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Systems of Powers of Conformal Mappings and Conjugate Systems of Functions

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Abstract. Applying the powers of conformal mappings from simply connected domains onto a disk, we construct biorthogonal systems of functions. Conditions for the expansion of analytic functions in given domains into series whose terms are the powers of these mappings are investigated.
Examples of the biorthogonal systems whose elements are the compositions of fractional rational and exponential functions are considered. We also construct solutions to the boundary value problems for the Helmholtz equation in the case where the boundary functions are defined by series in terms of biorthogonal systems of functions.

Key Words and Phrases: biorthogonal systems of functions, conformal mappings, the Helmholtz equation.

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1. Introduction

The systems of analytic functions that are biorthogonal on closed curves in simply connected domains of the complex plane form bases for the spaces of functions analytic in these domains.

The expansion of analytic functions into series whose terms are polynomials (e.g., Faber polynomials, Bernoulli polynomials, Euler polynomials) is investigated in [1, 2, 3, 4, 5, 6] by applying contour integration and conformal transformations. In [8, 9], biorthogonal systems of functions are used for constructing solutions to the boundary value problems for the Helmholtz equation on the plane and in the space.

In this paper we construct biorthogonal systems of functions by applying conformal mappings of simply connected domains onto a disk. Conditions for the expansion of analytic functions in given domains into series whose terms are the above biorthogonal

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systems are investigated. We also find solutions to the boundary value problems for the Helmholtz equation in the form of series whose terms are constructed using biorthogonal systems of functions.

2. Biorthogonal systems of functions

Let

$$w = \varphi(z) \tag{1}$$

be a conformal mapping of a simply connected domain D (assume that 0 belongs to D) in the extended complex plane onto the disk $K := \{w : |w| < 1\}$ in the complex plane such that $\varphi(0) = 0$, $\lim\limits_{z\to 0} \dfrac{\varphi(z)}{z} = 1$, and let $z = h(w)$ be an inverse mapping.

The system of functions $\{w^n\}_{n=0}^{\infty}$ and the conjugate system $\{1/w^{m+1}\}_{m=0}^{\infty}$ are biorthogonal on a closed contour; that is

$$\frac{1}{2\pi i} \int\limits_{C_r} \frac{w^n}{w^{m+1}} dw = \delta_{nm}, \tag{2}$$

where $C_r := \{w : |w| = r,\ 0 < r < 1\}$. As shown in [3, p.612], the system $\{w^n\}$ forms a basis for the space of functions analytic in the disk $K := \{w : |w| < 1\}$, while the system $\{1/w^{m+1}\}$ forms a basis for the space of functions analytic outside the disk $\overline{K} = \{w : |w| \leq 1\}$.

Substituting the transformation $w = \varphi(z)$ into the condition of biorthogonality (2), we obtain the following two relations

$$\frac{1}{2\pi i} \int\limits_{L_r} \varphi^n(z) \frac{\varphi'(z)}{\varphi^{m+1}(z)} dz = \delta_{nm}; \qquad \frac{1}{2\pi i} \int\limits_{L_r} \frac{d}{dz} \left(\frac{\varphi^{n+1}(z)}{n+1} \right) \frac{1}{\varphi^{m+1}(z)} dz = \delta_{nm}, \tag{3}$$

where $L_r \subset D$ is an inverse image of the circle C_r under mapping (1).

Let us introduce the systems of functions

$$\left\{ g_n(z) = \varphi^n(z) \right\}_{n=0}^{\infty}, \quad \left\{ g_n^*(z) = \frac{1}{n+1} \frac{d}{dz} \varphi^{n+1}(z) \right\}_{n=0}^{\infty}, \quad z \in D. \tag{4}$$

The functions $g_n(z)$ and $g_n^*(z)$ are analytic in a neighbourhood of zero because $\varphi(z)$ is analytic in D, so their expansions into the Laurent series do not contain the negative degree power series. According to (3), define the systems of functions $\{\omega_m(z)\}_{m=0}^{\infty}$, $\{\omega_m^*(z)\}_{m=0}^{\infty}$ conjugate to corresponding systems (4) as the principal parts (i.e., the series of terms with negative degree) of the Laurent series for the functions $\varphi'(z)\varphi^{-(m+1)}(z)$, $\varphi^{-(m+1)}(z)$, respectively, in a neighbourhood of zero (see [3]).

