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On Approximate Solution of Impedance Boundary Value Problem for Helmholtz Equation

A.R. Aliev*, R.J. Heydarov

Abstract. A sequence that converges to the exact solution of the impedance boundary value problem for the Helmholtz equation is built in this work and the error estimate is obtained.

Key Words and Phrases: collocation method, Helmholtz equation, impedance boundary value problem, cubature formula, surface integral.

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1. Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain with a twice continuously differentiable boundary S. Consider the impedance boundary value problem for the Helmholtz equation: find a function u which is twice continuously differentiable in $\mathbb{R}^3 \backslash \bar{D}$ and continuous on S, has a normal derivative in the sense of uniform convergence, satisfies the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbb{R}^3 \backslash \bar{D}$, satisfies the Sommerfeld radiation condition

$$\left(\frac{x}{|x|}, grad\, u\,(x) \right) - i\,k\,u\,(x) = o\,\left(\frac{1}{|x|} \right), \quad |x| \to \infty,$$

uniformly in all directions $x/\,|x|$, and satisfies the boundary condition

$$\frac{\partial u\,(x)}{\partial \vec{n}\,(x)} + \lambda\,(x)\;u\,(x) = g\;(x) \quad \text{on}\;\; S, \tag{1}$$

where Δ is a Laplace operator, k is a wave number with $Im\,k \geq 0$, $\vec{n}\,(x)$ is a unit outer normal at the point $x \in S$, and λ and g are the given continuous functions on S with $Im\,\left(\bar{k}\,\lambda\,(x)\right) \geq 0$, $x \in S$. In particular, if $\lambda\,(x) = 0$, $\forall x \in S$, then we have an external Neumann boundary value problem, and if $\lambda\,(x) = const$, $\forall x \in S$, then we have a mixed problem for the Helmholtz equation.

*Corresponding author.

It is known that one of the methods for solving impedance boundary value problem for Helmholtz equation is reducing it to the boundary integral equation (BIE). A number of works (see [1, 2, 3, 4, 5, 6, 7]) have been dedicated to the approximate solution of BIE of external Dirichlet and Neumann boundary value problems, and to the one of mixed problem for the Helmholtz equation. But so far there has been no research of approximate solution for the impedance boundary value problem for the Helmholtz equation. The presented work is just dedicated to this matter.

2. Justification of collocation method for BIE of impedance boundary value problem for Helmholtz equation

Let $\Phi_k(x,y) = e^{ik|x-y|}/\left(4\pi\,|x-y|\right)$, $x, y \in \mathbb{R}^3,\ x \neq y$. It is proved in [8] that the simple layer potential

$$u(x) = \int_S \Phi_k(x,y)\,\varphi(y)\,dS_y, x \in \mathbb{R}^3 \backslash \bar{D},$$

is a solution of the boundary value problem for the Helmholtz equation with the impedance condition (1) if the density $\varphi \in C(S)$ is a solution of BIE

$$\varphi + B\,\varphi = -2g, \tag{2}$$

where $C(S)$ is a space of continuous functions on S with the norm $\|\varphi\|_\infty = \max\limits_{x\in S} |\varphi(x)|$ and $B = -\tilde{K} - \lambda F$ is a linear compact operator in $C(S)$ with

$$(\tilde{K}\varphi)(x) = 2\int_S \frac{\partial\Phi_k(x,y)}{\partial\vec{n}(x)}\,\varphi(y)\,dS_y, (F\varphi)(x) = 2\int_S \Phi_k(x,y)\,\varphi(y)\,dS_y, x \in S.$$

Divide S into elementary domains $S = \bigcup_{l=1}^{N} S_l^N$:

(1) for every $l = \overline{1, N}$ the domain S_l^N is closed and the set of its internal points $\overset{0}{S_l^N}$ with respect to S is non-empty, with $mes\,\overset{0}{S_l^N} = mes S_l^N$ and $\overset{0}{S_l^N} \bigcap \overset{0}{S_j^N} = \varnothing$ for $j \in \{1, 2, ...N\}$, $j \neq l$;

(2) for every $l = \overline{1, N}$ the domain S_l^N is a connected piece of the surface S with a continuous boundary;

(3) for every $l = \overline{1, N}$ there exists a so-called control point $x_l \in S_l^N$ such that:

