neg c_1, \neg c_2)$.

Proof. 1), 2) and 3) are trivial.

4) (a) $\forall(1, c) = (c, c)$, $g^2 = (1, 0)$, $(c, c) \vee (0, 1) = (c, 1)$. So, $(C^2, \exists)$ is generated by $(1, c)$; (b) $2(\neg c, 0) = (1, 0)$, $\neg(\neg c, 0) = (c, 1)$, $(c, 1)^2 = (0, 1)$. So, $(C^2, \exists)$ is generated by $(\neg c, 0)$; (c) $2((c, \neg c)^2) = (0, 1)$, $\neg(0, 1) = (1, 0)$, $\forall(c, \neg c) = (c, c)$. So, $(C^2, \exists)$ is generated by $(c, \neg c)$;

5) $\exists(c, 0) = (c, c)$, $\neg(c, 0) = (\neg c, 1)$, $(c, c) \to (c, 0) = (1, \neg c)$, $\neg(1, \neg c) = (0, c)$. So, $(Rad^*(C^2), \exists)$ is generated by either $(c, 0)$ or $(\neg c, 1)$.

6) Let $g = (c_1, \neg c_2)$. $2g^2 = (0,1)$, $\neg(2g^2) = (1,0)$. $\forall g = (c_1.c_1$, $\neg\exists g = (c_2, c_2)$; $\forall g \wedge 2g^2 = (0, c_1)$; $\forall g \wedge \neg(2g^2) = (c_1, 0)$; $\neg\exists g \wedge 2g^2 = (0, c_2)$; $\neg\exists g \wedge \neg(2g^2) = (c_2, 0)$. In a similar way it is shown that $(C_2^2, \exists)$ is generated by $(\neg c_1, c_2)$ $(= \neg g)$.

7) Let $g = (c_1, c_2)$. From this element we obtain the following sequences of elements: $\forall g = (c_1, c_1)$, $\exists g = (c_2, c_2)$, $\neg\forall g = (\neg c_1, \neg c_1)$, $\neg\exists g = (\neg c_2, \neg c_2)$; $\neg\exists g \oplus g = (\neg c_2, \neg c_2) \oplus (c_1, c_2) = (\neg c_2 \oplus c_1, 1)$, $\neg g \oplus \forall g = (\neg c_1, \neg c_2) \oplus (c_1, c_1) = (1, \neg c_2 \oplus c_1)$; $(\neg\exists g \oplus g) \odot \forall g = (\neg c_2 \oplus c_1, 1) \odot (c_1, c_1) = (0, c_1)$, $(\neg g \oplus \forall g) \odot \forall g = (c_1, 0)$; $(\neg\exists g \oplus g) \odot \neg\forall g) = (c_2, c_1)$, $\neg(\neg(c_2, c_1) \oplus (0, c_1)) = (c_2, 0)$, $\neg(\neg(c_1, c_2) \oplus (c_1, 0)) = (0, c_2)$. From these elements we can obtain all elements of radical of $(C_2^2, \exists)$ and thereby the elements of perfect MV-algebra. ◄

Note that $(C_n, \exists)$ is not 1-generated for $n \geq 2$, since $\exists x = x$ for every $x \in C_n$ and C_n is not one-generated. It is clear that $(\mathbf{2}^n, \exists)$ is a homomorphic image of $(C^n, \exists)$. But $(\mathbf{2}^n, \exists)$ is not generated by one generator for $n \geq 3$. Indeed, for any element $x \in \mathbf{2}^n$ the operation $\exists$ is defined as follows: $\exists x = (1, ...1, 1) \in \mathbf{2}^n$ if $x \neq (0, 0, ..., 0) \in \mathbf{2^n}$ and $\exists x = (0, ...0, 0) \in \mathbf{2}^n$ in other case. So, $(\mathbf{2}^n, \exists)$ is one-generated if it is one-generated using only Boolean operations. But $\mathbf{2}^n$ is not generated by one generator if $n \geq 3$.

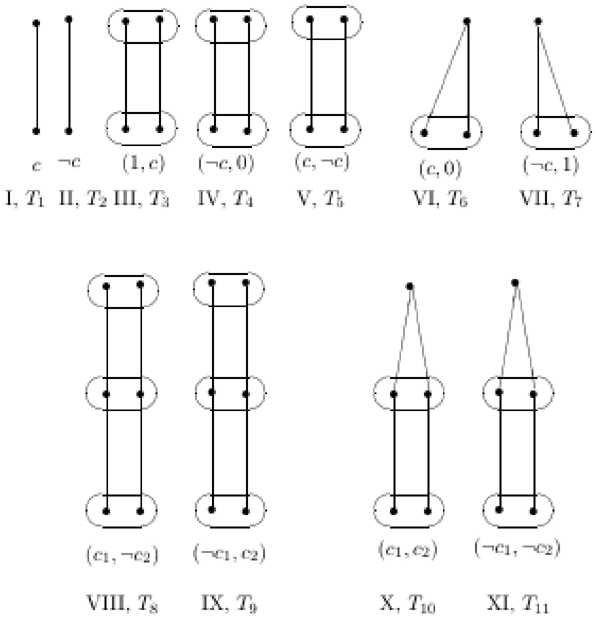

Fig. 1. Spectral spaces of one-generated subdirectly irreducible $MV(C)$-algebras

Fig. 1 presents the depicted ordered sets corresponding to the prime filter spaces for $(C, \exists)$ $(\cong T_1 \cong T_2)$, $(C^2, \exists)$ $(\cong T_3 \cong T_4 \cong T_5)$, $(Rad^*(C^2), \exists)$ $(\cong T_6 \cong T_7)$, $(C_2^2, \exists)$ $(\cong T_8 \cong T_9)$, $(Rad^*(C_2^2), \exists)$ $(\cong T_{10} \cong T_{11})$ with their generators. Note that the algebras

$T_1, T_2, ..., T_7$ have height 2 and the algebras T_8, T_9, T_{10}, T_{11} have height 3 (see the definition of height below).

Let us give some comments about the diagrams in Fig. 1. The posets I, II, VI, VII, X and XI have one maximal filter, i. e. they correspond to a perfect MV-algebra. As to III, IV, V, VII and IX, the elements inside ovals can be considered as equivalent elements and this equivalence relation corresponds to the $\exists$ operation on a corresponding MV-algebra which corresponds to the diagonal subalgebra of C^2 and C_2^2, respectively.

We say that MV-algebra A has *height* n if a maximal chain of the poset of prime filters (ordered by inclusion) contains n elements. Similarly, we say that $MMV(C)$-algebra A has *height* n if its MV-algebra reduct has height n. According to this definition, MV-algebra C_n has height $n+1$ $(n \geq 1)$.

Lemma 2. *If subdirectly irreducible $MMV(C)$-algebra, with non-trivial operation $\exists$, has height $n > 3$, then it is not one-generated.*

Proof. Let us suppose we have $MMV(C)$-algebra $(C_3^2, \exists)$. The optimal version to be a generator of $(C_3^2, \exists)$ is either (c_1, c_2), (c_2, c_3), (c_1, c_3), $(c_1, \neg c_2)$, $(c_1, \neg c_3)$, $(\neg c_1, c_2)$, $(\neg c_1, c_3)$, $(\neg c_2, c_3)$, $(c_2, \neg c_3)$, $(c_2, \neg c_1)$, $(c_3, \neg c_2)$. It is obvious that none of them generates the algebra $(C_3^2, \exists)$. Even more so $(C_n^k, \exists)$ is not generated by one generator for $k, n > 2$. ◄

The next lemma shows that there are infinitely many non-isomorphic one-generated subdirectly irreducible $MMV(C)$-algebras.

Lemma 3. *The $MMV(C)$-algebra $(Rad^*(C^n), \exists)$ is generated by the element $(c, 2c, ..., nc)$ for any positive integer n.*

Proof. Let $g = (c, 2c, ..., nc)$. Then $\forall g = (c, c, ..., c)$, $\neg\forall g = (\neg c, \neg c, ..., \neg c)$, $g \odot \neg\forall g = (0, c, 2c, ..., (n-1)c)$, $g \odot (\neg\forall g)^2 = (0, 0, c, 2c, ..., (n-2)c)$, ... , $g \odot (\neg\forall g)^{n-1} = (0, 0, ..., 0, c)$.

$(g \odot \neg\forall g) \wedge \forall g = (0, c, ..., c)$, $\neg(g \odot (\neg\forall g)^2) \odot ((g \odot \neg\forall g) \wedge \forall g) = (1, 1, \neg c, ..., (\neg c)^{n-2}) \odot (0, c, ..., c) = (0, c, 0, ..., 0)$.

