Azerbaijan Journal of Mathematics
V. 7, No 2, 2017, July
ISSN 2218-6816

Homogeneous Problem with Two-point Conditions in Time for Some Equations of Mathematical Physics

Z.M. Nytrebych, O.M. Malanchuk*, V.S. Il'kiv, P.Ya. Pukach

Abstract. We study the problem for homogeneous partial differential equations in two variables which are of second order with respect to time variable and of finite order with respect to another (spatial) variable, with homogeneous two-point conditions in time. We propose a method for the construction of nontrivial solutions of this problem when its characteristic determinant is nontrivial and the set of its zeroes is not empty. We apply this method to the construction of non-zero solutions of homogeneous two-point problems for some equations of mathematical physics.

Key Words and Phrases: differential-symbol method, two-point problem for partial differential equations, multipoint problem, equations of mathematical physics.

2010 Mathematics Subject Classifications: 35G15, 35K05

---

1. Introduction

Problems with two-point conditions in time for partial differential equations (PDE) are ill-posed ones [1, 2, 3]. As an example, we can mention the problem for string oscillation equation

$$\left[\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right] U(t,x) = 0, \quad a = \text{const} > 0, \tag{1}$$

with two-point conditions

$$U(0,x) = \varphi_1(x), \quad U(h,x) = \varphi_2(x), \quad h > 0. \tag{2}$$

Ill-posedness of the problem (1), (2) is caused by the fact that the set of nontrivial solutions of corresponding homogeneous problem is not empty. Nontrivial solutions of equation (1) which satisfy the homogeneous conditions

$$U(0,x) = U(h,x) = 0,$$

---

*Corresponding author.

are, in particular, the functions of the form

$$U_k\,(t,x) = \sin\frac{\pi k t}{h}\sin\frac{\pi k x}{ah}, \quad k\in\mathbb{N}.$$

Thus, the study of solutions of homogeneous problems for PDE is an actual task. Construction of nontrivial solutions of the homogeneous and nonhomogeneous partial differential equation of second order in time and, in general, infinite order in spatial variable with homogeneous Dirichlet conditions (two-point conditions) in the time strip using the differential-symbol method [4, 5] is described in [6]. The technique of using the differential-symbol method for constructing elements of the null space of the problem for evolution equation with nonlocal condition is proposed in [7]. The Dirichlet type problem in infinite strip for differential equation, which is homogeneous in the order of differentiation, is studied in [8].

The study of solutions of homogeneous PDE in two variables which are of second order with respect to time variable and, in general, of infinite order with respect to another (spatial) variable, with homogeneous two-point conditions in time, and which in particular cover string oscillation equation, Klein-Gordon-Fock equation, telegraph equation, etc., is an actual task.

2. Problem statement

In this work, we consider the sets of solutions of PDE

$$L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)U(t,x)\equiv\frac{\partial^2 U}{\partial t^2}+2\,a\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}+b\left(\frac{\partial}{\partial x}\right)U=0,\quad (t,x)\in\mathbb{R}^2, \tag{3}$$

that satisfy the following homogeneous two-point conditions:

$$\begin{aligned} A_1\left(\frac{\partial}{\partial x}\right)U(0,x)+A_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(0,x)&=0,\\ B_1\left(\frac{\partial}{\partial x}\right)U(h,x)+B_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(h,x)&=0,\qquad h>0,\quad x\in\mathbb{R}\,. \end{aligned} \tag{4}$$

In equation (3) and in conditions (4), operator coefficients $a\left(\frac{\partial}{\partial x}\right)$, $b\left(\frac{\partial}{\partial x}\right)$, $A_1\left(\frac{\partial}{\partial x}\right)$, $B_1\left(\frac{\partial}{\partial x}\right)$, $A_2\left(\frac{\partial}{\partial x}\right)$, $B_2\left(\frac{\partial}{\partial x}\right)$ are arbitrary differential polynomials with complex coefficients whose symbols are functions $a(\nu)$, $b(\nu)$, $A_1(\nu)$, $B_1\,(\nu)$, $A_2\,(\nu)$, $B_2\,(\nu)$. From now on we assume that $|A_1(\nu)|^2+|A_2(\nu)|^2\neq 0$, $|B_1(\nu)|^2+|B_2(\nu)|^2\neq 0$ for $\nu\in\mathbb{C}$.

The homogeneous problem (3), (4) obviously has the trivial solution. In [11], the condition was found under which there exists only a trivial solution of the problem — characteristic determinant of the problem for all values of the parameter ν is non-zero.

That work also considered the case where the set of zeroes of the characteristic determinant of the problem is the set of complex numbers.

In this paper, we construct nontrivial solutions of problem (3), (4) for other cases of the characteristic determinant and apply our results to some equations of mathematical physics (string oscillation equation, Klein-Gordon-Fock equation, telegraph equation, bicalorical equation, elasticity theory equation). Equation (3), in particular, describes the dynamic processes in the longitudinally moving environments (see [9]).

3. Main results

Consider the ordinary differential equation (ODE)

$$L\left(\frac{d}{dt},\nu\right)T(t,\nu) \equiv \left(\frac{d^2}{dt^2} + 2\,a(\nu)\frac{d}{dt} + b(\nu)\right)T(t,\nu) = 0, \quad \nu \in \mathbb{C}, \tag{5}$$

constructed on the basis of PDE (3).

The functions

$$\begin{aligned} T_0(t,\nu) &= e^{-a(\nu)t}\left\{a(\nu)\frac{\sinh\left[t\sqrt{a^2(\nu)-b(\nu)}\right]}{\sqrt{a^2(\nu)-b(\nu)}} + \cosh\left[t\sqrt{a^2(\nu)-b(\nu)}\right]\right\}, \\ T_1(t,\nu) &= e^{-a(\nu)t}\frac{\sinh\left[t\sqrt{a^2(\nu)-b(\nu)}\right]}{\sqrt{a^2(\nu)-b(\nu)}} \end{aligned} \tag{6}$$

form a normal in $t = 0$ fundamental system of solutions of equation (5) and, by the Poincare theorem on the analytic dependence of the solution of Cauchy problem on the parameter ([10], p. 59), they are entire with respect to the parameter ν. Note that they are entire in variable t, too.