Theorem 1. *Let $w = \varphi(z)$ be the conformal mapping* (1) *and $|z| < l$ (l is a positive real number) be the largest disk in which the Maclaurin series of this function converges and which lies in the domain D. Then the systems of functions* (4) *form bases for the space E_r, $0 < r < l$ of functions analytic in the disk $|z| < r$.*

Proof. Let us consider the function

$$F(\xi) = \frac{1}{\varphi(l/\xi)},$$

which is one-sheeted in the domain $|\xi| > l$ and satisfies the conditions

$$F(\infty)=\infty, \ \lim_{\xi\to\infty} \frac{F(\xi)}{\xi}=1.$$

Also let $\psi(\xi)$ be a one-sheeted function in the domain $|\xi| > l$ such that $\psi(\infty) = 1$. Then, according to Theorem 10 (see [3, p.616]), the system of polynomials $\{p_n(\xi)\}_{n=0}^{\infty}$ forms a basis for the space E_r, $r > l$, where $p_n(\xi)$ is the principal part of the Laurent series for the function $\dfrac{F^n(\xi)F'(\xi)}{\psi(\xi)}$ in a neighbourhood of the infinitely remote point. And the system of functions $\widetilde{\omega}_m(\xi) = \dfrac{\psi(\xi)}{F^{m+1}(\xi)}$, $m = 0, 1, \ldots$ is conjugate to these polynomials and also forms a basis for the space $\widetilde{E}_\rho$ of functions analytic in the domain $|\xi| > \rho$, $\rho \geq l$ and equal to zero in the infinitely remote point.

The condition of biorthogonality is as follows:

$$\frac{1}{2\pi i} \int\limits_{|\xi|=r'\geq r} p_n(\xi)\widetilde{\omega}_m(\xi)d\xi = \delta_{nm}.$$

Transform this condition using mapping $\xi = l/z$ to obtain

$$\frac{1}{2\pi i} \int\limits_{\Gamma^+} \widetilde{\omega}_m\left(\frac{l}{z}\right) p_n\left(\frac{l}{z}\right) \frac{l dz}{z^2} = \delta_{nm}. \tag{5}$$

Here $\Gamma^+ : |z| = \dfrac{l}{z}$ is a circumference oriented counter-clockwise.

The system $\left\{\dfrac{\widetilde{\omega}_m(l/z)}{z}\right\}$ forms a basis for the space of functions analytic in the disk $|z| < l$, while $\left\{\dfrac{l p_n(l/z)}{z}\right\}$ is the conjugate system of polynomials consisting of negative power functions

$$\frac{\widetilde{\omega}_m(l/z)}{z} = \frac{\psi(l/z)}{zF^{m+1}(l/z)} = \varphi^m(z)\, \frac{\varphi(z)\psi(l/z)}{z},$$

$$\frac{lp_n(l/z)}{z} = \Gamma\left[\frac{lF^n(l/z)}{z}\,\frac{F'(l/z)}{\psi(l/z)}\right] = \Gamma\left[\frac{\varphi'(z)}{\varphi^{n+1}(z)}\,\frac{z}{\varphi(z)\psi(l/z)}\right], \tag{6}$$

where $F'(\xi) = F'(l/z) = \dfrac{z^2\varphi'(z)}{l\varphi^2(z)}$, and $\Gamma[g(z)]$ is the principal part of the Laurent series for the function $g(z)$ in a neighbourhood of zero.

Choosing $\psi(l/z) = \dfrac{z}{\varphi(z)}$ and $\psi(l/z) = \dfrac{z\varphi'(z)}{\varphi(z)}$, from (5) we obtain the conditions of biorthogonality for the systems of functions (4) and the corresponding conjugate systems

$$\omega_n(z) = \frac{lp_n(l/z)}{z} = \Gamma\left[\frac{\varphi'(z)}{\varphi^{n+1}(z)}\,\frac{z}{\varphi(z)\psi(l/z)}\right] = \Gamma\left[\frac{\varphi'(z)}{\varphi^{n+1}(z)}\right],$$
$$\omega_n^*(z) = \frac{lp_n(l/z)}{z} = \Gamma\left[\frac{\varphi'(z)}{\varphi^{n+1}(z)}\,\frac{z}{\varphi(z)\psi(l/z)}\right] = \Gamma\left[\frac{1}{\varphi^{n+1}(z)}\right]. \tag{7}$$

Thus, if the function $f(z)$ is analytic in the disk $|z| < l$, then it can be expanded inside this disk into the series

$$f(z) = \sum_{n=0}^{\infty} a_n g_n(z), \qquad |z| \le r < l, \tag{8}$$

where $a_n = \dfrac{1}{2\pi i}\displaystyle\int_{|\xi|=r'\le r} f(\xi)\omega_n(\xi)d\xi$, $b_n = \dfrac{1}{2\pi i}\displaystyle\int_{|\xi|=r'\le r} f(\xi)\omega_n^*(\xi)d\xi$. ◄

Theorem 2. *The systems of functions* (4) *form bases for the space of functions analytic in the domain D.*

Proof. Let us show that the series (8) converges uniformly in each domain $\overline{D}' \subset D$ if the function $f(z)$ is analytic in D. Since (1) maps the domain D onto the unit disk, any point $z \in \overline{D}'$ lies on the line L_ρ, the inverse image of the circumference $|w| = \rho < 1$. Then we have the following estimates for the functions (4):

$$|g_n(z)| = |\varphi(z)|^n = \rho^n, \quad |g_n^*(z)| = |\varphi(z)|^n|\varphi'(z)| \le A_\varphi\rho^n, \; z \in L_\rho, \tag{9}$$

where $A_\varphi = \max\limits_{z\in L_\rho}|\varphi'(z)|$.