(3.1) $r_l(N) \sim R_l(N)$ ($r_l(N) \sim R_l(N) \;\Leftrightarrow\; C_1 \leq r_l(N)/R_l(N) \leq C_2$, C_1 and C_2 are positive constants independent of N), where $r_l(N) = \min\limits_{x\in\partial S_l^N} |x - x_l|$ and $R_l(N) = \max\limits_{x\in\partial S_l^N} |x - x_l|$;

(3.2.) $R_l(N) \leq d/2$, where d is the radius of a standard sphere (see [9]);

(3.3) for every $j = \overline{1\,,\,N}$ $r_j(N) \sim r_l(N)$.

It is clear that $r(N) \sim R(N)$ and $\lim\limits_{N\to\infty} R(N) = 0$, where $R(N) = \max\limits_{l=\overline{1,\,N}} R_l(N)$, $r(N) = \min\limits_{l=\overline{1,\,N}} r_l(N)$.

Such a partition, as well as the partition of the unit sphere into elementary parts, has been carried out earlier in [10].

Let $S_d(x)$ and $\Gamma_d(x)$ be the parts of the surface S and the tangential plane $\Gamma(x)$, respectively, at the point $x \in S$, contained inside the sphere $B_d(x)$ of radius d centered at the point x. Besides, let $\tilde{y} \in \Gamma(x)$ be the projection of the point $y \in S$. Then

$$|x - \tilde{y}| \leq |x - y| \leq c_1(S)\ |x - \tilde{y}| \text{ and } mesS_d(x) \leq c_2(S)\, mes\Gamma_d(x), \tag{3}$$

where $c_1(S)$ and $c_2(S)$ are positive constants depending only on S (if S is a sphere, then $c_1(S) = \sqrt{2}$ and $c_2(S) = 2$).

The following lemma is true.

Lemma 1. *(see [10]).There exist the constants $C_0' > 0$ and $C_1' > 0$, independent of N, such that for $\forall\ l, j \in \{1,\,2, ...,\,N\}$, $j \neq l$, and $\forall\ y \in S_j^N$ the inequality $C_0'\ |y - x_l| \leq |x_j - x_l| \leq C_1'\ |y - x_l|$ holds.*

For a function $\varphi\ \in C\,(S)$, we introduce the modulus of continuity of the following form:

$$\omega(\varphi,\,\tau) = \max_{\substack{|x-y|\leq\delta \\ x,\,y\in S}} |\varphi\,(x) - \varphi\,(y)|\,,\ \delta > 0.$$

Let

$b_{l\,j} = 2\ \ |sgn\,(l - j)\,|\ \ \left(-\frac{\partial\Phi_k(x_l,x_j)}{\partial\vec{n}\,(x_l)} - \lambda\ (x_l)\ \Phi_k\,(x_l, x_j)\right)\ mes\, S_j^N$ for $l\,,j = \overline{1\,,\,N}$.

Theorem 1. *Let $\varphi \in C\,(S)$. Then the expression*

$$\left(B^N \varphi\right)\ (x_l) = \sum_{j=1}^{N} b_{l\,j}\,\varphi\,(x_j) \tag{4}$$

is a cubature formula for $(B\varphi)(x)$ at the points x_l, $l = \overline{1\,,\,N}$, with

$$\max_{l=\overline{1,N}} \left|(B\varphi)\ (x_l) - \left(B^N\varphi\right)\ (x_l)\right| \leq$$

$$M^{\dagger}\,[\,\|\,\varphi\,\|_{\infty}\,R\,(N)\ |\ln R\,(N)\,| + \omega\ (\varphi, R\,(N))\,]\ . \tag{5}$$

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†Here and after, M denotes positive constants which can be different in different inequalities.