We write down the characteristic determinant of problem (3), (4):

$$\begin{aligned} \Delta(\nu) &= \begin{vmatrix} A_1(\nu) & A_2(\nu) \\ B_1(\nu)T_0(h,\nu) + B_2(\nu){T_0}'(h,\nu) & B_1(\nu)T_1(h,\nu) + B_2(\nu){T_1}'(h,\nu) \end{vmatrix} = \\ &= \begin{pmatrix} B_1(\nu) & B_2(\nu) \end{pmatrix} \begin{pmatrix} T_0(h,\nu) & T_1(h,\nu) \\ {T_0}'(h,\nu) & {T_1}'(h,\nu) \end{pmatrix} \begin{pmatrix} -A_2(\nu) \\ A_1(\nu) \end{pmatrix}, \end{aligned} \tag{7}$$

where ${T_0}'(h,\nu) = \frac{dT_0}{dt}(h,\nu)$, ${T_1}'(h,\nu) = \frac{dT_1}{dt}(h,\nu)$.

Three cases are possible for the determinant $\Delta(\nu)$:

1) $\Delta(\nu) \neq 0 \ \ \forall \nu \in \mathbb{C}$;

2) $\Delta(\nu) \equiv 0$;

3) $M \equiv \{\nu \in \mathbb{C} : \ \triangle(\nu) = 0\} \neq \varnothing$ and $M \neq \mathbb{C}$.

The first two cases as noted above were studied in [11]. We shall investigate the third case in this work.

The determinant $\Delta(\nu)$, as a superposition of entire functions, is an entire function. Therefore, the set M may contain finite number of multiple zeroes of the function $\triangle(\nu)$ or may consist of infinite number of distinct zeroes $\alpha_1, \alpha_2, \ldots$ of finite multiplicities $p_{\alpha_1}, p_{\alpha_2}, \ldots$, respectively. Moreover, $|\alpha_n| \to \infty$ as $n \to \infty$.

Let's introduce the following classes of functions:

$K_{\mathbb{C},L}$ is a class of quasi-polynomials of the form

$$U(t,x) = \sum_{j=1}^{m} \sum_{l=1}^{N} P_{lj}(t,x)\, e^{\beta_l t + \alpha_j x},$$

where $\beta_j \in \mathbb{C}$, $\alpha_j \in L \subseteq \mathbb{C}$, $P_{lj}(t,x)$, $j = \overline{1,m}$, $l = \overline{1,N}$, are arbitrary polynomials with complex coefficients, $m, N \in \mathbb{N}$;

K_L is a class of quasi-polynomials of the form

$$\psi(x) = \sum_{j=1}^{m} Q_j(x)\, e^{\alpha_j x}, \tag{8}$$

where $\alpha_j \in L \subseteq \mathbb{C}$, $Q_j(x)$, $j = \overline{1,m}$, are arbitrary polynomials with complex coefficients, $m \in \mathbb{N}$.

Remark 1. *For each quasi-polynomial* (8)*, we can put in correspondence the differential expression*

$$\psi\left(\frac{\partial}{\partial\nu}\right) = \sum_{j=1}^{m} Q_j\left(\frac{\partial}{\partial\nu}\right) e^{\alpha_j \frac{\partial}{\partial\nu}},$$

whose action onto entire function can be defined by the formula

$$\psi\left(\frac{\partial}{\partial\nu}\right)\chi(\nu) = \sum_{j=1}^{m} Q_j\left(\frac{\partial}{\partial\nu}\right)\chi(\alpha_j + \nu).$$

Theorem 1. *Let $\psi \in K_M$, with*

$$\psi(x) = x^r e^{\alpha x}, \tag{9}$$

where $\alpha \in M$, $\Delta(\alpha) = \Delta'(\alpha) = \Delta''(\alpha) = \ldots = \Delta^{(p_\alpha - 1)}(\alpha) = 0$, $\Delta^{(p_\alpha)}(\alpha) \neq 0$, $r \in \{0, 1, \ldots, p_\alpha - 1\}$, $p_\alpha \in \mathbb{N}$. Then the function

$$U(t,x) = \left(\frac{\partial}{\partial\nu}\right)^r \left\{ \left[A_2(\nu)\, T_0(t,\nu) - A_1(\nu)\, T_1(t,\nu)\right] e^{\nu x} \right\}\Bigg|_{\nu=\alpha}, \tag{10}$$

where $T_0(t,\nu)$, $T_1(t,\nu)$ are defined by (6)*, is a nontrivial solution of the problem* (3), (4) *from the class $K_{\mathbb{C},M}$.*

Proof. Let $\psi(x)$ be a quasi-polynomial of the form (9). Then the function

$$U(t,x) = \psi\left(\frac{\partial}{\partial\nu}\right)\Big\{[A_2(\nu)T_0(t,\nu) - A_1(\nu)T_1(t,\nu)]e^{\nu x}\Big\}\Big|_{\nu=0},$$

i.e. function (10) (see Remark 1), is a solution of equation (3). In fact, using the commutativity of action of operations $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial\nu}$ onto entire functions of variables t, x and ν, we have

$$L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)\left\{\left(\frac{\partial}{\partial\nu}\right)^r\Big\{[A_2(\nu)T_0(t,\nu) - A_1(\nu)T_1(t,\nu)]e^{\nu x}\Big\}\Big|_{\nu=\alpha}\right\} =$$

$$= \left(\frac{\partial}{\partial\nu}\right)^r\left\{L\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x}\right)\Big([A_2(\nu)T_0(t,\nu) - A_1(\nu)T_1(t,\nu)]e^{\nu x}\Big)\right\}\Big|_{\nu=\alpha} =$$

$$= \left(\frac{\partial}{\partial\nu}\right)^r\left\{e^{\nu x}L\left(\frac{d}{dt},\nu\right)[A_2(\nu)T_0(t,\nu) - A_1(\nu)T_1(t,\nu)]\right\}\Big|_{\nu=\alpha} =$$

$$= \left(\frac{\partial}{\partial\nu}\right)^r\left\{e^{\nu x}\left[A_2(\nu)L\left(\frac{d}{dt},\nu\right)T_0(t,\nu) - A_1(\nu)L\left(\frac{d}{dt},\nu\right)T_1(t,\nu)\right]\right\}\Big|_{\nu=\alpha} = 0.$$