Let L_{ρ_0} be the inverse image of the circumference $|w| = \rho_0$, $\rho < \rho_0 < 1$, and $F = \max\limits_{z\in L_{\rho_0}}|f(z)|$, $A_\psi = \max\limits_{|w|=\rho_0}|h'(w)|$. We transform expressions for the Fourier coefficients of the series (8) taking into account that the integral over the contour $L_{\rho_0} \subset D$ of a function analytical in D is equal to zero:

$$a_n = \frac{1}{2\pi i}\int_{|\xi|=L_{\rho_0}} f(\xi)\omega_n(\xi)d\xi = \frac{1}{2\pi i}\int_{|\xi|=L_{\rho_0}} \frac{f(\xi)\varphi'(\xi)d\xi}{\varphi^{n+1}(\xi)} = \frac{1}{2\pi i}\int_{|w|=\rho_0} \frac{f(h(w))dw}{w^{n+1}},$$

$$b_n = \frac{1}{2\pi i} \int\limits_{|\xi|=L_{\rho_0}} f(\xi)\omega_n^*(\xi)d\xi = \frac{1}{2\pi i} \int\limits_{|\xi|=L_{\rho_0}} \frac{f(\xi)d\xi}{\varphi^{n+1}(\xi)} = \frac{1}{2\pi i} \int\limits_{|w|=\rho_0} \frac{f(h(w))h'(w)dw}{w^{n+1}}.$$

Then we obtain the following estimates:

$$|a_n| \le \frac{1}{2\pi} \int\limits_{|w|=\rho_0} \frac{|f(h(w))||dw|}{|w|^{n+1}} \le \frac{F}{\rho_0^n}, \quad |b_n| \le \frac{1}{2\pi} \int\limits_{|w|=\rho_0} \frac{|f(h(w))||h'(w)||dw|}{|w|^{n+1}} \le \frac{FA_\psi}{\rho_0^n}. \tag{10}$$

By estimates (9), (10), and the inequality $\rho < \rho_0$, we see that the series (8) converges uniformly in each domain $\overline{D}' \subset D$:

$$\sum_{n=0}^{\infty} |a_n||g_n(z)| \le F \sum_{n=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^n = \frac{F\rho_0}{\rho_0 - \rho}, \quad \sum_{n=0}^{\infty} |b_n||g_n^*(z)| \le FA_\psi \sum_{n=0}^{\infty} \left(\frac{\rho}{\rho_0}\right)^n = \frac{FA_\psi \rho_0}{\rho_0 - \rho}. \blacktriangleleft$$

Let us consider examples of systems of functions that are constructed using exponential functions.

3. Basis in the space of functions analytic in a strip

Consider the domain D to be the strip $-\frac{\pi}{4} < \operatorname{Re} z < \frac{\pi}{4}$. Define a conformal mapping of D onto the unit disk $K : |w| < 1$ and the corresponding inverse mapping by (see [1])

$$w = \operatorname{tg} z = \frac{1}{i} \frac{e^{2zi} - 1}{e^{2zi} + 1}, \quad z \equiv \frac{1}{2} \ln \frac{1 + iw}{1 - iw}. \tag{11}$$

Then the points $z = \pi/4$, $z = -\pi/4$, $z = i\infty$, $z = -i\infty$ are mapped to the points $w = 1$, $w = -1$, $w = i$, $w = -i$, respectively, and the boundary L consisting of two straight lines $z = \pm\pi/4$ is mapped to the circumference $C : |w| = 1$.

Let us introduce the system of functions corresponding to the first system in (4):

$$g_0(z) = 1, \quad \left\{g_n(z) = \operatorname{tg}^n z\right\}_{n=1}^{\infty}, \; z \in D. \tag{12}$$

The functions $g_n(z)$ are analytic in a neighbourhood of zero, while the functions $\frac{\varphi'(z)}{\varphi^{m+1}(z)} = \frac{\operatorname{ctg}^{m+1}(z)}{\cos^2 z}$ have poles of order $m + 1$ at the origin.