Proof. It is proved in [11] that the expressions

$$\left(F^{N}\varphi\right)(x_l) = 2\sum_{\substack{j=1\\ j\neq l}}^{N} \Phi_k(x_l, x_j)\,\varphi(x_j)\; mesS_j^N$$

and

$$\left(\tilde{K}^{N}\varphi\right)(x_l) = 2\sum_{\substack{j=1\\ j\neq l}}^{N} \frac{\partial\Phi_k(x_l, x_j)}{\partial\vec{n}(x_l)}\,\varphi(x_j)\; mesS_j^N$$

are cubature formulas for the integrals $(F\varphi)(x)$ and $\left(\tilde{K}\varphi\right)(x)$, respectively, at the points $x_l, \quad l = \overline{1, N}$, with

$$\max_{l=\overline{1,N}} \left| (F\varphi)(x_l) - \left(F^{N}\varphi\right)(x_l) \right| \le M\,\left(\|\varphi\|_\infty R(N)\,|\ln R(N)\,| + \omega\,(\varphi, R(N)) \right),$$

$$\max_{l=\overline{1,N}} \left| \left(\tilde{K}\varphi\right)(x_l) - \left(\tilde{K}^{N}\varphi\right)(x_l) \right| \le M\,\left(\|\varphi\|_\infty R(N)\,|\ln R(N)\,| + \omega\,(\varphi, R(N)) \right).$$

Consequently, the expression $\left(B^N\varphi\right)(x_l) = \sum_{j=1}^{N} b_{l\,j}\,\varphi(x_j)$ is a cubature formula for the integral $(B\varphi)(x)$ at the points $x_l, \quad l = \overline{1, N}$. Besides, taking into account the error estimates for the cubature formulas for the integrals $(F\varphi)(x)$ and $\left(\tilde{K}\varphi\right)(x)$, we get the validity of the estimate (5). ◄

Let

$$B_l^N z^N = \sum_{j=1}^{N} b_{l\,j}\, z_j^N\,,\;\; l = \overline{1,N}\,,\;\; B^N z^N = \left(B_1^N z^N,\; B_2^N z^N\,, \ldots, B_N^N z^N\right)\,,$$

for $z^N \in \mathbb{C}^N$, where $\mathbb{C}^N$ is a space of N-dimensional vectors $z^N = \left(z_1^N, z_2^N, \ldots, z_N^N\right)$, $z_l^N \in \mathbb{C}$, $l = \overline{1,N}$, with the norm $\left\|z^N\right\| = \max_{l=\overline{1,N}} \left|z_l^N\right|$. Using cubature formula (4), we replace BIE (2) by the system of algebraic equations with respect to z_l^N, approximate values of $\varphi(x_l)$, $l = \overline{1,N}$, stated as follows:

$$z^N + B^N z^N = -2\,g^N, \tag{6}$$

where $g^N = p^N g = (g_1, g_2, \ldots, g_N)$, $g_l = g(x_l), l = \overline{1,N}$, p^N is a simple restriction operator acting boundedly from $C(S)$ to $\mathbb{C}^N$, and $B^N : \mathbb{C}^N \to \mathbb{C}^N$ is a linear bounded operator.

We will obtain justification for collocation method from Vainikko's convergence theorem for linear operator equations (see [12]). To formulate that theorem, we need some definitions and a theorem from [12].

Definition 1. *([12]) A system* $Q = \{ q^N \}$ *of operators* $q^N : C(S) \to \mathbb{C}^N$ *is called a connecting system for* $C(S)$ *and* $\mathbb{C}^N$ *if* $\| q^N \varphi \| \to \| \varphi \|_\infty$ *as* $N \to \infty$, $\forall \varphi \in C(S)$;
$\| q^N (a\varphi + a'\varphi') - (a\, q^N \varphi + a' q^N \varphi') \| \to 0$ *as* $N \to \infty$, $\forall \varphi, \varphi' \in C(S)$, $a, a' \in \mathbb{C}$.

Definition 2. *([12]) A sequence* $\{ \varphi_N \}$ *of elements* $\varphi_N \in \mathbb{C}^N$ *is called* $Q-$*convergent to* $\varphi \in C(S)$ *if* $\| \varphi_N - q^N \varphi \| \to 0$ *as* $N \to \infty$. *We denote this fact by* $\varphi_N \overset{Q}{\to} \varphi$.

Definition 3. *([12]) A sequence* $\{ \varphi_N \}$ *of elements* $\varphi_N \in \mathbb{C}^N$ *is called* $Q-$*compact if every subsequence of it* $\{ \varphi_{N_m} \}$ *contains a* $Q-$*convergent subsequence* $\left\{ \varphi_{N_{m_k}} \right\}$.