Now we will show that the function (10) satisfies the first of conditions (4):

$$A_1\left(\frac{\partial}{\partial x}\right)U(0,x) + A_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(0,x) =$$

$$= A_1\left(\frac{\partial}{\partial x}\right)\left\{\left(\frac{\partial}{\partial\nu}\right)^r\Big\{[A_2(\nu)T_0(0,\nu) - A_1(\nu)T_1(0,\nu)]e^{\nu x}\Big\}\Big|_{\nu=\alpha}\right\}+$$

$$+A_2\left(\frac{\partial}{\partial x}\right)\left\{\left(\frac{\partial}{\partial\nu}\right)^r\Big\{\Big[A_2(\nu)T_0{}'(0,\nu) - A_1(\nu)T_1{}'(0,\nu)\Big]e^{\nu x}\Big\}\Big|_{\nu=\alpha}\right\} =$$

$$= \left(\frac{\partial}{\partial\nu}\right)^r\left\{A_1\left(\frac{\partial}{\partial x}\right)\Big(A_2(\nu)e^{\nu x}\Big)\right\}\Big|_{\nu=\alpha} + \left(\frac{\partial}{\partial\nu}\right)^r\left\{A_2\left(\frac{\partial}{\partial x}\right)\Big(-A_1(\nu)e^{\nu x}\Big)\right\}\Big|_{\nu=\alpha} =$$

$$= \left(\frac{\partial}{\partial\nu}\right)^r\Big\{e^{\nu x}A_1(\nu)A_2(\nu)\Big\}\Big|_{\nu=\alpha} + \left(\frac{\partial}{\partial\nu}\right)^r\Big\{e^{\nu x}A_2(\nu)(-A_1(\nu))\Big\}\Big|_{\nu=\alpha} =$$

$$= \left(\frac{\partial}{\partial\nu}\right)^r\Big\{e^{\nu x}\big(A_1(\nu)A_2(\nu) - A_1(\nu)A_2(\nu)\big)\Big\}\Big|_{\nu=\alpha} = 0.$$

Furthermore

$$B_1\left(\frac{\partial}{\partial x}\right)U(h,x)+B_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(h,x) = \left(\frac{\partial}{\partial\nu}\right)^r\Big\{\Delta(\nu)e^{\nu x}\Big\}\Big|_{\nu=\alpha} = \sum_{j=0}^{r}C_r^j\Delta^{(j)}(\alpha)x^{r-j}e^{\alpha x},$$

where $C_r^j = \frac{r!}{j!(r-j)!}$.

As $\nu = \alpha$ is a zero of the function $\Delta(\nu)$ of multiplicity p_α, the second of conditions (4) is also satisfied. So, the function $U(t,x)$ is a solution of problem (3), (4) and belongs to $K_{\mathbb{C},M}$.

Let's calculate

$$U(0,x) = \left(\frac{\partial}{\partial\nu}\right)^r \left\{A_2(\nu)\, e^{\nu x}\right\}\Big|_{\nu=\alpha} = x^r A_2(\alpha) + \sum_{j=1}^{r} C_r^j A_2^{(j)}(\alpha) x^{r-j} e^{\alpha x},$$

$$\frac{\partial U}{\partial t}(0,x) = -\left(\frac{\partial}{\partial\nu}\right)^r \left\{A_1(\nu)\, e^{\nu x}\right\}\Big|_{\nu=\alpha} = -x^r A_1(\alpha) - \sum_{j=1}^{r} C_r^j A_1^{(j)}(\alpha) x^{r-j} e^{\alpha x}.$$

The condition $|A_1(\nu)|^2 + |A_2(\nu)|^2 \neq 0$ implies that at least one of the functions $U(0,x)$ or $\frac{\partial U}{\partial t}(0,x)$ is nonzero. So, the function (10) is a nontrivial solution of problem (3), (4). This completes our proof.◄

Corollary 1. *If $Q(x)\, e^{\alpha x}$ is a quasi-polynomial with $\Delta(\alpha) = \Delta'(\alpha) = \ldots = \Delta^{(p_\alpha-1)}(\alpha) = 0$, $\Delta^{(p_\alpha)}(\alpha) \neq 0$, $Q(x)$ is a nontrivial polynomial and $\deg Q(x) \leq p_\alpha - 1$, then the formula*

$$U(t,x) = Q\left(\frac{\partial}{\partial\nu}\right)\left\{[A_2(\nu)\, T_0(t,\nu) - A_1(\nu)\, T_1(t,\nu)]\, e^{\nu x}\right\}\Big|_{\nu=\alpha}$$

defines the nontrivial solution of problem (3), (4) *in the class $K_{\mathbb{C},M}$.*

4. Null spaces of two-point problems for some equations of mathematical physics

Let's consider the two-point in time problems for some classical equations of mathematical physics, and find the elements of their null spaces using Theorem 1.

4.1. Vibrating string equation

Consider the problem of finding nontrivial solutions of PDE

$$\frac{\partial^2 U}{\partial t^2} = \gamma^2 \frac{\partial^2 U}{\partial x^2}, \quad (t,x) \in \mathbb{R}^2, \ \ \gamma = \text{const} > 0, \tag{11}$$

which satisfy two-point conditions

$$A\left(\frac{\partial}{\partial x}\right)\left\{U(0,x) + \frac{\partial U}{\partial t}(0,x)\right\} = 0, \quad B\left(\frac{\partial}{\partial x}\right)\left\{U(h,x) + \frac{\partial U}{\partial t}(h,x)\right\} = 0, \ \ x \in \mathbb{R}. \tag{12}$$

Problem (11), (12) is the one (3), (4), with $a(\nu) = 0$, $b(\nu) = -\gamma^2\nu^2$, $A_1(\nu) = A_2(\nu) = A(\nu)$, $B_1(\nu) = B_2(\nu) = B(\nu)$. The normal fundamental system of solutions of ODE

$$\frac{d^2T}{dt^2} = \gamma^2\nu^2 T$$

has the form

$$T_0(t,\nu) = \cosh[\gamma\nu t], \quad T_1(t,\nu) = \frac{\sinh[\gamma\nu t]}{\gamma\nu}.$$

Note that $T_0(t,0) = 1,\ T_1(t,0) = t$.