First, let us expand the functions $g_n(z)$ into the Maclaurin series converging in the disk $|z| < \pi/2$. Using the expansion

$$\frac{2^n e^{xt}}{(e^t+1)^n} = \frac{e^{(x-n/2)t}}{\operatorname{ch}^n(t/2)} = \sum_{k=0}^{\infty} E_k^{(n)}(x)\frac{t^k}{k!},$$

where $E_k^{(n)} = \sum\limits_{l=0}^{k} C_k^l E_l^{(n)} x^{k-l}$ are the Euler polynomials of order n and degree l, we find

$$\frac{1}{\cos^n t} = \sum_{k=0}^{\infty}(-1)^k 2^{2k} E_{2k}^{(n)}\left(\frac{n}{2}\right)\frac{t^{2k}}{(2k)!},\quad E_{2k+1}^{(n)}\left(\frac{n}{2}\right) = 0,\ |t| < \frac{\pi}{2}. \tag{13}$$

Note that $E_k^{(1)}(x) = E_k(x)$ are the Euler polynomials, while $E_k = 2^k E_k^{(1)}\left(\frac{1}{2}\right)$ are Euler numbers. For the first four coefficients in this series we have expressions

$$E_0^{(n)}\left(\frac{n}{2}\right) = 1,\quad E_2^{(n)}\left(\frac{n}{2}\right) = -\frac{1}{4}n,$$
$$E_4^{(n)}\left(\frac{n}{2}\right) = \frac{1}{16}n(3n+2),\quad E_6^{(n)}\left(\frac{n}{2}\right) = -\frac{1}{64}n\left[15(n+1)^2+1\right].$$

We also have formulas for the series expansion of the n-th power of trigonometric functions

$$\sin^{2n+1} t = \sum_{l=n}^{\infty}(-1)^l G_{2l+1}^{-(2n+1)}\frac{t^{2l+1}}{(2l+1)!},\quad \sin^{2n} t = \sum_{l=n}^{\infty}(-1)^l G_{2l}^{-(2n)}\frac{t^{2l}}{(2l)!},$$
$$\cos^n t = \sum_{l=0}^{\infty}(-1)^l G_{2l}^{+(n)}\frac{t^{2l}}{(2l)!}, \tag{14}$$

where

$$G_{2l}^{\pm(2n)} = \frac{1}{2^{2n-1}}\sum_{k=1}^{n}(\pm 1)^k C_{2n}^{n-k}(2k)^{2l},\ n \ge 1,\ l \ge 1;$$
$$G_p^{\pm(2n+1)} = \frac{1}{2^{2n}}\sum_{k=0}^{n}(\pm 1)^k C_{2n+1}^{n-k}(2k+1)^p,$$
$$G_0^{+(2n+1)} = 1,\quad G_0^{+(2n)} = 1,\ n \ge 1;\qquad G_{2n}^{-(2n)} = (-1)^n(2n)!,$$
$$G_{2n+1}^{-(2n+1)} = (-1)^n(2n+1)!,\quad G_p^{-(2n+1)} = 0,\ p < n.$$

Using corresponding formulas (13)–(15), we find the power series expansion of the function (12):

$$\operatorname{tg}^{2n}t = \frac{\sin^{2n} t}{\cos^{2n} t} = \sum_{l=n}^{\infty}\sum_{k=0}^{\infty}(-1)^{l+k}2^{2k}G_{2l}^{-(2n)}E_{2k}^{(2n)}(n)\,\frac{t^{2(l+k)}}{(2l)!(2k)!} =$$

$$= \sum_{l=n}^{\infty}\sum_{k=l}^{\infty}(-1)^{k}2^{2k-2l}G_{2l}^{-(2n)}E_{2k-2l}^{(2n)}(n)\,\frac{t^{2k}}{(2k-2l)!(2l)!} =$$

$$= \sum_{k=n}^{\infty}(-1)^{k}\frac{t^{2k}}{(2k)!}\sum_{l=n}^{k}2^{2k-2l}C_{2k}^{2l}G_{2l}^{-(2n)}E_{2k-2l}^{(2n)}(n),$$

$$\operatorname{tg}^{2n+1}t = \frac{\sin^{2n+1} t}{\cos^{2n+1} t} = \sum_{l=n}^{\infty}\sum_{k=0}^{\infty}(-1)^{l+k}2^{2k}G_{2l+1}^{-(2n+1)}E_{2k}^{(2n+1)}\left(\frac{2n+1}{2}\right)\frac{t^{2(l+k)+1}}{(2l+1)!(2k)!} =$$

$$= \sum_{l=n}^{\infty}\sum_{k=l}^{\infty}(-1)^{k}2^{2k-2l}G_{2l+1}^{-(2n+1)}E_{2k-2l}^{(2n+1)}\left(\frac{2n+1}{2}\right)\frac{t^{2k+1}}{(2k-2l)!(2l+1)!} =$$

$$= \sum_{k=n}^{\infty}(-1)^{k}\frac{t^{2k+1}}{(2k+1)!}\sum_{l=n}^{k}2^{2k-2l}C_{2k+1}^{2l+1}G_{2l+1}^{-(2n+1)}E_{2k-2l}^{(2n+1)}\left(\frac{2n+1}{2}\right).$$