Proposition 1. *([12]) Let* $q^N : C(S) \to \mathbb{C}^N$ *be linear and bounded. Then the following conditions are equivalent:*

1. *the sequence* $\{ \varphi_N \}$ *is* $Q-$*compact and the set of its* $Q-$*limit points is compact in* $C(S)$;
2. *there exists a relatively compact sequence* $\left\{ \varphi^{(N)} \right\} \subset C(S)$ *such that*

$$\left\| \varphi_N - q^N \varphi^{(N)} \right\| \to 0 \quad \text{as } N \to \infty.$$

Definition 4. *([12]) A sequence of operators* $B^N : \mathbb{C}^N \to \mathbb{C}^N$ *is called* $QQ-$*convergent to the operator* $B : C(S) \to C(S)$ *if for every* $Q-$*convergent sequence* $\{ \varphi_N \}$ *the relation* $\varphi_N \overset{Q}{\to} \varphi \Rightarrow B^N \varphi_N \overset{Q}{\to} B\varphi$ *holds. We denote this fact by* $B^N \overset{QQ}{\to} B$.

Definition 5. *([12]) A sequence of operators* B^N *acting boundedly in* $\mathbb{C}^N$ *is called compactly convergent to the operator* B*, which is bounded in* $C(S)$*, if* $B^N \overset{QQ}{\to} B$ *and the following compactness condition holds:* $\varphi_N \in \mathbb{C}^N$, $\| \varphi_N \| \le M \Rightarrow \{ B^N \varphi_N \}$ *is Q-compact.*

Theorem 2. *([12]) Let the following conditions hold:*

1. $Ker\ (I + B) = \{ 0 \}$*, where* I *is a unit operator in the space* $C(S)$;
2. $I^N + B^N \quad (N \ge N_0)$ *are Fredholm operators of index zero, where* I^N *is a unit operator in the space* $\mathbb{C}^N$;
3. $\psi_N \overset{Q}{\to} \psi$, $\psi_N \in \mathbb{C}^N$, $\psi \in C(S)$;
4. $B^N \to B$ *compactly.*

Then the equation $(I + B)\, \varphi = \psi$ *has a unique solution* $\tilde{\varphi} \in C(S)$*, the equation* $\left(I^N + B^N \right) \varphi_N = \psi_N \ (N \ge N_0)$ *has a unique solution* $\tilde{\varphi}_N \in \mathbb{C}^N$*, and* $\tilde{\varphi}_N \overset{Q}{\to} \tilde{\varphi}$ *with an estimate*

$$c_1 \left\| \left(I^N + B^N\right) q^N \tilde{\varphi} - \psi_N \right\| \le \left\| \tilde{\varphi}_N - q^N \tilde{\varphi} \right\| \le c_2 \left\| \left(I^N + B^N\right) q^N \tilde{\varphi} - \psi_N \right\| ,$$

where

$$c_1 = 1/ \sup_{N \ge N_0} \left\| I^N + B^N \right\| > 0\,, \quad c_2 = \sup_{N \ge N_0} \left\| \left(I^N + B^N\right)^{-1} \right\| < +\infty.$$

Theorem 3. *Let* $Im\, k > 0$*. Then the equations* (2) *and* (6) *have unique solutions* $\varphi_* \in C(S)$ *and* $z_*^N \in \mathbb{C}^N$ $(N \ge N_0)$*, respectively, and* $\left\| z_*^N - p^N \varphi_* \right\| \to 0$ *as* $N \to \infty$ *with an estimate*

$$\left\| z_*^N - p^N \varphi_* \right\| \le M \left[\left\| g \right\|_\infty (R(N))^\alpha + \omega\left(\lambda, R(N)\right) + \omega\left(g, R(N)\right) \right],$$

where $\alpha \in (0\,,\,1)$.

Proof. It is proved in [8] that if $Im\, k > 0$, then $Ker\ (I + B) = \{0\}$. It is clear that the system of simple restriction operators $P = \left\{ p^N \right\}$ is a connecting system for the spaces $C(S)$ and $\mathbb{C}^N$, and the operators $I^N + B^N$ are Fredholm operators of index zero. Then $g^N \stackrel{P}{\to} g$, and from Theorem 1 we obtain $I^N + B^N \stackrel{PP}{\to} I + B$. By Definition 5, it remains only to verify the compactness condition, which in view of Proposition 1 is equivalent to the following one: $\forall \left\{ z^N \right\}$, $z^N \in \mathbb{C}^N$, $\left\| z^N \right\| \le M$ there exists a relatively compact sequence $\left\{ B_N z^N \right\} \subset C(S)$ such that $\left\| B^N z^N - p^N \left(B_N z^N \right) \right\| \to 0$ as $N \to \infty$.