$(g \odot (\neg\forall g)^2) \wedge \forall g = (0, 0, c, ..., c)$. $\neg(g \odot (\neg\forall g)^3) = (1, 1, 1, \neg c, (\neg c)^2, ..., (\neg c)^{n-3})$. $(1, 1, 1, \neg c, (\neg c)^2, ..., (\neg c)^{n-3}) \odot (0, 0, c, ..., c) = (0, 0, c, 0, ..., 0)$, and so on. Moreover, $\neg g \odot 2\forall g = (c, 0, ..., 0)$. This finishes the proof of the theorem. ◄

Lemma 4. *$MMV(C)$-algebra*

$$U_1 = Rad^*((Rad^*(C^2, \exists) \times (C, \exists)))(= Rad^*(T_7 \times T_1))$$

is generated by $((c, 0), c)$ $(((\neg c, 1), \neg c))$, which is a perfect MV-algebra. Moreover, the subalgebra of $Rad^*(C^2, \exists) \times (C, \exists))$ generated by $((c, 0), c)$ $(((\neg c, 1), \neg c))$ is isomorphic to $Rad^*((Rad^*(C^2, \exists) \times (C, \exists)))$.

Proof. It is clear that by means of elements $((0,0),c)$, $((0,c),0)$, $((c,0),0)$ and the operations $\oplus$, $\vee$ we can obtain all elements of $Rad((Rad^*(C^2,\exists)\times(C,\exists))$, and thereby all elements of $Rad^*((Rad^*(C^2,\exists)\times(C,\exists)))$. Let $g=((c,0),c)$. Then $\forall g=((0,0),c)$; $\exists g=((c,c),c)$; $\neg\forall g=((1,1),\neg c)$; $\neg g=((\neg c,1),\neg c)$; $\neg\forall g\odot\exists g=((c,c),0)$; $(\neg\forall g\odot\exists g)\to g=((1,\neg c),1)$; $\neg\forall g\odot\exists g=((c,c),0)$; $\neg((\neg\forall g\odot\exists g)\to g)=((0,c),0)$; $((c,0)c)\wedge((c,c),0)=((c,0),0)$. So we have obtained the elements $((0,0),c)$, $((0,c),0)$, $((c,0),0)$. Hence, $Rad^*((Rad^*(C^2,\exists)\times(C,\exists)))$ is generated by $((c,0),c)$, and thereby by element $(\neg c,1),\neg c)$.

Observe that the element $((c,0),c)$ $((\neg c,1),\neg c))$ belongs to radical (co-radical). So, the subalgebra generated by this element is perfect and isomorphic to $Rad^*((Rad^*(C^2,\exists)\times(C,\exists)))$. ◄

Lemma 5. *$MMV(C)$-algebra*

$$U_1^2=Rad^*((Rad^*(C^2,\exists)\times(C,\exists)))\times Rad^*((Rad^*(C^2,\exists)\times(C,\exists)))$$

is generated by $((c,0),c),(\neg c,1),\neg c))$.

Proof. Indeed, from the generator $((c,0),c),(\neg c,1),\neg c))$ we can obtain the elements $((0,0),0),(1,1),1))=2(((c,0),c),(\neg c,1),\neg c))^2)$ and

$$((1,1),1),(0,0),0))=\neg((0,0),0),(1,1),1)).$$

So, $Rad^*((Rad^*(C^2,\exists)\times(C,\exists)))\times Rad^*((Rad^*(C^2,\exists)\times(C,\exists)))$ is generated by

$$((c,0),c),(\neg c,1),\neg c))$$

◄

Lemma 6. *The subalgebra U_2 of $MMV(C)$-algebra $(C^2,\exists)^3$ generated by*

$$t=((c,\neg c),(1,c),(\neg c,0))$$

is a proper subalgebra with one maximal monadic filter.

Proof. Let us note that one-generated non-trivial monadic Boolean algebra is isomorphic to $(2^2,\exists)$ with generator $(0,1)$. Note also that $(2^2,\exists)\cong(C^2,\exists)/((c,c)]$, where $((c,c)]$ is the monadic ideal generated by (c,c) which is maximal at the same time. So, since $(2^2,\exists)$ should be homomorphic image of the subalgebra of $MMV(C)$-algebra $(C^2,\exists)^3$ generated by $((c,\neg c),((c,0),c),((\neg c,1),\neg c))$, the subalgebra must have one maximal monadic ideal. Moreover, U_2 is a subdirect product of subdirectly irreducible copies of algebra $(C^2,\exists)$, since $(C^2,\exists)$ is generated separately by $(c,\neg c),(1,c),(\neg c,0)$. ◄

Lemma 7. *$U_2/J_i \cong (C^2, \exists)$ $(i = 1, 2, 3)$, where $J_1 = (((c,c),(0,0),(0,0))]$, $J_2 = (((0,0),(c,c),(0,0))]$, $J_3 = (((0,0),(0,0),(c,c))]$, are monadic ideals generated by $((c,c),(0,0),(0,0))$, $((0,0),(c,c),(0,0))$, $((0,0),(0,0),(c,c))$, respectively.*

Proof. Now we show that the elements can be obtained by the generator t. Indeed, $\neg\exists t \wedge \forall t = ((c,c),(0,0),(0,0))$; $(\neg\exists t \vee \forall t) \wedge \exists\neg t = ((0,0),(0,0),(c,c))$; $(\neg\exists t \oplus (\neg\exists t \vee \forall t)) \odot (\neg(\exists\neg t \odot (\neg\exists t \wedge \forall t) \wedge (\neg\exists t \wedge \forall t)^2 = ((0,0),(c,c),(0,0))$. ◄

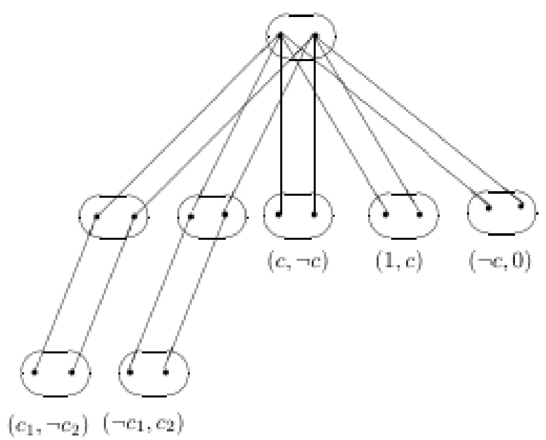

Fig. 2

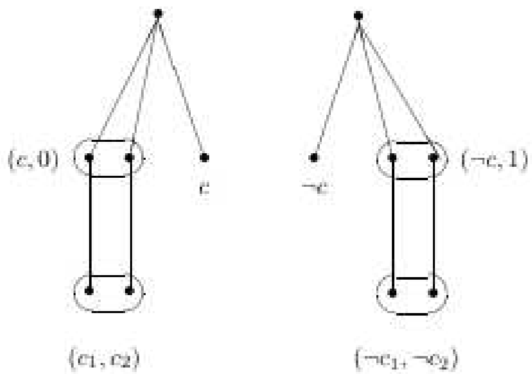

Fig. 3

The ordered set corresponding to the prime filter space of algebras $T_8 \times T_9 \times T_3 \times T_4 \times T_5$ generated by $(c_1, \neg c_2), (\neg c_1, c_2), (c, \neg c), (1, c), (\neg c, 0)$ is depicted in Fig. 2 and the ordered set corresponding to the prime filter space of algebras generated by $(c_1, c_2), c, \neg c, (\neg c_1, \neg c_2)$ is depicted in Fig. 3.

Theorem 1. *Let $A = \prod_{i\in I} A_i$ be a direct product of the family of all subdirectly irreducible one-generated $MMV(C)$-algebras A_i with generators $g_i \in A_i$ $(i \in I)$. Let $F_{\mathbf{MMV(C)}}(1)$ be the subalgebra of A generated by the generator $g = (g_i)_{i\in I} \in A$. Then*

1) the algebra $F_{\mathbf{MMV(C)}}(1)$ is a subdirect product of the family $\{A_i : i \in I\}$;

2) any subdirectly irreducible one-generated $MMV(C)$-algebra is a homomorphic image of $F_{\mathbf{MMV(C)}}(1)$;

3) the algebra $F_{\mathbf{MMV(C)}}(1)$ generated by the generator $g = (g_i)_{i\in I} \in A$ is one-generated free $MMV(C)$-algebra with free generator $g = (g_i)_{i\in I}$;

4) the algebra $F_{\mathbf{MMV(C)}}(1)$ has height 3;

5) the poset of prime filters of the algebra $F_{\mathbf{MMV(C)}}(1)$ contains only four maximal elements and this four elements form the poset of $MMV(C)$-algebra $(\mathbf{2}^2, \exists) \times (\mathbf{2}, \exists)^2$, where $\mathbf{2}$ is two-element Boolean algebra.

Proof. 1). It is obvious that for any projection π_i $(i \in I)$ $\pi_i(g) = g_i$ that generates A_i. So, $F_{\mathbf{MMV(C)}}(1)$ is a subdirect product of the family $\{A_i : i \in I\}$.