Thus, we obtain the power series for (12), which is convergent in the domain $|z| < \pi/2$, in the following form:

$$\operatorname{tg}^{2n}(z) = \sum_{k=n}^{\infty}R_{2k}^{(2n)}z^{2k}, \quad \operatorname{tg}^{2n+1}(z) = \sum_{k=n}^{\infty}R_{2k+1}^{(2n+1)}z^{2k+1}, \tag{15}$$

where

$$R_{2k}^{(2n)} = \frac{(-1)^k}{(2k)!}\sum_{l=n}^{k}2^{2k-2l}C_{2k}^{2k-2l}G_{2l}^{-(2n)}E_{2k-2l}^{(2n)}(n),$$

$$R_{2k+1}^{(2n+1)} = \frac{(-1)^k}{(2k+1)!}\sum_{l=n}^{k}2^{2k-2l}C_{2k+1}^{2k-2l}G_{2l+1}^{-(2n+1)}E_{2k-2l}^{(2n+1)}\left(\frac{2n+1}{2}\right).$$

Now we find the system of functions conjugate to (15). First, let us construct the power series expansion of the function $(z\operatorname{ctg} z)^n$ in a neighbourhood of zero. We have (see [7])

$$\frac{t^n e^{xt}}{(e^t-1)^n} = \sum_{k=0}^{\infty}B_k^{(n)}(x)\,\frac{t^k}{k!},$$

where $B_k^{(n)}(x)=\sum\limits_{l=0}^{k} C_k^l B_l^{(n)} x^{k-l}$ are the Bernoulli polynomials of order n and degree k, while $B_k^{(n)}(0)=B_l^{(n)}$ are Bernoulli's numbers. Consequently, assuming $x = n/2$, we obtain

$$\frac{t^n}{\sin^n t} = \sum_{k=0}^{\infty}(-1)^k 2^{2k} B_{2k}^{(n)}\left(\frac{n}{2}\right)\frac{t^{2k}}{(2k)!}, \quad B_{2k+1}^{(n)}\left(\frac{n}{2}\right) = 0, \ |t| < \pi. \tag{16}$$

For the first coefficients in this series, we have the following representation:

$$B_0^{(n)}\left(\frac{n}{2}\right) = 1, \quad B_2^{(n)}\left(\frac{n}{2}\right) = -\frac{1}{12}n,$$
$$B_4^{(n)}\left(\frac{n}{2}\right) = \frac{1}{240}n(5n+2), \quad B_6^{(n)}\left(\frac{n}{2}\right) = -\frac{1}{64}n\left[\frac{1}{9}(n+1)(5n+1)+\frac{1}{7}\right].$$

Taking into account the formulas (3)–(16), we obtain

$$(t\operatorname{ctg} t)^n = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{l+k}2^{2k}G_{2l}^{+(n)}B_{2k}^{(n)}\left(\frac{n}{2}\right)\frac{t^{2(l+k)}}{(2l)!(2k)!} =$$
$$= \sum_{l=0}^{\infty}\sum_{k=l}^{\infty}(-1)^k 2^{2k-2l}G_{2l}^{+(n)}B_{2k-2l}^{(n)}\left(\frac{n}{2}\right)\frac{t^{2k}}{(2k-2l)!(2l)!} =$$
$$= \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}\sum_{l=0}^{k}2^{2k-2l}C_{2k}^{2l}G_{2l}^{+(n)}B_{2k-2l}^{(n)}\left(\frac{n}{2}\right), \quad |t| < \frac{\pi}{2}.$$

Hence, we find the principal part of the Laurent series for the function $\operatorname{ctg}^n t$ in a neighbourhood of zero:

$$F_n(t) = \sum_{l=0}^{[(n-1)/2]} Q_{2l}^{(n)}\frac{1}{t^{n-2l}}, \tag{17}$$

where $Q_{2k}^{(n)} = \frac{(-1)^k}{(2k)!}\sum\limits_{l=0}^{k}2^{2k-2l}C_{2k}^{2l}G_{2l}^{+(n)}B_{2k-2l}^{(n)}\left(\frac{n}{2}\right)$. And similarly, we find the principal part of the Laurent series for the function $\frac{-1}{m}\frac{d}{dz}\varphi^{-m}(z) = \varphi^{-(m+1)}(z)\varphi'(z), \ m \geq 1$, which defines the system of functions conjugate to (15):

$$\omega_m(z) = \frac{-1}{m}\frac{d}{dz}F_m(z) = \sum_{l=0}^{[(m-1)/2]}(-1)^l\frac{(m-2l)}{m}Q_{2l}^{(m)}\frac{1}{z^{m-2l+1}}.$$

In the case $m = 0$, $\varphi^{-1}(z)\varphi'(z) = (\cos z \sin z)^{-1} = 2\sin^{-1} 2z$, according to (16), we have $\omega_0(z) = 1/z$. So, for the the system of functions $\{\omega_m(z)\}_{m=0}^{\infty}$, we obtain the following formulas:

$$\omega_0(z) = \frac{1}{z}; \;\; \omega_m(z) = \sum_{l=0}^{[(m-1)/2]} (-1)^l \frac{(m-2l)}{m} Q_{2l}^{(m)} \frac{1}{z^{m-2l+1}}, \;\; m \geq 1. \tag{18}$$

The systems of functions (15) and (18) are biorthogonal on an arbitrary circumference $L: \; |z| = r, \; 0 < r < \pi/4$, that is

$$\frac{1}{2\pi i} \int\limits_{|z|=r_0<r} g_n(z)\omega_m(z)dz = \delta_{nm},$$

and, by Theorem 1, system (15) serves as a basis for the space $E_r, \; 0 < r < \pi/4$.