As $\left\{ B_N z^N \right\}$, we choose the sequence

$$\left(B_N z^N\right)(x) = -\left(\tilde{K}_N z^N\right)(x) - \lambda(x) \left(F_N z^N\right)(x),$$

where

$$\left(\tilde{K}_N z^N\right)(x) = 2 \sum_{j=1}^{N} z_j^N \int_{S_j^N} \frac{\partial \Phi_k(x,y)}{\partial \vec{n}(x)} dS_y,$$

$$\left(F_N z^N\right)(x) = 2 \sum_{j=1}^{N} z_j^N \int_{S_j^N} \Phi_k(x,y)\, dS_y.$$

Take arbitrary points $x',\ x'' \in S$ such that $|x' - x''| = \delta < d/2$. Then

$$\left| \left(\tilde{K}_N z^N\right)(x') - \left(\tilde{K}_N z^N\right)(x'') \right| \le M \left\| z^N \right\| \int_S \left| \frac{\partial \Phi_k(x',y)}{\partial \vec{n}(x')} - \frac{\partial \Phi_k(x'',y)}{\partial \vec{n}(x'')} \right| dS_y \le$$

$$M \left\|z^N\right\| \int_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k(x', y)}{\partial \vec{n}(x')} \right| dS_y + M \left\|z^N\right\| \int_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k(x'', y)}{\partial \vec{n}(x'')} \right| dS_y +$$

$$M \left\|z^N\right\| \int_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k(x'', y)}{\partial \vec{n}(x'')} \right| dS_y +$$

$$M \left\|z^N\right\| \int_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k(x', y)}{\partial \vec{n}(x')} \right| dS_y +$$

$$M \left\|z^N\right\| \int_{S \backslash \left(S_{\delta/2}(x') \bigcup S_{\delta/2}(x'')\right)} \left| \frac{\partial \Phi_k(x', y)}{\partial \vec{n}(x')} - \frac{\partial \Phi_k(x'', y)}{\partial \vec{n}(x'')} \right| dS_y.$$

Using the inequality

$$\left| \frac{\partial \Phi_k(x, y)}{\partial \vec{n}(x)} \right| \leq \frac{M}{|x-y|}, \forall\, x\,, y \in S,\ x \neq y,$$

and the formula for reducing a surface integral to a double integral, we obtain:

$$\int_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k(x', y)}{\partial \vec{n}(x')} \right| dS_y \leq M \int_{S_{\delta/2}(x')} \frac{1}{|x'-y|} dS_y \leq M\,\delta,$$

$$\int_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k(x'', y)}{\partial \vec{n}(x'')} \right| dS_y \leq M\,\delta.$$

Besides, taking into account the inequalities $|x''-y| \geq \delta/2\,,\ \forall y \in S_{\delta/2}(x')$ and $|x_1-y| \geq \delta/2\,,\ \ \forall y \in S_{\delta/2}(x_2)$, we have

$$\int_{S_{\delta/2}(x')} \left| \frac{\partial \Phi_k(x'', y)}{\partial \vec{n}(x'')} \right| dS_y \leq M \int_{S_{\delta/2}(x')} \frac{1}{|x''-y|} dS_y \leq \frac{2M}{\delta} mes\left(S_{\delta/2}(x')\right) \leq M\,\delta,$$

$$\int_{S_{\delta/2}(x'')} \left| \frac{\partial \Phi_k(x', y)}{\partial \vec{n}(x')} \right| dS_y \leq M\,\delta.$$

It is not difficult to show that

$$\left| \frac{\partial \Phi_k(x', y)}{\partial \vec{n}(x')} - \frac{\partial \Phi_k(x'', y)}{\partial \vec{n}(x'')} \right| \leq \frac{M\,\delta}{|x'-y|^2}\,,\ \forall y \in S \backslash \left(S_{\delta/2}(x') \bigcup S_{\delta/2}(x'')\right).$$