2) Since $F_{\mathbf{MMV(C)}}(1)$ is a subdirect product of all subdirectly irreducible one-generated $MMV(C)$-algebras A_i, any subdirectly irreducible one-generated $MMV(C)$-algebra is a homomorphic image of $F_{\mathbf{MMV(C)}}(1)$

3) Let us suppose that an identity $P(x) = Q(x)$ does not hold in the variety $\mathbf{MMV(C)}$. Then it does not hold in some subdirectly irreducible one-generated $MMV(C)$-algebras A_i on the generator g_i. So, it does not hold in $F_{\mathbf{MMV(C)}}(1)$ on the generator g. From here we conclude that $F_{\mathbf{MMV(C)}}(1)$ generated by the generator $g = (g_i)_{i\in I} \in A$ is one-generated free $MMV(C)$-algebra with free generator $g = (g_i)_{i\in I}$.

4) The assertion follows from Lemma 2.

5) This item follows from the fact that the algebra $(\mathbf{2}^2, \exists) \times (\mathbf{2}, \exists)^2$ is a free one-generated monadic Boolean algebra and the variety of monadic Boolean algebras is a subvariety of the variety $\mathbf{MMV(C)}$. ◄

4. m-generated free monadic $MMV(C)$-algebras

We can easily generalize the results of one-generated $MMV(C)$-algebras on m-generated ones. Since the prime filter space of 1-generated free $MMV(C)$-algebra and, also, m-generated free $MV(C)$-algebra $(m > 1)$ is infinite [8], the prime filter space of m-generated free $MMV(C)$-algebra is also infinite. But the number of the prime filter spaces of m-generated subdirectly irreducible $MMV(C)$-algebra is finite.

Note that the smallest subvariety of the variety $\mathbf{MMV(C)}$, different from the variety of Boolean algebras with trivial monadic operator, is the variety of monadic Boolean algebras. So, any m-generated free monadic Boolean algebra is a homomorphic image of m-generated free $MMV(C)$-algebra. The following proposition is true.

Proposition 1. *[2, 13, 14]. m-generated free monadic Boolean algebra* $(B(m), \exists)$ *is isomorphic to*

$$\prod_{k=1}^{2^m} (\mathbf{2}^k, \exists)^{\binom{k}{2^m}}.$$

Corollary 1. *There exists exactly* $\sum_{k=1}^{2^m} \binom{k}{2^m} (= 2^{2^m} - 1)$ *number of maximal monadic filters of* $(B(m), \exists)$*. These maximal monadic filters are generated by* $(0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^m})$*, where* 1^k *is the top element of* $(\mathbf{2}^k, \exists)$ *(*$1 \leq k \leq 2^m$*),* 0^i *is the bottom element of* $(\mathbf{2}^i, \exists)$ *(*$1 \leq i \leq 2^m$*).*

Note that monadic Boolean algebras are also monadic $MV(C)$-algebras, but of height 1.

As for one-generated case, as an obvious fact we have the following

Lemma 8. *The height of an m-generated subdirectly irreducible* $MMV(C)$*-algebra is limited by some natural number* $k > 0$*. In other words, a maximal chain of the poset of prime filters of a subdirectly irreducible* $MMV(C)$*-algebra is limited by some natural number* $k > 0$*.*

Since we have infinitely many subdirectly irreducible one-generated $MMV(C)$-algebras, it holds

Lemma 9. *There are infinitely many subdirectly irreducible m-generated* $MMV(C)$*-algebras for* $m > 1$*.*

Theorem 2. *The m-generated subdirectly irreducible* $MMV(C)$*-algebras for* $m \geq 2$ *are:*
1) $(\mathbf{2}^{2^m}, \exists)$*,*
2) $(C_m, \exists)$*,*
3) $(C^{2^m}, \exists)$*,*
4) $(Rad^*(C^m), \exists)$*,*
5) $(C_m^m, \exists)$*.*

Proof. 1) and 2) are trivial. 3). It is obvious that $(C^{2^m}, \exists)$ has as a subalgebra the monadic Boolean algebra $(\mathbf{2}^{2^m}, \exists)$ the generators of which are the generators of the free m-generated Boolean algebra $\mathbf{2}^{2^m}$. If we change in every free generator of $\mathbf{2}^{2^m}$ the element 0 by c and 1 by $\neg c$, then we will get m generators of $(C^{2^m}, \exists)$. 4). It is obvious that $(c, 0, ..., 0), (0, c, , ..., 0), ..., (0, ..., c)$ generate $(Rad^*(C^m), \exists)$. 5). The generators of $(C_m^m, \exists)$ are $g_1 = (\neg c_1, c_2, ..., c_m$, $g_2 = (c_1, \neg c_2, ..., c_m$, ... , $g_m = (c_1, c_2, ..., \neg c_m$. Indeed, $\neg\exists g_1 = (c_1, c_1, ..., c_1)$, $\neg\exists g_2 = (c_2, c_2, ..., c_2)$, ... , $\neg\exists g_m = (c_m, c_m, ..., c_m)$; $2g_1^2 = (1, 0, ..., 0)$, $2g_2^2 = (0, 1, ..., 0)$, ... , $2g_m^2 = (0, 0, ..., 1)$. And these elements generate $(C_m^m, \exists)$. ◀

Theorem 3. *Let $A = \prod_{i \in I} A_i$ be a direct product of the family of all subdirectly irreducible m-generated $MMV(C)$-algebras A_i with generators $g_i^{(1)}, g_i^{(2)}, ..., g_i^{(m)} \in A_i$ $(i \in I)$, where $\{g_i^{(1)}, g_i^{(2)}, ..., g_i^{(m)}\} \neq \{g_j^{(1)}, g_j^{(2)}, ..., g_j^{(m)}\}$ for $i \neq j$. Let $F_{\boldsymbol{MMV(C)}}(m)$ be the subalgebra of A generated by the generators $g_1 = (g_i^{(1)})_{i \in I} \in A$, ... $g_m = (g_i^{(m)})_{i \in I} \in A$. Then*

1) the algebra $F_{\boldsymbol{MMV(C)}}(m)$ is a subdirect product of the family $\{A_i : i \in I\}$;

2) any subdirectly irreducible m-generated $MMV(C)$-algebra is a homomorphic image of $F_{\boldsymbol{MMV(C)}}(m)$;

3) the algebra $F_{\boldsymbol{MMV(C)}}(m)$ generated by the generator $g_1 = (g_i^{(1)})_{i \in I} \in A$, ... $g_m = (g_i^{(m)})_{i \in I} \in A$ is m-generated free $MMV(C)$-algebra with free generator $g_1 = (g_i^{(1)})_{i \in I} \in A$, ... $g_m = (g_i^{(m)})_{i \in I} \in A$.

Proof. The theorem is proved as in one-generated case. ◄

Theorem 4. *Free algebra $F_{\boldsymbol{MMV(C)}}(m)$ is isomorphic to the finite product of monadic $MV(C)$-algebras D_k $(1 \leq k \leq 2^{2^m} - 1)$ the homomorphic image by maximal monadic filter of which is isomorphic to the subdirectly irreducible monadic Boolean algebra $(\mathbf{2}^{m(k)}, \exists)$, where $m(k) \leq 2^m$. The number of subdirectly irreducible $MMVC)$-algebras having the algebra $\boldsymbol{2}^{m(k)}$ as a maximal homomorphic image is equal to $\binom{m(k)}{2^m}$.*

Proof. Note that m-generated monadic Boolean algebra $(B(m), \exists)$ is a homomorphic image of $F_{\mathbf{MMV(C)}}(m)$. The algebra $(B(m), \exists)$ contains $2^{2^m} - 1$ maximal monadic filters. The intersection of all maximal monadic filters of $(B(m), \exists)$ is equal to $[1_{B(m)})$. According to Corollary 1, these maximal monadic filters of $(B(m), \exists)$ are generated by $(0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^m})$ where 1^k is the top element of $(\mathbf{2}^k, \exists)$ $(1 \leq k \leq 2^m)$, 0^i is the bottom element of $(\mathbf{2}^i, \exists)$ $(1 \leq i \leq 2^m)$. Denote the maximal monadic filters of $(B(m), \exists)$ generated by $(0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^m})$ by F_k. The factor algebra $(B(m)/F_k, \exists)$ is isomorphic to $(\mathbf{2}^k, \exists)$ that is subdirectly irreducible the number of which is equal to $\binom{k}{2^m}$. Let F_k^M be the monadic filter of $F_{\mathbf{MMV(C)}}(m)$ generated in $F_{\mathbf{MMV(C)}}(m)$ by F_k. It is obvious that the intersection of all such kind of the monadic filters of $F_{\mathbf{MMV(C)}}(m)$ is also equal to the unit element of $F_{\mathbf{MMV(C)}}(m)$. So, $F_{\mathbf{MMV(C)}}(m)$ is isomorphic to the finite product of algebras $D_k = F_{\mathbf{MMV(C)}}(m)/F_k^M$, where $1 \leq k \leq 2^{2^m} - 1$. ◄

5. Finitely generated projective $MMV(C)$-algebras

In this section, we first prove auxiliary assertions.