Let's write the characteristic determinant $\Delta(\nu)$ of the above problem:

$$\Delta(\nu) = \begin{pmatrix} B(\nu) & B(\nu) \end{pmatrix} \begin{pmatrix} \cosh[\gamma\nu h] & \frac{\sinh[\gamma\nu h]}{\gamma\nu} \\ \gamma\nu\sinh[\gamma\nu h] & \cosh[\gamma\nu h] \end{pmatrix} \begin{pmatrix} -A(\nu) \\ A(\nu) \end{pmatrix} =$$

$$= A(\nu)B(\nu)\frac{\sinh[\gamma\nu h]}{\gamma\nu}\left[1 - \gamma^2\nu^2\right].$$

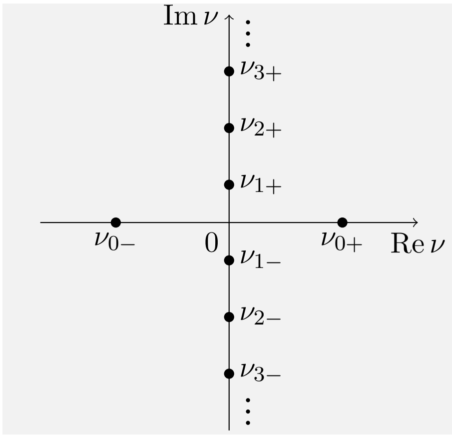
Note that the zeroes of the functions $A(\nu)$ and $B(\nu)$ are not considered, as the conditions $|A_1(\nu)|^2 + |A_2(\nu)|^2 = 2|A(\nu)|^2 \neq 0$, $|B_1(\nu)|^2 + |B_2(\nu)|^2 = 2|B(\nu)|^2 \neq 0$ do not hold.

Therefore, the set M has the form

$$M = M_1 \cup M_2,$$

$$M_1 = \left\{\nu_{k\pm}: \ \nu_{k\pm} = \pm\frac{\pi k}{\gamma h}i,\ k \in \mathbb{N}\right\},$$

$$M_2 = \left\{\nu_{0\pm}: \ \nu_{0\pm} = \pm\frac{1}{\gamma}\right\},\ i^2 = 1.$$



The numbers $\nu_{k\pm}$ in M_1 are the simple zeroes of the function $\Delta(\nu)$, so, due to Theorem 1, nontrivial solutions of problem (11), (12) are as follows:

$$U_{k\pm}(t,x) = \left\{A(\nu)\left[T_0(t,\nu) - T_1(t,\nu)\right]e^{\nu x}\right\}\Big|_{\nu=\pm i\frac{\pi k}{\gamma h}}$$

or

$$U_{k\pm}(t,x) = A\left(\pm i\frac{\pi k}{\gamma h}\right)\left(\cos\frac{\pi k t}{h} - \frac{\sin\frac{\pi k t}{h}}{\frac{\pi k}{h}}\right)e^{\pm i\frac{\pi k}{h\gamma}x}, \quad k \in \mathbb{N}.$$

Note, that for all $k \in N$ the last formulas contain two sets of solutions of the problem (11), (12): the first with the (upper) sign „+", and the second with the (lower) sign „–".

The numbers $\nu_{0\pm}$ in M_2 are also the simple zeroes of the function $\Delta(\nu)$, so, due to Theorem 1, nontrivial solutions of problem (11), (12) are as follows:

$$U_{0\pm}(t,x) = A\left(\frac{1}{\gamma}\right) e^{-t\pm\frac{1}{\gamma}x}.$$

4.2. Klein-Gordon-Fock equation

Let's consider two-point problem for equation

$$\left[\frac{\partial^2}{\partial t^2} - \gamma^2\frac{\partial^2}{\partial x^2} + m^2\right] U(t,x) = 0, \quad (t,x) \in \mathbb{R}^2, \;\; \gamma = \text{const} > 0, \; m = \text{const} > 0, \qquad (13)$$

with conditions (12).

Problem (13), (12) is the one (3), (4), with $a(\nu) = 0$, $b(\nu) = -\gamma^2\nu^2 + m^2$, $A_1(\nu) = A_2(\nu) = A(\nu)$, $B_1(\nu) = B_2(\nu) = B(\nu)$.

The normal fundamental system of solutions of ODE

$$\frac{d^2T}{dt^2} = \left(\gamma^2\nu^2 - m^2\right) T$$

has the form

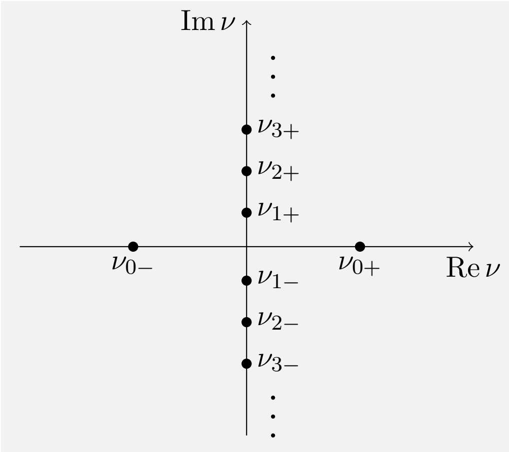
$$T_0(t,\nu) = \cosh\left[t\sqrt{\gamma^2\nu^2 - m^2}\right], \quad T_1(t,\nu) = \frac{\sinh\left[t\sqrt{\gamma^2\nu^2 - m^2}\right]}{\sqrt{\gamma^2\nu^2 - m^2}}.$$

Note that $T_0(t,0) = \cos[mt]$, $T_1(t,0) = \frac{\sin[mt]}{m}$.

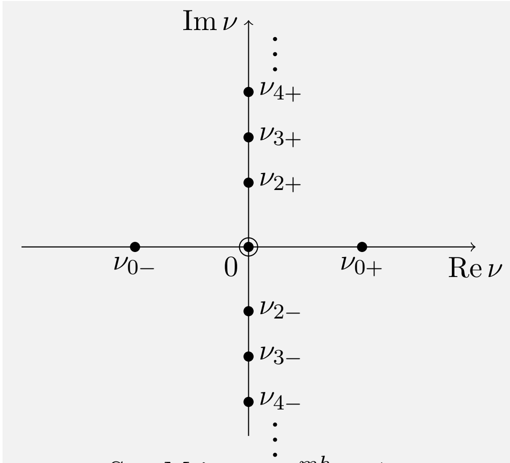
Characteristic determinant of the problem (13), (12) and the set M are as follows:

$$\Delta(\nu) = A(\nu)\, B(\nu)\, \frac{\sinh\left[h\sqrt{\gamma^2\nu^2 - m^2}\right]}{\sqrt{\gamma^2\nu^2 - m^2}} \left[1 - \gamma^2\nu^2 + m^2\right],$$

$$M = \left\{\nu_{k\pm}: \;\; \nu_{k\pm} = \pm\frac{\sqrt{m^2h^2 - \pi^2k^2}}{\gamma h}, \; k \in \mathbb{N}\right\} \bigcup \left\{\nu_{0\pm}: \;\; \nu_{0\pm} = \pm\frac{\sqrt{1+m^2}}{\gamma}\right\}.$$



Set M in case $\frac{mh}{\pi} < 1$



Set M in case $\frac{mh}{\pi} = 1$

The elements of M are symmetric with respect to the point $(0,0)$ of the complex plane. The distance between ν_{k+} and $\nu_{k+1,+}$ for $k \to \infty$ tends to $\frac{\pi}{\gamma h}$.