Example 1. *By formulas* (15) *and* (18)*, we can find explicit expressions for the first few functions of the systems considered above:*

$$g_0(z) = 1; \quad g_1(z) = z + \frac{z^3}{3} + \frac{2}{3\cdot 5}z^5 + \frac{17}{5\cdot 7\cdot 9}z^7 + \ldots;$$
$$g_2(z) = z^2 + \frac{2}{3}z^4 + \frac{17}{5\cdot 9}z^6 + \frac{62}{5\cdot 7\cdot 9}z^8 + \ldots; \quad g_3(z) = z^3 + z^5 + \frac{11}{3\cdot 5}z^7 + \ldots;$$
$$g_4(z) = z^4 + \frac{4}{3}z^6 + \frac{6}{5}z^8 + \ldots; \quad g_5(z) = z^5 + \frac{5}{3}z^7 + \frac{8}{9}z^9 + \ldots;$$
$$\omega_0(z) = \frac{1}{z}; \quad \omega_1(z) = \frac{1}{z^2}; \quad \omega_2(z) = \frac{1}{z^3}; \quad \omega_3(z) = -\frac{1}{3}\frac{1}{z^2} + \frac{1}{z^4}; \quad \omega_4(z) = -\frac{2}{3}\frac{1}{z^3} + \frac{1}{z^5};$$
$$\omega_5(z) = \frac{1}{5}\frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6}; \quad \omega_6(z) = \frac{23}{5\cdot 9}\frac{1}{z^3} - \frac{4}{3}\frac{1}{z^5} + \frac{1}{z^7}.$$

It is easy to see that the orthogonality conditions are met.

By Theorem 2, system (15) forms a basis for the space of functions analytic in D, while the system (18) is conjugated to it.

Example 2. *The expansions of power functions in terms of the system* (12) *have the form*

$$z^{2p} = \sum_{n=p}^{\infty} a_{2n} g_{2n}(z), \qquad z^{2p+1} = \sum_{n=p}^{\infty} a_{2n+1} g_{2n+1}(z), \quad z \in D,$$

where

$$a_{2n} = \frac{1}{2\pi i}\int\limits_{L_r} z^{2p}\omega_{2n}(z)dz = \frac{p}{n}Q_{2n-2p}^{(2n)},$$
$$a_{2n+1} = \frac{1}{2\pi i}\int\limits_{L_r} z^{2p+1}\omega_{2n+1}(z)dz = \frac{2p+1}{2n+1}Q_{2n-2p}^{(2n+1)}.$$

Using the second relation in (3), we obtain the following systems of functions:

$$\left\{g_n^*(z) = \frac{1}{n+1}\frac{d}{dz}g_{n+1}(z) = \frac{\mathrm{tg}^n z}{\cos^2 z}\right\}_{n=0}^{\infty}; \quad \left\{\omega_m^*(z) = F_{m+1}(z)\right\}_{m=0}^{\infty}, \quad z \in D, \tag{19}$$

which are biorthogonal on the closed curve $L_r \subset D$ around the origin. By Theorem 2, the system of functions $\{g_n^*(z)\}$ is a basis for the space of functions analytic in D.

4. Basis in the space of functions analytic in a domain bounded by parabola

A conformal mapping of domain D, where D is assumed to be bounded by the parabola $z = \rho e^{i\varphi}$, $\rho \cos^2 \frac{\varphi}{2} = \frac{\pi^2}{16}$, onto the unit disk K and the inverse mapping are given by (see [7])

$$w = \varphi(z) \equiv \mathrm{tg}^2\sqrt{z}, \qquad z \equiv -\frac{1}{4}\ln\frac{1+i\sqrt{w}}{1-i\sqrt{w}}. \tag{20}$$

The points of parabola $z = \frac{\pi^2}{16}$, $z = \frac{\pi^2}{8}i$, $z = -\frac{\pi^2}{8}i$, $z = -\infty$, are mapped to the points of circumference $w = 1$, $w = e^{i\theta}$, $w = e^{-i\theta}$, $w = -1$, respectively, where $\theta = 2\mathrm{arctgsh}\left(\frac{\pi}{2}\right)$.