Let $\mathbf{V}$ be a variety. Recall that an algebra $A \in \mathbf{V}$ is said to be a *free algebra* over $\mathbf{V}$, if there exists a set $A_0 \subset A$ such that A_0 generates A and every mapping f from A_0 to any algebra $B \in \mathbf{V}$ is extended to a homomorphism h from A to B. In this case A_0 is said to be *the set of free generators* of A. If the set of free generators is finite, then A is said to be a *free algebra of finitely many generators.* We denote a free algebra A with $m \in (\omega + 1)$ free generators by $F_{\mathbf{V}}(m)$. We shall omit the subscript $\mathbf{V}$ if the variety $\mathbf{V}$ is known.

An algebra A is called *projective* if for any algebra epimorfism (=homomorphism onto) $f : D \to B$ and homomorphism $h : A \to B$ there is a homomorphism $g : A \to D$ such that $fg = h$. An algebra H is a retract of an algebra A if there are homomorphisms $f : A \to H$ and $g : H \to A$ such that $fg = Id_H$, where Id_H is an identity mapping of the set H. It is well-known that in varieties the projective algebras are just the retracts of the free algebras. So, a $MMV(C)$-algebra is projective if and only if it is a retract of a free $MMV(C)$-algebra. We say that the subalgebra A of $F_{\mathbf{V}}(m)$ is *projective* if there exists endomorphism $h : F_{\mathbf{V}}(m) \to F_{\mathbf{V}}(m)$ such that $h(x) = x$ for every $x \in A$.

An algebra in a variety $\mathbf{V}$ is said to be *finitely presented* if for some $m \in \omega$ it is isomorphic to $F_{\mathbf{V}}(m)/\theta$, where θ is a principal congruence relation.

Proposition 2. *[17, 7]. An m-generated algebra A in a variety $\mathbf{V}$ is projective if and only if there exist polynomials $P_1, \ldots, P_m$ such that, denoting by $g_1, ..., g_m$ the free generators of $F_{\mathbf{V}}(m)$,*

$P_i(P_1(g_1, \ldots, g_m), \ldots, P_m(g_1, \ldots, g_m)) = P_i(g_1, \ldots, g_m)$, *for each* $1 \leq i \leq m$

and

$P_1(g_1, \ldots, g_m), \ldots, P_m(g_1, \ldots, g_m)$ *generate an algebra isomorphic to A.*

Theorem 5. *If A is n-generated projective $MMV(C)$-algebra, then A is finitely presented.*

Proof. Since A is n-generated projective $MMV(C)$-algebra, A is retract of $F_{\mathbf{MMV(C)}}(n)$, i. e. there exist homomorphisms $h : F_{\mathbf{MMV(C)}}(n) \to A$ and $\varepsilon : A \to F_{\mathbf{MMV(C)}}(n)$ such that $h\varepsilon = Id_A$, and moreover, there exist n polynomials $P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n)$ such that

$$P_i(g_1, \ldots, g_n) = \varepsilon(a_i) = \varepsilon h(g_i)$$

and

$$P_i(P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n)) = P_i(x_1, \ldots, x_n),\ i = 1, \ldots, n,$$

where $g_1, \ldots, g_n$ are free generators of $F_{\mathbf{MMV(C)}}(n)$. Observe that $h(g_1), \ldots, h(g_n)$ are generators of A which we denote by $a_1, \ldots, a_n$, respectively. Let e be the endomorphism $\varepsilon h : F_{\mathbf{MMV(C)}}(n) \to F_{\mathbf{MMV(C)}}(n)$. This endomorphism has properties : $ee = e$ and $e(x) = x$ for every $x \in \varepsilon(A)$.

Let us consider the set of equations $\Omega = \{P_i(x_1, \ldots, x_n) \leftrightarrow x_i = 1 : i = 1, \ldots, n\}$ and let $u = \bigwedge_{i=1}^{n}((P_i(g_1, \ldots, g_n) \leftrightarrow g_i) \in F(n)$, where $x \leftrightarrow y$ is abbreviation of $(x \to y) \wedge (y \to x)$. Observe that the equations from Ω are true in A on the elements $\varepsilon(a_i) = e(g_i),\ i = 1, \ldots, n$. Indeed, since e is an endomorphism

$$e(u) = \bigwedge_{i=1}^{n} e(g_i) \leftrightarrow P_i(e(g_1), \ldots, e(g_n)).$$

But $P_i(e(g_1),\ldots,e(g_n)) = P_i(P_1(g_1,\ldots,g_n),\ldots,P_n(g_1,\ldots,g_n)) = P_i(g_1,$ $\ldots,g_n) = \varepsilon h(g_i) = e(g_i), i = 1,\ldots,n$. Hence $e(u) = 1$ and $u \in e^{-1}(1)$, i. e. $[u] \subseteq e^{-1}(1)$. Therefore there exists homomorphism $f : F(n)/[u] \to \varepsilon(A)$ such that the diagram

$$\begin{tikzcd} F(n) \arrow[r, "e"] \arrow[dr, "r"'] & \varepsilon(A) \\ & F(n)/[u] \arrow[u, "f"'] \end{tikzcd}$$

commutes, i. e. $rf = e$, where r is a natural homomorphism sending x to $x/[u]$. Now consider the restrictions e' and r' on $\varepsilon(A) \subseteq F(n)$ of e and r, respectively Then $fr' = e'$. But $e' = Id_{\varepsilon(A)}$. Therefore $fr' = Id_{\varepsilon(A)}$. From here we conclude that r' is an injection. Moreover, r' is a surjection, since $r(\varepsilon(a_i)) = r(g_i)$. Indeed, $e(g_i) = P_i(g_1,\ldots,g_n)$ and $g_i \leftrightarrow P_i(g_1,\ldots,g_n) = g_i \leftrightarrow e(g_i)$, where $e(g_i) = \varepsilon h(g_i)$. So $g_i \leftrightarrow P_i(g_1,\ldots,g_n) \geq \bigwedge_{i=1}^{n} g_i \leftrightarrow P_i(g_1,\ldots,g_n)$, i. e. $g_i \leftrightarrow P_i(g_1,\ldots,g_n) \in [u]$. Hence r' is an isomorphism between $\varepsilon(A)$ and $F(n)/[u]$. Consequently, $A(\cong \varepsilon(A))$ is finitely presented. ◄

It is easy to prove the following

Lemma 10. *Any m-generated non-Boolean subdirectly irreducible $MMV(C)$-algebra A contains $(C,\exists)$ as a subalgebra.*

Lemma 11. *Any subdirectly irreducible m-generated $MMV(C)$-algebra $(A,\exists)$ is a subalgebra of $(C_n^k,\exists)$ for some $n,k \in \omega$ and $n \leq m$.*

Proof. Let $(A,\exists)$ be subdirectly irreducible m-generated $MMV(C)$-algebra. Since $(A,\exists)$ is subdirectly irreducible, it follows that $\exists A$ is totally ordered which is isomorphic to $(C_n,\exists)$ for some $n \leq m$. Then A as $MV(C)$-algebra is subdirect product of copies of C_n, i .e. A is a subalgebra of C_n^k for some $n,k \in \omega$ and $n \leq m$. Therefore, $(A,\exists)$ is a subalgebra of $(C_n^k,\exists)$, where the operation $\exists$ in $(A,\exists)$ is defined in the same way as in $(C_n^k,\exists)$. ◄

Lemma 12. *The algebra $(C_m^k,\exists)$ is a retract of $(C_n^k,\exists)$ for any positive integer k, $1 \leq m \leq n$.*

Proof. Note that $(C_m,\exists)$ is a subalgebra of $(C_n,\exists)$. So, we can define the embedding $\varepsilon : C_m^k \to C_n^k$ in the following way: $\varepsilon(a_1,...,a_k) = (\varepsilon(a_1),...,\varepsilon(a_k))$, where $\varepsilon(c_i) = c_{n-m+i}$ for $i = 1,...,m$.