In the case $\frac{mh}{\pi} < 1$ the numbers $\nu_{k\pm} = \pm\frac{\sqrt{m^2h^2-\pi^2k^2}}{\gamma h}$, $k \in \mathbb{N}$, are simple zeroes for function $\Delta(\nu)$ and, by Theorem 1, we obtain the following series of quasi-polynomial solutions of problem (13), (12) (in the sequel, constant factors are ignored):

$$U_{k\pm}(t,x) = \left(\cos\frac{\pi kt}{h} - \frac{\sin\frac{\pi kt}{h}}{\frac{\pi k}{h}}\right) e^{\pm i\frac{\sqrt{\pi^2k^2-m^2h^2}}{\gamma h}x}. \tag{14}$$

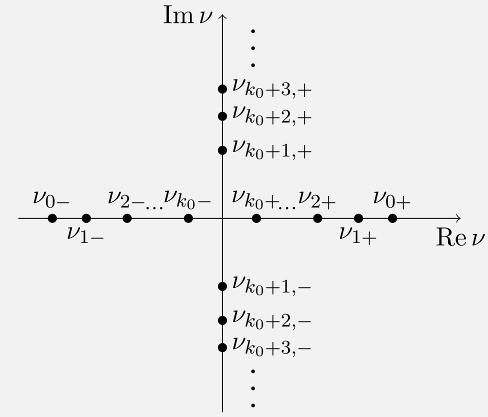
In case $\frac{mh}{\pi} = 1$, the number $\nu_{1\pm} = 0$ is a double zero of the function $\Delta(\nu)$, so we calculate

$$\begin{aligned} U_{0+}(t,x) &= \left\{A(\nu)\left[T_0(t,\nu) - T_1(t,\nu)\right]e^{\nu x}\right\}\Big|_{\nu=0} = \cos[mt] - \frac{\sin[mt]}{m}, \\ U_{0-}(t,x) &= \frac{\partial}{\partial\nu}\left\{A(\nu)\left[T_0(t,\nu) - T_1(t,\nu)\right]e^{\nu x}\right\}\Big|_{\nu=0} = x\left(\cos[mt] - \frac{\sin[mt]}{m}\right). \end{aligned} \tag{15}$$

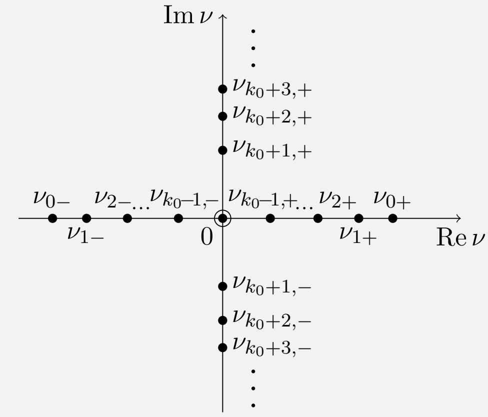
For the numbers $\nu_{0\pm} = \pm\frac{\sqrt{1+m^2}}{\gamma}$, which are simple zeroes of the function $\Delta(\nu)$, we obtain the following solutions of the problem (13), (12):

$$U_{\pm}(t,x) = e^{-t\pm\frac{\sqrt{1+m^2}}{\gamma}x}.$$

For $\frac{mh}{\pi} > 1$ we have:



Set M in case $\frac{mh}{\pi} \notin \mathbb{N}$, $[\frac{mh}{\pi}] = k_0 \in \mathbb{N}$



Set M in case $\frac{mh}{\pi} = k_0 \in \mathbb{N}\backslash\{1\}$

If $\frac{mh}{\pi} \notin \mathbb{N}$, $\frac{mh}{\pi} > 1$ and $[\frac{mh}{\pi}] = k_0 \in \mathbb{N}$, for $1 \leq k \leq k_0$ we have the following solutions of the problem (13), (12):

$$U_{k\pm}(t,x) = \left(\cos\frac{\pi k t}{h} - \frac{\sin\frac{\pi k t}{h}}{\frac{\pi k}{h}}\right) e^{\pm\frac{\sqrt{m^2h^2-\pi^2k^2}}{\gamma h}x}. \tag{16}$$

In the case $\frac{mh}{\pi} = k_0 \in \mathbb{N}\backslash\{1\}$, the solutions of the problem (13), (12) coincide with:
— solutions (14) for $k > k_0$;
— solutions (15) for $k = k_0$;
— solutions (16) for $k < k_0$.

4.3. Telegraph equation

Let's find nontrivial solutions of equation

$$\left[\frac{\partial^2}{\partial t^2} + 2m\frac{\partial}{\partial t} - \gamma^2\frac{\partial^2}{\partial x^2}\right] U(t,x) = 0, \qquad (t,x) \in \mathbb{R}^2, \quad \gamma = \text{const} > 0, \ m = \text{const} > 0, \tag{17}$$

which would satisfy the conditions

$$\alpha U(0,x) + \beta\frac{\partial U}{\partial t}(0,x) = 0, \quad \alpha U(h,x) + \beta\frac{\partial U}{\partial t}(h,x) = 0, \quad x \in \mathbb{R}, \tag{18}$$

where α, $\beta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$.