Hence

$$\int_{S\setminus\left(S_{\delta/2}(x')\bigcup S_{\delta/2}(x'')\right)}\left|\frac{\partial\Phi_k\left(x',y\right)}{\partial\vec{n}\left(x'\right)}-\frac{\partial\Phi_k\left(x'',y\right)}{\partial\vec{n}\left(x''\right)}\right|\,dS_y\le M\,\delta\,\left|\,\ln\delta\right|.$$

As a result, we obtain

$$\left|\left(\tilde{K}_N\,z^N\right)\,\left(x'\right)-\left(\tilde{K}_N\,z^N\right)\,\left(x''\right)\right|\le M\,\left\|z^N\right\|\,\delta\,\left|\,\ln\delta\right|.$$

We can similarly prove this estimate for the sequence $\left\{F_N\,z^N\right\}$. Consequently

$$\left|\,\left(B_N\,z^N\right)\,\left(x'\right)-\left(B_N\,z^N\right)\,\left(x''\right)\,\right|\le M\,\left\|z^N\right\|\,\delta\,\left|\,\ln\delta\right|, \tag{7}$$

and hence, $\left\{B_N\,z^N\right\}\subset C\left(S\right)$.

The relative compactness of the sequence $\left\{\,B_N\,z^N\right\}$ follows from the Arzela theorem. In fact, the uniform boundedness follows directly from the condition $\left\|z^N\right\|\le M$, and the equicontinuity follows from the estimate (7). Besides, taking into account the way the surface has been divided into elementary parts and using Lemma 1, it is not difficult to prove that $\left\|B^N\,z^N-p^N\left(B_N\,z^N\right)\,\right\|\to 0$ as $N\to\infty$. Then, applying Theorem 2, we obtain that the equations (2) and (6) have unique solutions $\varphi_*\in C\left(S\right)$ and $z_*^N\in\mathbb{C}^N$ $\left(N\ge N_0\right)$, respectively, with

$$c_1\,\delta_N\le\left\|\,z_*^N-p^N\varphi_*\right\|\le\,c_2\,\delta_N,$$

where

$$c_1\,=1/\sup_{N\ge N_0}\,\left\|\,I^N+B^N\right\|>0\,,c_2\,=\,\sup_{N\ge N_0}\,\left\|\,\left(I^N+B^N\right)^{-1}\right\|<+\infty,$$

$$\delta_N=\max_{l=\overline{1,N}}\,\left|B_l^N\,\left(p^N\varphi_*\right)-\left(B\,\varphi_*\right)\,\left(x_l\right)\right|.$$

From Theorem 1 we obtain

$$\delta_N\le M\,\left[\,\left\|\varphi_*\right\|_\infty\,\left(R\left(N\right)\right)^\alpha+\omega\,\left(\varphi_*,R\left(N\right)\right)\right]$$

for $\forall\alpha\in\left(0\,,\,1\right)$. As $\varphi_*=-2\,\left(I+B\right)^{-1}g$, we have

$$\left\|\varphi_*\right\|_\infty\le 2\,\left\|\left(I+B\right)^{-1}\right\|\,\left\|g\right\|_\infty. \tag{8}$$

Besides, in view of

$$\omega\,\left(B\varphi_*,R\left(N\right)\right)\le M\,\left\|\varphi_*\right\|_\infty\left(\left(R\left(N\right)\right)^\alpha+\omega\left(\lambda,R\left(N\right)\right)\right),$$

we have

$$\omega\ (\varphi_*\,,\ R\,(N)) = \omega\ (-2\,g - B\varphi_*, R\,(N)) \leq 2\,\omega\ (g, R\,(N)) + \omega\ (B\varphi_*, R\,(N)) \leq$$

$$M\ [\omega\ (g, R\,(N)) + \omega\,(\lambda, R\,(N)) + \|\,g\,\|_\infty\ (R\,(N))^\alpha]\ ,\ \forall\,\alpha \in (0\,,\,1)\,. \qquad (9)$$

As a result, from the obtained estimates we find that

$$\delta_N \leq M\ [\,\|\,g\,\|_\infty\,(R\,(N))^\alpha + \omega\,(\lambda, R\,(N)) + \omega\ (g, R\,(N))\,]\,, \forall\,\alpha \in (0\,,\,1)\,.$$

◄

3. Main result

Now let's state the main result of this work.