Let $h : C_n^k \to C_m^k$ be the homomorphism corresponding to the principal ideal generated by $(c_{n-m}, ..., c_{n-m})$. By this homomorphism we have $h(0) = h(c_i) = 0$ for $i = 1, ..., n-m$ and $h(c_{n-m+1}) = c_1,\ h(c_{n-m+2}) = c_2, ..., h(c_n) = c_m$. Then it is easy to check that $h\varepsilon = Id_{C_m^k}$, i. e. $(C_m^k, \exists)$ is a retract of $(C_n^k, \exists)$. ◄

Lemma 13. *Let $(A, \exists)$ be m-generated subdirectly irreducible $MMV(C)$-algebra and $(u] \subset A$ be principal monadic ideal generated by $u \in A$. Then $(A, \exists)/(u]$ is a retract of $(A, \exists)$.*

Proof. The algebra $(A, \exists)$ is a subalgebra of $(C_n^k, \exists)$ for some $n, k \in \omega$ and $n \leq m$ (Lemma 11) and as an MV-algebra A is a subdirect product of copies of C_n, $n \leq m$. Then for some $m \leq n$, we have $u = (c_{m-n}, ..., c_{m-n}) \in C_n^k$, since $(c_{m-n}, ..., c_{m-n}) \in \exists A$. Let h be the homomorphism corresponding to the principal ideal $(u]$. So, we have a homomorphism $h : C_n^k \to C_m^k$ such that $h(0) = h(c_i) = 0$ for $i = 1, ..., m-n$ and $h(c_{m-n+1}) = c_1,\ h(c_{m-n+2}) = c_2, ..., h(c_m) = c_n$.

Define the embedding $\varepsilon : C_n^k \to C_m^k$ in the following way: $\varepsilon(a_1, ..., a_k) = (\varepsilon(a_1), ..., \varepsilon(a_k))$, where $\varepsilon(c_i) = c_{m-n+i}$ for $i = 1, ..., m$. Then it is easy to check that $h\varepsilon = Id_A/9(u]$, i. e. $(A, \exists)/(u]$ is a retract of $(A, \exists)$. ◄

Lemma 14. *Let $A \subset \prod_{i \in I} A_i$ be m-generated $MMV(C)$-algebra which is subdirect product of the family $\{A_i\}_{\in I}$ of the subdirectly irreducible algebras A_i $(i \in I)$ and $A_i' \subset A$, which is a retract of A_i for $i \in I$. Then subalgebra $A' = A \cap \prod_{i \in I} A_i'$ is a retract of A.*

Proof. Since A_i' is a retract of A_i, there exist homomorphisms $\varepsilon_i : A_i' \to A_i$ and $h_i : A_i \to A_i'$ such that $h_i \varepsilon_i = Id_{A_i'}$. It is obvious that $\prod_{i \in I} A_i'$ is a retract of $\prod_{i \in I} A_i$. Indeed, there exist homomorphisms $h = (h_i)_{i \in I} : \prod_{i \in I} A_i \to \prod_{i \in I} A_i'$ and $\varepsilon = (\varepsilon_i)_{i \in I} : \prod_{i \in I} A_i' \to \prod_{i \in I} A_i$ such that $h\varepsilon = Id_{\prod_{i \in I} A_i'}$. Then the restriction of the homomorphism h on A, denoted by h_A, and the restriction of the homomorphism ε on A', denoted by ε_A, give $h_A \varepsilon_{A'} = Id_{A'}$. ◄

Proposition 3. *[18]. m-generated monadic Boolean algebra $(B, \exists)$ is projective in the variety of monadic Boolean algebras if and only if $(B, \exists) \cong (\mathbf{2}, \exists) \times (B', \exists)$ for some m-generated monadic Boolean algebra $(B', \exists)$.*

Lemma 15. *The Boolean envelope $(B(m), \exists)$ of the algebra $F_{\mathbf{MMV(C)}}(m)$, where $B(m) = \{2x^2 : x \in F_{\mathbf{MMV(C)}}(m)\}$, is a retract of the algebra $F_{\mathbf{MMV(C)}}(m)$. In other words, the m-generated monadic Boolean algebra $(B(m), \exists)$ is a projective algebra in $\mathbf{MMV(C)}$.*

Proof. Firstly we show that $(\mathbf{2}^k, \exists)$ is a retract of D_k. Recall that $(\mathbf{2}^k, \exists)$ is a homomorphic image by maximal monadic filter. Denote this homomorphism by $h : D_K \to (\mathbf{2}^k, \exists)$. Note that the maximal monadic filter is generated by the set $\{x \in \exists D_k : 2x = 1\}$. On the other hand, the Boolean envelope $(B(D_k), \exists)$, where $B(D_k) = \{2x^2 : x \in D_k\}$, is a subalgebra of D_k, which is isomorphic to $(\mathbf{2}^k, \exists)$. Denote by $\varepsilon : (B(D_k), \exists) \to D_k$ this embedding. It is obvious that $h\varepsilon = Id_{B(D_k)}$. ◄

Corollary 2. $(\mathbf{2}^{k_1}, \exists) \times ... \times (\mathbf{2}^{k_n}, \exists)$ *is a retract of* $D_{k_1} \times ... \times D_{k_n}$.

Proof. Let A_1, A_2 be any algebras and, respectively, B_1, B_2 be retracts of them, i. e. we have homomorphisms $h_i : A_i \to B_i$ and $\varepsilon_i : B_i \to A_i$ such that $h_i\varepsilon_i = Id_{B_i}$ $(i = 1, 2)$. Then $B_1 \times B_2$ is a retract of $A_1 \times A_2$. Indeed, $h = (h_1, h_2n) : A_1 \times A_2 \to B_1 \times B_2$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ are homomorphisms such that $h\varepsilon = Id_{B_1 \times B_2}$. From here we get the validity of Corollary. ◄

Lemma 16. *For any* $k \in \{1, ..., 2^{2^m} - 1\}$ *there exists principal monadic filter* $[u)$ *of* m*-generated free* $MMV(C)$*-algebra* $F_{\boldsymbol{MMV(C)}}(m)$ $(= \prod_{k=1}^{2^{2^m}-1} D_k)$ *such that* $\pi_k(F_{\boldsymbol{MMV(C)}}(m)) \cong F_{\boldsymbol{MMV(C)}}(m)/[u)$, *where* $\pi_k : F_{\boldsymbol{MMV(C)}}(m) \to D_k$ *is a projection on* k*-th component* D_k *and* $u \in F_{\boldsymbol{MMV(C)}}(m)$.

Proof. Let $u = (0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^{2^m}-1}) \in F_{\mathbf{MMV(C)}}(m)$, where 1^k is the top element of D_k, 0^i is the bottom element of D_i. Note that $(0^1, ..., 0^{k-1}, 1^k, 0^{k+1}, ..., 0^{2^{2^m}-1})$ is Boolean element that belongs to $F_{\mathbf{MMV(C)}}(m)$. Then $[u)$ will be a monadic filter such that $F_{\mathbf{MMV(C)}}(m)/[u) \cong D_k$. So lemma is proved. ◄

Lemma 17. *The algebra* $D_1 \times D_{k_1} \times ... \times D_{k_n}$ *is a projective* $MMV(C)$*-algebra, where* $1 < k_i \leq 2^{2^m} - 1$, $1 \leq i \leq n$ *and* D_1 *is* m*-generated subdirectly irreducible perfect* $MMV(C)$*-algebra.*

Proof. Let $\pi_{1k_1...k_n} : F_{\mathbf{MMV(C)}}(m) \to D_1 \times D_{k_1} \times ... \times D_{k_n}$ be a projection onto $D_1 \times D_{k_1} \times ... \times D_{k_n}$. Let $\{r_1, ..., r_p\} = \{1, ..., 2^{2^m} - 1\} - \{1, k_1, ..., k_n\}$. So, $F_{\mathbf{MMV(C)}}(m) = D_1 \times \prod_{i=1}^n D_{k_i} \times \prod_{i=1}^p D_{r_i}$. Then $D_1 \times \prod_{i=1}^n D_{k_i} \times (\mathbf{2}, \exists)$ is a subalgebra of $D_1 \times \prod_{i=1}^n D_{k_i} \times \prod_{i=1}^p D_{r_i}$. Observe that $(D, \exists)$, where $D = \{(x, 1) : x \in \neg RadD_1\} \cup \{(x, 0) : x \in RadD_1\}$, is a subalgebra of $D_1 \times (\mathbf{2}, \exists)$, which is isomorphic to D_1. So, $D_1 \times \prod_{i=1}^n D_{k_i}$ is a subalgebra of $D_1 \times \prod_{i=1}^n D_{k_i} \times (\mathbf{2}, \exists)$. Then there exists the embedding $\varepsilon : D_1 \times D_{k_1} \times ... \times D_{k_n} \to D_1 \times \prod_{i=1}^n D_{k_i} \times \prod_{i=1}^p D_{r_i}$. Now, it is easy to check that $\pi_{1k_1...k_n}\varepsilon = Id_{D_1 \times D_{k_1} \times ... \times D_{k_n}}$. So lemma is proved. ◄

As in the variety $\mathbf{MV(C)}$ of $MV(C)$-algebras we have

Theorem 6. m*-generated subalgebra* $(A, \exists)$ *of* $F_{\boldsymbol{MMV(C)}}(m)$ *is projective if and only if* $(A, \exists)$ *is finitely presented and* $A \cong A_0 \times A_1$, *where* A_0 *is a perfect* MV*-algebra.*