Consider the problem (17), (18) as the one (3), (4), with $a(\nu) = m$, $b(\nu) = -\gamma^2\nu^2$, $A_1(\nu) = B_1(\nu) = \alpha$, $A_2(\nu) = B_2(\nu) = \beta$. The normal fundamental system of the solutions of ODE

$$\frac{d^2T}{dt^2} + 2m\frac{dT}{dt} - \gamma^2\nu^2 T = 0$$

has the following form:

$$T_0\left(t,\nu\right)=e^{-mt}\left\{m\frac{\sinh\left[t\sqrt{\gamma^2\nu^2+m^2}\right]}{\sqrt{\gamma^2\nu^2+m^2}}+\cosh\left[t\sqrt{\gamma^2\nu^2+m^2}\right]\right\},$$

$$T_1\left(t,\nu\right)=e^{-mt}\frac{\sinh\left[t\sqrt{\gamma^2\nu^2+m^2}\right]}{\sqrt{\gamma^2\nu^2+m^2}}.$$

Let's write down the characteristic determinant of the problem (17), (18):

$$\Delta(\nu)=\begin{pmatrix}\alpha & \beta\end{pmatrix}\begin{pmatrix}T_0(h,\nu) & T_1(h,\nu)\\ T_0{}'(h,\nu) & T_1{}'(h,\nu)\end{pmatrix}\begin{pmatrix}-\beta\\ \alpha\end{pmatrix}=$$

$$=e^{-mt}\frac{\sinh\left[h\sqrt{\gamma^2\nu^2+m^2}\right]}{\sqrt{\gamma^2\nu^2+m^2}}\left(\alpha^2-2\alpha\beta m-\beta^2\nu^2\gamma^2\right).$$

Consider the set of zeroes of $\Delta(\nu)$:

$$M=\begin{cases} M_1, & \beta=0;\\ M_1\cup M_2, & \beta\neq 0,\ \alpha^2>2\alpha\beta m;\\ M_1\cup M_3, & \beta\neq 0,\ \alpha^2=2\alpha\beta m;\\ M_1\cup M_4, & \beta\neq 0,\ \alpha^2<2\alpha\beta m,\end{cases}$$

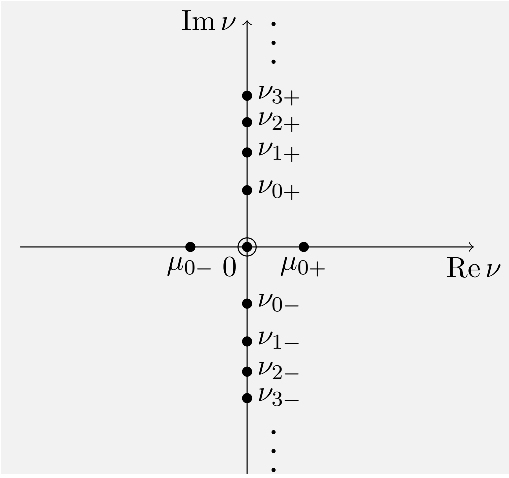
where

$$M_1=\left\{\nu_{k\pm}:\ \ \nu_{k\pm}=\pm i\frac{\sqrt{m^2h^2+\pi^2k^2}}{\gamma h},\ k\in\mathbb{N}\right\},$$

$$M_2=\left\{\mu_{0\pm}:\ \ \mu_{0\pm}=\pm\frac{\sqrt{\alpha^2-2\alpha\beta m}}{\beta\gamma}\right\},$$

$$M_3=\{0\},\quad M_4=\left\{\nu_{0\pm}:\ \ \nu_{0\pm}=\pm i\frac{\sqrt{2\alpha\beta m-\alpha^2}}{\beta\gamma}\right\}.$$

Sets M_1, M_2, M_3 and M_4 on the complex plane are as follows:



The elements of the set M are symmetric with respect to the point $(0,0)$ of the complex plane. The distance between ν_{k+} and $\nu_{k+1,+}$ from M_1 tends to $\frac{\pi}{\gamma h}$ as $k \to \infty$.

The numbers $\nu_{k\pm}$ from M_1 are simple zeroes. Then according to Theorem 1, the solution of the problem (17), (18) is as follows:

$$U_{k\pm}(t,x) = \left(\beta \cos \frac{\pi k t}{h} + (\beta m - \alpha) \frac{\sin \frac{\pi k t}{h}}{\frac{\pi k}{h}}\right) e^{-mt \pm \frac{\sqrt{\pi^2 k^2 + m^2 h^2}}{\gamma h} i x}, \quad k \in \mathbb{N}.$$

For the simple zeroes $\mu_{0\pm}$ from M_2 , we have the following solutions for the problem (17), (18):

$$U_{0\pm}(t,x) = e^{-mt \pm \frac{\sqrt{\alpha^2 - 2\alpha\beta m}}{\beta\gamma} x}.$$

In the case $\beta \neq 0$ and $\alpha^2 = 2\alpha\beta m$, the number $\nu = 0$ is a double zero of the function $\Delta(\nu)$, so we have the following solutions for the problem (17), (18):

$$U_1(t,x) = \left\{ \big(\beta T_0(t,\nu) - \alpha T_1(t,\nu)\big) e^{\nu x} \right\}\Big|_{\nu=0} = \left(\beta \cosh[mt] + (\beta m - \alpha) \frac{\sinh[mt]}{m}\right) e^{-mt},$$

$$U_2(t,x) = \frac{\partial}{\partial \nu} \left\{ \big(\beta T_0(t,\nu) - \alpha T_1(t,\nu)\big) e^{\nu x} \right\}\Big|_{\nu=0} = x \left(\beta \cosh[mt] + (\beta m - \alpha) \frac{\sinh[mt]}{m}\right) e^{-mt}.$$

For $\nu_{0\pm}$ from M_4, we have the following solutions of the problem (17), (18):

$$U_{00\pm}(t,x) = e^{-mt \pm \frac{\sqrt{2\alpha\beta m - \alpha^2}}{\beta\gamma} x}.$$

4.4. Bicalorical equation

Consider the problem of finding nontrivial solutions for the equation

$$\left[\frac{\partial}{\partial t} - a^2 \frac{\partial^2}{\partial x^2}\right]^2 U\,(t,x) = 0, \quad (t,x) \in \mathbb{R}^2, \;\; a = \mathrm{const} > 0, \tag{19}$$

with two-point conditions

$$\frac{\partial U}{\partial x}(0,x) + \frac{\partial U}{\partial t}(0,x) = 0, \quad \frac{\partial U}{\partial x}(1,x) + \frac{\partial U}{\partial t}(1,x) = 0, \;\; x \in \mathbb{R}. \tag{20}$$

Problem (19), (20) is the one (3), (4), with $a\,(\nu) = -a^2\nu^2$, $b\,(\nu) = a^4\nu^4$, $A_1(\nu) = \nu$, $A_2(\nu) = 1$, $B_1(\nu) = \nu$, $B_2(\nu) = 1$, $h = 1$.