Theorem 4. *Let $Imk > 0$ and $z_*^N = (z_1^*, z_2^*, \ldots, z_N^*)^{\mathrm{T}}$ be a solution of the system of algebraic equations* (6)*. Then the sequence*

$$u^N(x_0) = \sum_{j=1}^{N} \Phi_k\,(x_0, x_j)\ z_j^*\,mesS_j, x_0 \in \mathbb{R}^3 \backslash \bar{D},$$

converges to the value of the solution $u\,(x)$ of the boundary value problem for the Helmholtz equation with impedance condition (1) *at the point x_0, with*

$$\left|\ u^N\,(x_0) - u\,(x_0)\right| \leq M\ \ [\ \|g\|_\infty\ R\,(N)\ \ |\ln R\,(N)\,| + \omega\,(\lambda, R\,(N)) + \omega\ (g, R\,(N))\,]\,.$$

Proof. Let the function $\varphi_* \in C\,(S)$ be a solution of the equation (2). Then the function

$$u\,(x) = \int_S \Phi_k\,(x, y)\ \varphi_*\,(y)\ dS_y, x \in \mathbb{R}^3 \backslash \bar{D},$$

is a solution of the boundary value problem for the Helmholtz equation with impedance condition (1). Obviously

$$u\,(x_0) - u^N\,(x_0) = \sum_{j=1}^{N} \int_{S_j} \Phi_k\,(x_0, y)\ (\varphi_*\,(y) - \varphi_*\,(x_j))\ dS_y +$$

$$\sum_{j=1}^{N} \int_{S_j} \Phi_k\,(x_0, y)\ \left(\varphi_*\,(x_j) - z_j^*\right)\ dS_y + \sum_{j=1}^{N} \int_{S_j} \left(\Phi_k\,(x_0, y)\ - \Phi_k\,(x_0, x_j)\right)\ \varphi_*\,(y)\ dS_y +$$

$$\sum_{j=1}^{N}\int_{S_j}\left(\Phi_k(x_0,x_j)-\Phi_k(x_0,y)\right)\left(\varphi_*(y)-\varphi_*(x_j)\right)dS_y+$$

$$\sum_{j=1}^{N}\int_{S_j}\left(\Phi_k(x_0,x_j)-\Phi_k(x_0,y)\right)\left(\varphi_*(x_j)-z_j^*\right)dS_y.$$

As $x_0\notin S$, it is clear that the function $\psi(y)=\Phi_k(x_0,y)$ is continuously differentiable on the surface S, and, consequently,

$$\max_{j=\overline{1,N}}\left|\psi(y)-\psi(x_j)\right|\leq\|grad\,\psi\|_\infty\,R(N),\forall\,y\in S.$$

By Theorem 3, we find

$$\left|u(x_0)-u^N(x_0)\right|\leq M\,mesS\,\left(\|\psi\|_\infty\left(\omega(\varphi_*,R(N))+\|g\|_\infty(R(N))^\alpha+\right.\right.$$

$$\omega(\lambda,R(N))+\omega(g,R(N)))+$$

$$\|grad\,\psi\|_\infty R(N)\,\left(\|\varphi_*\|_\infty+\omega\,(\varphi_*,R(N))+\|g\|_\infty\,(R(N))^\alpha+\right.$$

$$\left.\left.\omega\,(g,R(N))\right)\right),\alpha\in(0\,,\,1)\,.$$

As a result, in view of the inequalities (8) and (9), we obtain

$$\left|u(x_0)-u^N(x_0)\right|\leq M\left(\|\psi\|_\infty+\|grad\,\psi\|_\infty\right)\left(\|g\|_\infty\,(R(N))^\alpha+\right.$$

$$\left.\omega\,(\lambda,R(N))+\omega\,(g,R(N))\right),\alpha\in(0\,,\,1)\,.$$

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Araz R. Aliev
Azerbaijan State Oil and Industry University
20, Azadlig ave., AZ1010, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B. Vahabzadeh str., AZ1141, Baku, Azerbaijan
E-mail: alievaraz@yahoo.com

Rahib J. Heydarov
Ganja State University
429, H. Aliyev ave., AZ2000, Ganja, Azerbaijan
E-mail: heyderov.rahib@gmail.com

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