Proof. First of all note that if A is not represented as $A_0 \times A_1$, where A_0 is a perfect MV-algebra, then A can not be a subalgebra of $F_{\mathbf{MMV(C)}}(m)$ and thereby it will not be a retract of $F_{\mathbf{MMV(C)}}(m)$. Indeed, let A_0 be a retract of $F_{\mathbf{MMV(C)}}(m)$, i.e. let there exist homomorphisms $h_1 : F_{\mathbf{MMV(C)}}(m) \to A_0$ and $\varepsilon_1 : A_0 \to F_{\mathbf{MMV(C)}}(m)$ such that $h_1\varepsilon_1 = Id_{A_0}$. Since the variety **MB** of monadic Boolean algebras is a subvariety of **MMV(C)**, there exists a homomorphism $f : F_{\mathbf{MMV(C)}}(m) \to F_{\mathbf{MB}}(m)$. Let $B(A_0) = f\varepsilon_1(A_0)$. Denote the composition $f\varepsilon_1$ by k. So, for homomorphisms $f : F_{\mathbf{MMV(C)}}(m) \to F_{\mathbf{MB}}(m)$ and $kh_1 : F_{\mathbf{MMV(C)}}(m) \to B(A_0)$ there exists homomorphism $h_2 : F_{\mathbf{MB}}(m) \to B(A_0)$ such that $h_2 f = kh_1$. For $f\varepsilon_1 : A_0 \to F_{\mathbf{MMV(C)}}(m)$ and $k : A_0 \to B(A)$ there exists a homomorphism $\varepsilon_2 : B(A_0) \to F_{\mathbf{MB}}(m)$ such that $f\varepsilon_1 = \varepsilon_2 k$. From $h_2 f = kh_1$ we have $h_2 f\varepsilon_1 = kh_1\varepsilon_1$, and hence $h_2 f\varepsilon_1 = k$, since $h_1\varepsilon_1 = Id_{A_0}$. Then $h_2\varepsilon_2 k = k$, because $f\varepsilon_1 = \varepsilon_2 k$. Since k is a surjective homomorphism, we have $h_2\varepsilon_2 = Id_{B(A_0)}$. So, $B(A_0)$ is a retract of $F_{\mathbf{MB}}(m)$ and, hence, it is projective. According to Proposition 3, m-generated monadic Boolean algebra $(B, \exists)$ is projective in the variety of monadic Boolean algebras if and only if $(B, \exists) \cong (\mathbf{2}, \exists) \times (B', \exists)$ for some m-generated monadic Boolean algebra $(B', \exists)$. But $(\mathbf{2}, \exists)$ is a homomorphic image of perfect monadic $MV(C)$-algebra. Note also that any m-generated projective $MMV(C)$-algebra is finitely presented.

Now suppose that $(A, \exists)$ is finitely presented and $A \cong A_0 \times A_1$, where A_0 is a perfect MV-algebra. Then $(A, \exists)$ is a homomorphic image of $F_{\mathbf{MMV(C)}}(m)$ by some principal monadic filter $[u)$ for some $u \in F_{\mathbf{MMV(C)}}(m)$.

According to Theorem 4, free algebra $F_{\mathbf{MMV(C)}}(m)$ is isomorphic to the finite product of monadic $MV(C)$-algebras D_k $(1 \leq k \leq 2^{2^m} - 1)$ the homomorphic image by maximal monadic filter of which is isomorphic to the subdirectly irreducible monadic Boolean algebra $(\mathbf{2}^k, \exists)$. Then $(A, \exists)$ is a homomorphic image of $D_1 \times D_{k_1} \times ... \times D_{k_n}$ which is projective (Lemma 17), where D_1 is a perfect $MMV(C)$-algebra. So, there exists principal monadic filter $[u')$ of $F_{\mathbf{MMV(C)}}(m)$ such that $F_{\mathbf{MMV(C)}}(m)/[u') \cong D_1 \times D_{k_1} \times ... \times D_{k_n}$. Then there exists principal monadic filter $[u_A) = [\pi_{1k_1...k_n}(u'))$ of the algebra $D_1 \times D_{k_1} \times ... \times D_{k_n}$ such that $D_1 \times D_{k_1} \times ... \times D_{k_n}/[\pi_{1k_1...k_n}(u')) \cong A$, where $\pi_{1k_1...k_n} : F_{\mathbf{MMV(C)}}(m) \to D_1 \times D_{k_1} \times ... \times D_{k_n}$ is a projection of $F_{\mathbf{MMV(C)}}(m)$ onto $D_1 \times D_{k_1} \times ... \times D_{k_n}$. Let $u_1 = \pi_1(u_A)$, $u_{k_i} = \pi_{k_i}(u_A)$ be projections of the element u_A on corresponding components $D_1, D_{k_1}, ..., D_{k_n}$, respectively. Then $D_1/[u_1)$, $D_{k_i}/[u_{k_i})$ are retracts of D_1, D_{k_i} $(i = 1, ..., n)$, respectively (Lemma 13). Then $D_1/[u_1) \times \prod_{i=1}^n D_{k_i}/[u_{k_i})$ is a retract of $D_1 \times \prod_{i=1}^n D_{k_i}$. Therefore A is projective (Lemmas 13, 16, 17). ◀

6. Projective formulas

Let us denote by $\mathcal{P}_m$ a fixed set $x_1, ..., x_m$ of propositional variables and by Φ_m the set of all propositional formulas in L_P with variables in $\mathcal{P}_m$. Note that the m-generated free $MV(C)$-algebra $F_{\mathbf{MV(C)}}(m)$ is isomorphic to $\Phi_m/\equiv$, where $\alpha \equiv \beta$ if and only if $\vdash (\alpha \leftrightarrow \beta)$ and $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$. Subsequently we do not distinguish between the formulas and their equivalence classes. Hence we simply write Φ_m for $F_{\mathbf{MV(C)}}(m)$, and $\mathcal{P}_m$ plays the role of the set of free generators. Since Φ_m is a lattice, we have an order $\leq$ on Φ_m . It follows from the definition of $\to$ that for all $\alpha, \beta \in \Phi_m$, $\alpha \leq \beta$ iff $\vdash (\alpha \to \beta)$.

Let α be a formula of the logic L_P and consider a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ and extend it to all of Φ_m by $\sigma(\alpha(x_1, ..., x_m)) = \alpha(\sigma(x_1), ..., \sigma(x_m))$. We can consider this substitution as an endomorphism $\sigma : \Phi_m \to \Phi_m$ of the free algebra Φ_m.

Definition 1. *A formula $\alpha \in \Phi_m$ is called* projective *if there exists a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ such that $\vdash \sigma(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \sigma(\beta)$, for all $\beta \in \Phi_m$.*

Note that the notion of projective formula was introduced for intuitionistic logic in [10].

Observe that we can rewrite any equation $P(x_1, ..., x_m) = Q(x_1, ..., x_m)$ in the variety **MV(C)** into an equivalent one $P(x_1, ..., x_m) \leftrightarrow Q(x_1, ..., x_m) = 1$. So, for **MV(C)** we can replace n equations by one:

$$\bigwedge_{i=1}^{n} P_i(x_1, ..., x_m) \leftrightarrow Q_i(x_1, ..., x_m) = 1.$$

Now we are ready to show a close connection between projective formulas and projective subalgebras of the free algebra Φ_m.

Theorem 7. *Let A be an m-generated projective subalgebra of the free algebra Φ_m. Then there exists a projective formula α of m variables, such that A is isomorphic to $\Phi_m/[\alpha)$, where $[\alpha)$ is the principal filter generated by $\alpha \in \Phi_m$.*

Proof. Suppose A is an m-generated projective subalgebra of Φ_m with generators $a_1, ..., a_m$. Then A is a retract of Φ_m, and there exist homomorphisms $\varepsilon : A \to \Phi_m$, $h : \Phi_m \to A$ such that $h\varepsilon = Id_A$, where $\varepsilon(x) = x$ for every $x \in A \subset \Phi_m$. Observe that εh is an endomorphism of Φ_m. We will show now that $\alpha = \bigwedge_{j=1}^{m}(x_j \leftrightarrow \varepsilon h(x_j))$ is a projective formula, namely, that $\vdash \varepsilon h(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \varepsilon h(\beta)$, for all $\beta \in \Phi_m$.