The normal fundamental system of solutions of the equation

$$\frac{d^2T}{dt^2} - 2a^2\nu^2\frac{dT}{dt} + a^4\nu^4 T = 0,$$

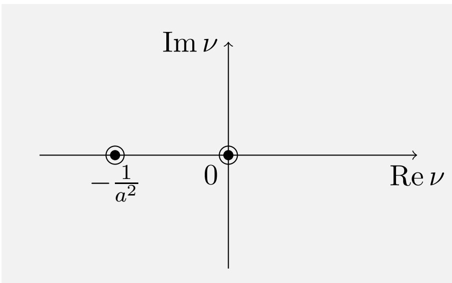
has the following form:

$$T_0\,(t,\nu) = e^{a^2\nu^2 t}(-a^2\nu^2 t + 1), \quad T_1\,(t,\nu) = t\,e^{a^2\nu^2 t}.$$

Let's write the characteristic determinant of this problem and the set M:

$$\Delta\,(\nu) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} T_0(1,\nu) & T_1(1,\nu) \\ T_0{}'(1,\nu) & T_1{}'(1,\nu) \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = e^{a^2\nu^2}\left(a^2\nu^2 + \nu\right)^2,$$

$$M = \left\{\nu \in \mathbb{C} : \;\; \left(a^2\nu^2 + \nu\right)^2 = 0\right\}.$$



The numbers $\nu = 0$ and $\nu = -\frac{1}{a^2}$ are the double zeroes of the function $\Delta\,(\nu)$, so, due to Theorem 1, nontrivial solutions of problem (19), (20) are as follows:

$$U_1(t,x) = \left\{\left[T_0\,(t,\nu) - \nu T_1\,(t,\nu)\right] e^{\nu x}\right\}\Big|_{\nu=0} = 1,$$

$$U_2(t,x) = \frac{\partial}{\partial \nu}\Big\{ \left[T_0\left(t,\nu\right) - \nu T_1\left(t,\nu\right)\right] e^{\nu x}\Big\}\Big|_{\nu=0} = x-t,$$

and

$$U_3(t,x) = \Big\{ \left[T_0\left(t,\nu\right) - \nu T_1\left(t,\nu\right)\right] e^{\nu x}\Big\}\Big|_{\nu=-\frac{1}{a^2}} = e^{\frac{t-x}{a^2}},$$

$$U_4(t,x) = \frac{\partial}{\partial \nu}\Big\{ \left[T_0\left(t,\nu\right) - \nu T_1\left(t,\nu\right)\right] e^{\nu x}\Big\}\Big|_{\nu=-\frac{1}{a^2}} = (x-t)\, e^{\frac{t-x}{a^2}}.$$

4.5. Equation of elasticity theory

Let's consider the problem of finding nontrivial solutions of the equation

$$\left[\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial x^4}\right] U(t,x) = 0, \qquad (t,x) \in \mathbb{R}^2, \tag{21}$$

which satisfy conditions (18).

Problem (21), (18) is the one (3), (4), with $a\left(\nu\right) = 0$, $b\left(\nu\right) = \nu^4$.

The normal fundamental system of solutions of the equation

$$\frac{d^2T}{dt^2} + \nu^4 T = 0$$

has the form

$$T_0\left(t,\nu\right) = \cos[\nu^2 t], \quad T_1\left(t,\nu\right) = \begin{cases} \frac{\sin[\nu^2 t]}{\nu^2}, & \nu \neq 0, \\ t, & \nu = 0. \end{cases}$$

Characteristic determinant of this problem and the set M are as follows:

$$\Delta\left(\nu\right) = \begin{pmatrix}\alpha & \beta\end{pmatrix} \begin{pmatrix} T_0(h,\nu) & T_1(h,\nu) \\ T_0{}'(h,\nu) & T_1{}'(h,\nu)\end{pmatrix} \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} = \frac{\sin[\nu^2 h]}{\nu^2}\left(\alpha^2 + \nu^4\beta^2\right).$$

Consider the set of zeroes of $\Delta(\nu)$:

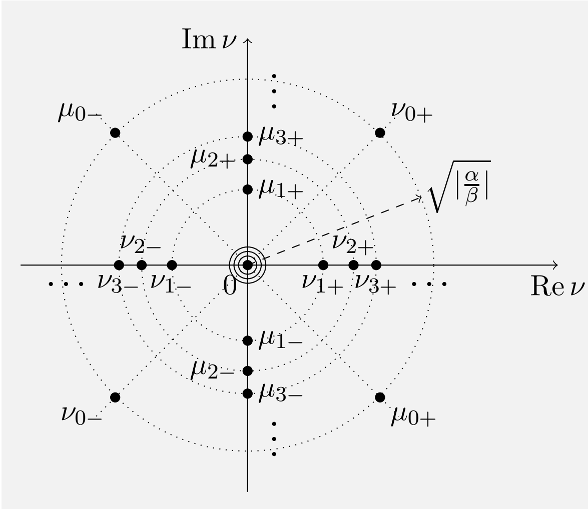
$$M = \begin{cases} M_1 \cup M_2 \cup M_3, & \alpha\beta \neq 0; \\ M_1 \cup M_2, & \alpha \neq 0,\ \beta = 0; \\ M_1 \cup M_2 \cup M_4, & \alpha = 0,\ \beta \neq 0, \end{cases}$$

where

$$M_1 = \left\{\nu_{k\pm}:\ \nu_{k\pm} = \pm\sqrt{\frac{\pi k}{h}},\ k \in \mathbb{N}\right\}, \quad M_2 = \left\{\mu_{k\pm}:\ \mu_{k\pm} = \pm i\sqrt{\frac{\pi k}{h}},\ k \in \mathbb{N}\right\},$$

$$M_3 = \left\{\nu_{0\pm}:\ \nu_{0\pm} = \sqrt{\left|\frac{\alpha}{2\beta}\right|}\left(\pm 1 \pm i\right),\right\} \bigcup \left\{\mu_{0\pm}:\ \mu_{0\pm} = \sqrt{\left|\frac{\alpha}{2\beta}\right|}\left(\pm 1 \mp i\right)\right\},\ M_4 = \{0\}.$$

Sets M_1, M_2, M_3 and M_4 on the complex plane are as follows:



The elements of set M are symmetric with respect to the point $(0,0)$ of the complex plane. The distances between ν_{k+} and $\nu_{k+1,+}$ from M_1 and μ_{k+} and $\mu_{k+1,+}$ from M_2 tend to 0 as $k\to\infty$.