Let us introduce the system of functions

$$\left\{g_n(z) = \mathrm{tg}^{2n}\sqrt{z}\right\}_{n=0}^{\infty}, \quad z \in D. \tag{21}$$

Using (15), we find the power series expansions of functions (21):

$$g_n(z) = \sum_{l=0}^{\infty}(-1)^l R_{2l}^{(2n)} z^l, \quad |t| < \frac{\pi^2}{16}. \tag{22}$$

Let us construct the system of functions conjugate to (21) and (22). First, we find the power series expansion of the function $(\sqrt{z}\mathrm{ctg}\sqrt{z})^{2n}$ in a neighborhood of the origin. Using the first formula in (16), we obtain

$$\left(\sqrt{t}\mathrm{ctg}\sqrt{t}\right)^{2n} = t^n\left(\mathrm{ctg}\sqrt{t}\right)^{2n} = \sum_{l=0}^{\infty}(-1)^l Q_{2l}^{(2n)} t^l.$$

Further, we find the principal part of the Laurent series for the function in a neighborhood of the origin:

$$F_n(t) = \sum_{l=0}^{n-1}(-1)^l Q_{2l}^{(2n)} \frac{1}{t^{n-l}}.$$

Similarly, we find the principal part of Laurent series for the function $\frac{-1}{m}\frac{d}{dz}\varphi^{-m}(z) = \varphi^{-(m+1)}(z)\varphi'(z)$, $m \geq 1$ (here $\varphi(z) = \mathrm{tg}^2\sqrt{z}$), which gives the corresponding conjugate system $\{\omega_m(z)\}_{m=0}^{\infty}$. Thus

$$\omega_m(z) = \frac{-1}{m}\frac{d}{dz}F_m(z) = \sum_{l=0}^{m-1}(-1)^l\frac{m-l}{m}Q_{2l}^{(2m)}\frac{1}{z^{m-l+1}}. \tag{23}$$

In the case $m = 0$, $\varphi^{-1}(z)\varphi'(z) = \left(\sqrt{z}\cos z\sin z\right)^{-1} = 2\left(\sqrt{z}\sin 2\sqrt{z}\right)^{-1}$, according to (16), we have $\omega_0(z) = \frac{1}{z}$. By Theorem 1, this system of functions is a basis for the space E_r, $0 < r \leq \frac{\pi^2}{16}$, and, by Theorem 2, system (21) forms a basis for the space of functions analytic in D.

According to the second relation in (3), we find the systems of functions similar to (19). By Theorem 2, the system $\{g_n^*\}_{n=0}^{\infty}$ forms a basis for the space of functions analytic in D.

5. Basis in the space of functions analytic in a domain bounded by catenary

Assume that D is a domain bounded by catenary. A conformal mapping of D onto the unit disk K and the inverse mapping are given by the functions (see [10])

$$w = \varphi(z) \equiv \frac{\mathrm{e}^z - 1}{\mathrm{e}^z}, \qquad z = h(z) \equiv \ln\frac{1}{1-w}. \tag{24}$$

The curve $L : z = \ln\frac{1}{2\sin(\theta/2)} + i\frac{\pi-\theta}{2}$, $0 \leq \theta < 2\pi$, which is the boundary of D, is mapped to the circumference $C := \{w : \ |w| = 1\}$. Setting $z = x + iy$, we may write an equation for the curve L in the form $y = -\ln(2\cos x)$.

Let us consider the system of functions

$$\left\{g_n(z) = \left(\frac{\mathrm{e}^z - 1}{\mathrm{e}^z}\right)^n\right\}_{n=0}^{\infty}, \qquad z \in D \tag{25}$$

and construct the conjugate system $\{\omega_m(z)\}_{m=0}^{\infty}$.

First, we find the expansion of functions $g_n(z)$ into the Maclaurin series

$$g_n(z) = \left(1 - \mathrm{e}^{-z}\right)^n = \sum_{l=0}^{n}C_n^l(-1)^l\mathrm{e}^{-zl} =$$

$$= \sum_{l=0}^{n} (-1)^l C_n^l \sum_{k=0}^{\infty} (-1)^k \frac{z^k l^k}{k!} = \sum_{k=0}^{\infty} \sum_{l=0}^{n} (-1)^{l+k} C_n^l \frac{l^k}{k!} z^k.$$

Denoting $P_n^{(k)} = \sum\limits_{l=0}^{n} (-1)^{l+k} C_n^l \frac{l^k}{k!}$, we obtain

$$g_n = \sum_{k=0}^{\infty} P_n^{(k)} z^k, \quad |z| < \ln 2. \tag{26}$$

Now we find the Laurent series in a neighborhood of the origin for the function $\varphi^{-m}(z) = \left(\frac{\mathrm{e}^z}{\mathrm{e}^z - 1}\right)^m$. Using the expansions of Bernoulli polynomials, we have