Indeed, $\varepsilon h(\bigwedge_{j=1}^{m}(p_j \leftrightarrow \varepsilon h(p_j))) = \bigwedge_{j=1}^{m}(\varepsilon h(x_j) \leftrightarrow \varepsilon h \varepsilon h(x_j))$, and since $h\varepsilon = Id_A$, we have $\varepsilon h(\bigwedge_{j=1}^{m}(x_j \leftrightarrow \varepsilon h(x_j))) = \bigwedge_{j=1}^{m}(\varepsilon h(x_j) \leftrightarrow \varepsilon h(x_j))$. Thus $\vdash \varepsilon h(\alpha)$. Further, for any $\beta \in \Phi_m$, $\varepsilon h(\beta(x_1, ..., x_m)) = \beta(\varepsilon h(x_1), ..., \varepsilon h(x_m))$, and since $\alpha \vdash x_j \leftrightarrow \varepsilon h(x_j)$, $j = 1, ..., m$, we have $\alpha \vdash \beta \leftrightarrow \varepsilon h(\beta)$.

Since A is an m-generated projective $MV(C)$-algebra, according to the Proposition 2, there exist m polynomials $P_1(x_1, ..., x_m), ..., P_m(x_1, ..., x_m)$ such that

$$P_i(x_1, ..., x_m) = \varepsilon(a_i) = \varepsilon h(x_i)$$

and

$$P_i(P_1(x_1, ..., x_m), ..., P_m(x_1, ..., x_m)) = P_i(x_1, ..., x_m),\ i = 1, ..., m.$$

Observe that $h(x_i) = a_i$. Since the m-generated projective MV-algebra A is finitely presented by the equation $\bigwedge_{j=1}^{m}(x_j \leftrightarrow \varepsilon h(x_j)) = 1$, we have $A \cong \Phi_m/[\alpha)$. ◀

Theorem 8. *If α is a projective formula of m variables, then $\Phi_m/[\alpha)$ is a projective algebra which is isomorphic to a projective subalgebra of Φ_m.*

Proof. Suppose that α is a projective formula of m variables. Then there exists a substitution $\sigma : \mathcal{P}_m \to \Phi_m$ such that $\vdash \sigma(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \sigma(\beta)$, for all $\beta \in \Phi_m$. Since σ is an endomorphism of Φ_m, $\sigma(\Phi_m)$ is a subalgebra of Φ_m. Now we will show that $\sigma(\Phi_m)$ is a retract of Φ_m, i.e. $\sigma^2 = \sigma$. Indeed, since α is a projective formula, $\sigma(\alpha) = 1_{\Phi_m}$, and $\alpha \leq \beta \leftrightarrow \sigma(\beta)$ for all $\beta \in \Phi_m$. But then $\sigma(\alpha) \leq \sigma(\beta) \leftrightarrow \sigma^2(\beta)$, $\sigma(\beta) \leftrightarrow \sigma^2(\beta) = 1_{\Phi_m}$, $\sigma(\beta) = \sigma^2(\beta)$, and $\sigma^2 = \sigma$. Hence $\sigma(\Phi_m)$ is a retract of Φ_m. So, $\sigma(\Phi_m)$ is isomorphic to $\Phi_m/[\alpha)$. ◀

Thus we have the following correspondence between projective formulas and projective subalgebras of Φ_m. To each m-generated projective subalgebra of m-generated free $MV(C)$-algebra there corresponds an m-variable projective formula and to two non-isomorphic m-generated projective subalgebra of m-generated free $MV(C)$-algebra there correspond non-equivalent m-variable projective formulas. And to two non-equivalent m-variable projective formulas there correspond two different m-generated projective subalgebras of m-generated free $MV(C)$-algebra (but they can be isomorphic).

Therefore we arrive at the following

Corollary 3. *There exists a one-to-one correspondence between projective formulas with m variables and m-generated projective subalgebras of Φ_m.*

7. Unification problem

Let E be an equational theory. The E-unification problem is: given two terms s, t (built from function symbols and variables), to find a unifier for them, that is, a uniform replacement of the variables occurring in s and t by other terms that makes s and t equal by modulo E. For detailed information on unification problem we refer to [10, 11].

Let us be more precise. Let Φ be a set of functional symbols and V be a set of variables. Let $T_V(\Phi)$ be the term algebra built from Φ and V, and $T_V(\Phi_m)$ be the term algebra of m-variable terms. Let E be a set of equations $p(x) = q(x)$, where $p(x), q(x) \in T_V(\Phi_m)$.

Let $\mathbf{V}$ be the variety of algebras over Φ axiomatized by the equations in E.

A unification problem modulo E is a finite set of pairs

$$\mathcal{E} = \{(s_j, t_j) : s_j, t_j \in T_V(\Phi_m), j \in J\}$$

for some finite set J. *A solution to* (or *a unifier for*) $\mathcal{E}$ is a substitution (or an endomorphism of the term algebra $T_V(\Phi_m)$) σ (which is extension of the map $s : V_m \to T_V(\Phi)$, where V_m $(= \{x_1, ..., x_m\})$ is the set of m variables) such that the equality $\sigma(s_j) = \sigma(t_j)$ holds in every algebra of the variety $\mathbf{V}$. The problem $\mathcal{E}$ is *solvable* (or *unifiable*) if it admits at least one unifier.

Let $(X, \preceq)$ be a quasi-ordered set (i.e. a reflexive and transitive relation). A μ-set [11] for $(X, \preceq)$ is a subset $M \subseteq X$ such that: (1) every $x \in X$ is less or equal to some $m \in M$; (2) all elements of M are mutually $\preceq$-incomparable. There might be no μ-set for $(X, \preceq)$ (in this case we say that $(X, \preceq)$ has *type* 0) or there might be many of them, due to the lack of antisymmetry. However, all μ-sets for $(X, \preceq)$, if any, must have the same cardinality. We say that $(X, \preceq)$ has *type* $1, \omega, \infty$ if and only if it has a μ-set of cardinality 1, of finite (greater than 1) cardinality or of infinite cardinality, respectively.

Substitutions are compared by instantiation in the following way: we say that $\sigma : T_V(\Phi_m) \to T_V(\Phi_m)$ is *more general than* $\tau : T_V(\Phi_m) \to T_V(\Phi_m)$ (written as $\tau \preceq \sigma$) if and only if there is a substitution $\eta : T_V(\Phi_m) \to T_V(\Phi_m)$ such that for all $x \in V_m$ we have $E \vdash \eta(\sigma(x)) = \tau(x)$. The relation $\preceq$ is quasi-order.

Let $U_E(\mathcal{E})$ be the set of unifiers for the unification problem $\mathcal{E}$. Then $(U_E(\mathcal{E}), \preceq)$ is a quasi-ordered set.

We say that an equational theory E has:

1. Unification type 1 if and only if for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type 1;
2. Unification type ω if and only if for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type ω;
3. Unification type ∞ if and only if for every solvable unification problem $\mathcal{E}$, $U_E(\mathcal{E})$ has type 1 or ω or ∞ - and there is a solvable unification problem $\mathcal{E}$ such that $U_E(\mathcal{E})$ has type ∞;
4. Unification type nullary, if none of the preceding cases applies.

An algebra A is called *finitely presented* if A is finitely generated, with the generators $a_1, ..., a_m \in A$, and there exist a finite number of equations $P_1(x_1, ..., x_m) = Q_1(x_1, ..., x_m), ..., P_n(x_1, ..., x_m) = Q_n(x_1, ..., x_m)$ holding in A on the generators $a_1, ..., a_m \in A$ such that if there exists an m-generated algebra B, with generators $b_1, ..., b_m \in B$, such that the equations $P_1(x_1, ..., x_m) = Q_1(x_1, ..., x_m), ..., P_n(x_1, ..., x_m) = Q_n(x_1, ..., x_m)$ hold in B on the generators $b_1, ..., b_m \in B$, then there exists a homomorphism $h : A \to B$ sending a_i to b_i.

Now we will give characterization of finitely presented $MMV(C)$-algebras.

Recall that filter F of an algebra $(A, \exists) \in \mathbf{MMV(C)}$ is called a monadic filter (which is dual to an ideal, see [19]) if for every $a \in A$ we have $a \in F \Rightarrow \forall a \in F$.

For any set $X \subseteq A$, let $[X)$ denote the monadic filter generated by X. It is easy to check that $[X) = \{a \in A : a \geq \forall x_1 \odot \ldots \odot \forall x_n : x_1, \ldots, x_n \in X\}$.

Theorem 9. *Let p be an m-ary term. Then there is a principal monadic filter F such that $F_{\mathbf{MMV(C)}}(m, p = 1) \cong F_{\mathbf{MMV(C)}}(m)/F$.*

Proof. Any $\mathbf{MMV(C)}$-equation $p = q$ is equivalent to an equation of the form $r = 1$. Indeed, $p = q$ if and only if $(p^* \oplus q) \wedge (q^* \oplus p) = 1$. If we have a finite set of equations $\{r_i = 1 : i = 1, \ldots, n\}$, then we can represent this set as one equation $r_1 \wedge \ldots \wedge r_n = 1$.