The numbers $\nu_{k\pm}$ from M_1 and $\mu_{k\pm}$ from M_2 are simple zeroes of the function $\Delta(\nu)$ and, by Theorem 1, we obtain the following series of quasi-polynomial solutions of the problem (21), (18):

$$U_{k\pm}(t,x)=\left(\beta\cos\frac{\pi kt}{h}-\alpha\frac{\sin\frac{\pi kt}{h}}{\frac{\pi k}{h}}\right)e^{\pm\sqrt{\frac{\pi k}{h}}x},$$

$$U_{k\pm}(t,x)=\left(\beta\cos\frac{\pi kt}{h}-\alpha\frac{\sin\frac{\pi kt}{h}}{\frac{\pi k}{h}}\right)e^{\pm i\sqrt{\frac{\pi k}{h}}x}.$$

In the case $\alpha\beta\neq 0$, the numbers $\nu_{0\pm}$ and $\mu_{0\pm}$ from M_3 are simple zeroes of the function $\Delta(\nu)$, so we obtain the following solutions of the problem (21), (18):

$$U_{1\pm}(t,x)=e^{\sqrt{|\frac{\alpha}{2\beta}|}(\pm1\pm i)x-\frac{\alpha}{\beta}t},$$

$$U_{2\mp}(t,x)=e^{\sqrt{|\frac{\alpha}{2\beta}|}(\pm1\mp i)x-\frac{\alpha}{\beta}t}.$$

In the case $\alpha=0$ and $\beta\neq 0$, the number $\nu=0$ from M_4 is fourfold zero for the function $\triangle(\nu)$ and, by Theorem 1, we obtain the following solution of the problem (18), (15):

$$U_{01}(t,x)=\left\{\beta\,T_0(t,\nu)\,e^{\nu x}\right\}\Big|_{\nu=0}=1,$$

$$U_{02}\,(t,x) = \frac{\partial}{\partial \nu}\Big\{\beta\, T_0\,(t,\nu)\, e^{\nu x}\Big\}\Big|_{\nu=0} = x,$$

$$U_{03}\,(t,x) = \frac{\partial^2}{\partial \nu^2}\Big\{\beta\, T_0\,(t,\nu)\, e^{\nu x}\Big\}\Big|_{\nu=0} = x^2,$$

$$U_{04}\,(t,x) = \frac{\partial^3}{\partial \nu^3}\Big\{\beta\, T_0\,(t,\nu)\, e^{\nu x}\Big\}\Big|_{\nu=0} = x^3.$$

5. Conclusions

We investigated the homogeneous problem for PDE in two variables which are of second order in time variable and of finite order in another (spatial) variable with two-point conditions in time. We considered the case where the set of zeros of the characteristic determinant of the problem is not empty and does not coincide with $\mathbb{C}$ as well. We proved the existence of nontrivial solutions of the problem and constructed them. The results of the research were used to construct non-zero quasi-polynomial solutions of homogeneous two-point problems for some equations of mathematical physics.

References

[1] V.M. Borok, *Uniqueness classes for the solution of a boundary problem in an infinite layer.*, Dokl. Akad. Nauk SSSR, **183(5)**, 1968, 995–998.

[2] B.Yo. Ptashnyk, *Ill-posed boundary value problems for partial differential equations*, Nauk. Dumka, 1984.

[3] B.Yo. Ptashnyk, V.S. Il'kiv, I. Ya. Kmit, V. M. Polishchuk, *Nonlocal boundary value problems for partial differential equations*, Nauk. Dumka, 2002.

[4] P.I. Kalenyuk, Z.M. Nytrebych, *Generalized scheme of separation of variables. Differential-symbol method*, Publishing House of Lviv Polytechnic National University, 2002.

[5] P.I. Kalenyuk, Ya.Ye. Baranetskyi, Z. N. Nytrebych, *Generalized method of separation of variables*, Nauk. Dumka, 1983.

[6] V.S. Il'kiv, Z.M. Nytrebych, *On solutions of the homogeneous Dirihlet problem in the time strip for the partial differential equation of the second order with respect to time variable*, Visn. Lviv. Univ., Ser. mekh-mat., **78**, 2013, 65–77.

[7] P.I. Kalenyuk, I.V. Kogut, Z.M. Nytrebych, *Differential-symbol method of solving the problem with nonlocal boundary condition for partial differential equation*, Mat. Met. Fiz.-Meh. Polya, **45(2)**, 2002, 7–15.

[8] Z.M. Nytrebych, *A boundary-value problem in an unbounded strip*, J. Math. Sci., **79(6)**, 1996, 1388–1392.

[9] P.Ya. Pukach, *Qualitative research methods of mathematical model of nonlinear vibrations of conveyor*, J. Math. Sci., **198(1)**, 2014, 31–36.

[10] A.N. Tihonov, A.B. Vasil'eva, A.G. Sveshnikov, *Differential equations*, Nauka, 1980.

[11] Z.M. Nytrebych, O.M. Malanchuk, *The homogeneous problem with boundary conditions on the border of the strip for partial differential equation of second order in time*, Nauk. Visn. Uzhorod. Univ, **2(27)**, 2015, 98–108.

Zinovij M. Nytrebych
Lviv Polytechnic National University, Lviv, Ukraine
E-mail: znytrebych@gmail.com

Oksana M. Malanchuk
Lviv Polytechnic National University,
Danylo Halytsky Lviv National Medical University, Lviv, Ukraine
E-mail: Oksana.Malan@gmail.com

Volodymyr S. Il'kiv
Lviv Polytechnic National University, Lviv, Ukraine
E-mail: ilkivv@i.ua

Petro Ya. Pukach
Lviv Polytechnic National University, Lviv, Ukraine
E-mail: ppukach@gmail.com

Received 08 June 2016
Accepted 10 March 2017