Now let $F = \{x : x \in F_{\mathbf{MMV(C)}}(m)$ and $x \geq \forall p^n(g_1, ..., g_m), n \in \omega\}$, where $g_1, \ldots, g_m$ are free generators of $F_{\mathbf{MMV(C)}}(m)$. Then $g_1/F, ..., g_m/F$ are generators of $F_{\mathbf{MMV(C)}}(m)/F$. Let $\pi_F : F_{\mathbf{MMV(C)}}(m) \to F_{\mathbf{MMV(C)}}(m)/F$ be the natural homomorphism. Let also A be an $MMV(C)$-algebra generated by $\{a_1, \ldots, a_m\}$, $p(a_1, ..., a_m) = 1$ and $f : F_{\mathbf{MMV(C)}}(m) \to A$ be a homomorphism such that $f(g_i) = a_i, \quad i = 1, \ldots, m$. Then $\forall p^n(g_1, \ldots, g_m) \in f^{-1}(1)$, $n \in \omega$ and therefore $F \subseteq f^{-1}(1)$. By the homomorphism theorem, there is a homomorphism $f' : F_{\mathbf{MMV(C)}}(m)/F \to A$ such that $\pi_F f' = f$. It should be clear that f' is the needed homomorphism extending the map $g_i/F \mapsto a_i$. ◀

From this theorem it follows that if an algebra A is finitely presented, then there exists a principal monadic filter F of the free algebra $F_{\mathbf{MMV(C)}}(m)$ such that $A \cong F_{\mathbf{MMV(C)}}(m)/F$.

Theorem 10. *Let $u \in F_{\mathbf{MMV(C)}}(m)$ be such that $\forall u^n \neq 0$ for any $n \in \omega$. Then $F = \{x : x \geq \forall u^n, \ n \in \omega\}$ is a proper principal monadic filter in $F_{\mathbf{MMV(C)}}(m)$ such that $F_{\mathbf{MMV(C)}}(m)/F \cong F_{\mathbf{MMV(C)}}(m, p = 1)$ for some m-ary term p.*

Proof. Let F be a monadic filter satisfying the condition of the theorem. Then $u = p(g_1, \ldots, g_m)$ for some term p, where $g_1, \ldots, g_m$ are free generators of $F_{\mathbf{MMV(C)}}(m)$. We have that $F_{\mathbf{MMV(C)}}(m)/F$ is generated by $g_1/F, \ldots, g_m/F$, and that $p(g_1/F, \ldots, g_m/F) = p(g_1, \ldots, g_m)/F = 1_{F(m)/F}$. The rest can be verified as in the proof of Theorem 9. ◀

Combining the two theorems we arrive at

Theorem 11. *An m-generated $MMV(C)$-algebra A is finitely presented if and only if there exists a principal monadic filter F of $F_{\mathbf{MMV(C)}}(m)$ such that*

$$F_{\mathbf{MMV(C)}}(m)/F \cong A$$

.

Following Ghilardi [10], who has introduced the relevant definitions for E-unification from an algebraic point of view, by an algebraic unification problem we mean a finitely presented algebra A of $\mathbf{V}$. In this context an E-unification problem is simply a finitely presented algebra A, and a solution for it (also called a unifier for A) is a pair given by a projective algebra P and a homomorphism $u : A \to P$. The set of unifiers for A is denoted by $U_E(A)$. A is said to be *unifiable* or solvable if and only if $U_E(A)$ is not empty. Given another algebraic unifier $w : A \to Q$, we say that u is more general than w, written $w \preceq u$, if there is a homomorphism $g : P \to Q$ such that $w = gu$.

The set of all algebraic unifiers $U_E(A)$ of a finitely presented algebra A forms a quasi-ordered set with the quasi-ordering $\preceq$.

The algebraic unification type of an algebraically unifiable finitely presented algebra A in the variety $\mathbf{V}$ is now defined exactly as in the symbolic case, using the quasi-ordering set $(U_E(A), \preceq)$.

Theorem 12. *The unification type of the equational class* $\mathbf{MMV(C)}$ *is* 1*, i.e. unitary.*

Proof. According to Theorem 6, finitely generated projective $MMV(C)$-algebras are exactly those of the kind $D_0 \times D$, where D_0 is perfect MV-algebra. We show that $MMV(C)$-algebra A is unifiable if and only if it is projective (thus identity morphisms act as mgu's in the algebraic setting). Let A be unifiable. Then there is a homomorphism from A into an algebra of the kind $D_0 \times D$, hence also a homomorphism $h : A \to D_0$. So, A is a retract of $A \times D_0$ (which is projective by the above remark). Indeed, we have homomorphisms $\varepsilon : A \to D_0$ and $\pi_1 : A \times D_0 \to A$, where $\varepsilon(a) = (a, h(a))$, π_1 is a projection on the first component and $\pi_1\varepsilon = Id_A$. Since A is a retract of a projective algebra, it follows that A is also projective. ◀

References

[1] L.P. Belluce, *Further results on infinite valued predicate logic*, J. Symbolic Logic, **29**, 1964, 69–78.

[2] H. Bass, *Finite monadic algebras*, Proceedings of the American Mathematical Society, **9**, 1958, 258–268.

[3] L.P. Belluce, C.C. Chang, *A weak completeness theorem for infinite valued 2rst-order logic*, J. Symbolic Logic, **28**, 1963, 43–50.

[4] L.P. Belluce, A. Di Nola, B. Gerla, *Perfect MV-algebras and their Logic*, Applied Categorical Structures, **15(1-2**, 2007, 135–151.

[5] L.P. Belluce, R. Grigolia, A. Lettieri, *Representations of monadic MV- algebras*, Studia Logica, **81**, 2005, 125–144.

[6] C.C. Chang, *Algebraic analysis of many-valued logics*, Trans. Amer. Math. Soc., **88**, 1958, 467–490.

[7] A. Di Nola, R. Grigolia, *On monadic MV-algebras*, APAL, **128(1-3)**, 2004, 125–139.

[8] A. Di Nola, R. Grigolia, *Gödel spaces and perfect MV-algebras*, Journal of Applied Logic, **13(3)**, 2015, 270–284.

[9] A. Di Nola, A. Lettieri, *Perfect MV-algebras are categorically equivalent to Abelian ℓ-Groups*, Studia Logica, **53**, 1994, 417–432.

[10] S. Ghilardi, *Unification through projectivity*, J. Logic Comput., **7(6)**,1997, 733–752.

[11] S. Ghilardi, *Unification, finite duality and projectivityin varieties of Heyting algebras*, APAL, **127**, 2004, 99–115.

[12] G. Georgescu, A. Iurgulescu, I. Leustean, *Monadic and closure MV-algebras*, Multi. Val. Logic, **3**, 1998, 235–257.

[13] R. Grigolia, *Free algebras of non-classical logics*, "Metsniereba", Tbilisi,1987, 110 pages. (in Russian)

[14] R. Grigolia, *Finitely generated free $S4.3$-algebras, Studies on non-classical logics and formal systems*, Nauka, Moscow, 1983, 281-286. (in Russian)

[15] L.S. Hay, *An axiomatization of the infinitely many-valued calculus*, M.S. Thesis, Cornell University, 1958.

[16] J. Łukasiewicz, A. Tarski, *Unntersuchungen Ouber den Aussagenkalkul*, Comptes Rendus des seances de la Societe des Sciences et des Lettres de Varsovie, **23(3)**, 1930, 30–50.

[17] R. McKenzie, *An algebraic version of categorical equivalence for varieties and more general algebraic categories*, Logic and Algebra, edited by P. Aglianò and A. Ursini, volume 180 of *Lectures Notes in Pure and Applied Mathematics*, Marcel Dekker, Dedicated to Roberto Magari, 1996, 221–243.

[18] R. W. Quackenbush, *Demi-Semi-Primal algebras and Malcev-type conditions*, Mathematische Zeithschrift, **122**, 1971, 177–188.

[19] J.D. Rutledge, *A preliminary investigation of the infinitely many-valued predicate calculus*, Ph.D. Thesis, Cornell University, 1959.

[20] B. Scarpellini, *Die Nichtaxiomatisierbarkeit des unendlichwertigen PrOadikatenkalkulus von Łukasiewicz,*J. Symbolic Logic, **27**, 1962, 159–170.

[21] D. Schwartz, *Theorie der polyadischen MV-Algebren endlicher Ordnung*, Math. Nachr., **78**, 1977, 131–138.

[22] D. Schwartz, *Polyadic MV-algebras*, Zeit. f. math. Logik und Grundlagen d. Math., **26**, 1980, 561–564.

Antonio Di Nola
Department of Mathematics, University of Salerno
E-mail: adinola@unisa.it

Revaz Grigolia
Tbilisi State University
E-mail: revaz.grigolia@tsu.ge

Ramaz Liparteliani
Georgian Technical University, Institute of Cybernetics
E-mail: r.liparteliani@yahoo.com

Received 13 October 2016
Accepted 24 December 2016