$$\varphi^{-m}(z) = \left(\frac{\mathrm{e}^z}{\mathrm{e}^z - 1}\right)^m = \frac{1}{z^m} \mathrm{e}^{\frac{mz}{2}} \sum_{k=0}^{\infty} B_{2k}^{(m)} \left(\frac{m}{2}\right) \frac{z^{2k}}{(2k)!} =$$
$$= \frac{1}{z^m} \sum_{l=0}^{\infty} \frac{(mz)^l}{2^l l!} \sum_{k=0}^{\infty} B_{2k}^{(m)} \left(\frac{m}{2}\right) \frac{z^{2k}}{(2k)!} = \frac{1}{z^m} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{m^l}{2^l} B_{2k}^{(m)} \left(\frac{m}{2}\right) \frac{z^{2k+l}}{l!(2k)!} =$$
$$= \frac{1}{z^m} \sum_{k=0}^{\infty} \sum_{l=k}^{\infty} \frac{m^{l-2k}}{2^{l-2k}} B_{2k}^{(m)} \left(\frac{m}{2}\right) \frac{z^l}{(l-2k)!(2k)!}.$$

Defining $G_l^{(m)} = \frac{1}{l!} \sum\limits_{k=0}^{l} \left(\frac{m}{2}\right)^{l-2k} C_l^{2k} B_{2k}^{(m)} \left(\frac{m}{2}\right)$, we obtain

$$\varphi^{-m}(z) = \sum_{l=0}^{\infty} G_l^{(m)} \frac{1}{z^{m-l}}.$$

Hence, we find the principal part of the Laurent series for the function $\varphi^{-m}(z)$ in a neighborhood of zero

$$F_m = \sum_{l=0}^{m-1} G_l^{(m)} \frac{1}{z^{m-l}}. \tag{27}$$

Similarly, we find the principal part of the Laurent series for the function $\frac{-1}{m} \frac{d}{dz} \varphi^{-m}(z) = \varphi^{-(m+1)}(z) \varphi'(z)$, $m \geq 1$, which gives the corresponding conjugate system of functions

$$\omega_m(z) = \frac{-1}{m} \frac{d}{dz} F_m(z) = \sum_{l=0}^{m-1} \frac{(m-l)}{m} G_l^{(m)} \frac{1}{z^{m-l+1}}.$$

Thus, for the system $\{\omega_m(z)\}_{m=0}^{\infty}$ we obtain the formulas

$$\omega_0(z) = \frac{1}{z}; \quad \omega_m = \sum_{l=0}^{m-1} \frac{(m-l)}{m} G_l^{(m)} \frac{1}{z^{m-l+1}}, \quad m \geq 1. \tag{28}$$

Example 3. *For the first few functions of the systems considered above, we have the following expressions:*

$$g_0(z) = 1; \quad g_1(z) = z - \frac{z^2}{2} + \frac{z^3}{6} - \frac{z^4}{24} + \ldots;$$
$$g_2(z) = z^2 - z^3 + \frac{7}{12} z^4 - \ldots; \quad g_3(z) = z^3 - \frac{3}{2} z^4 + \ldots;$$
$$\omega_0(z) = \frac{1}{z}; \quad \omega_1(z) = \frac{1}{z^2}; \quad \omega_2(z) = \frac{1}{2}\frac{1}{z^2} + \frac{1}{z^3}; \quad \omega_3(z) = \frac{1}{3}\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}.$$

By Theorem 1, the system of function (26) forms a basis for the space of functions analytic in the disk $|z| < \ln 2$, and, by Theorem 2, the system of function (25) is a basis for the space of functions analytic in D.

From the second relation in (3), we obtain systems of functions analogous to (19).

6. The construction of a solution to the Dirichlet problem for the Helmholtz equation

Let $w = \varphi(z)$ be the conformal mapping (1). Let us write the Helmholtz equation using the variables $w, \overline{w}$:

$$4 \frac{\partial^2 U}{\partial w \partial \overline{w}} + \kappa U = 0, \tag{29}$$

where $\kappa = const$ and $U = U(w, \overline{w})$ is a real-valued function.

The set of solutions to this equation in the disk can be written in the form (see [7, 8])

$$U = \mathrm{Re} \sum_{m=0}^{\infty} c_m w^m J_m^*(w\overline{w}), \tag{30}$$

where $J_m^*(w\overline{w}) = J_m^*\left(|w|^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \kappa^n |w|^{2n}}{2^{2n+m}(n+m)!n!}$ and c_m are arbitrary complex constants. The functions $J_m^*(w\overline{w})$, for $\kappa > 0$, can be expressed through the Bessel functions of order m: $J_m\left(\sqrt{\kappa}|w|\right) = \left(\sqrt{\kappa}|w|\right)^m J_m^*\left(|w|^2\right)$.

Making the change of variables $z = h(w)$, $\overline{z} = \overline{h(w)}$, and using $\varphi'(z) \neq 0$, $z \in D$, we derive the following equation:

$$4\frac{\partial^2 U}{\partial z \partial \overline{z}} + \kappa \varphi'(z)\overline{\varphi'(z)}U = 0. \tag{31}$$

Solutions of equation (31) can be obtained from (30) using the change of variables $w = \varphi(z)$, $\overline{w} = \overline{\varphi(z)}$:

$$U(z, \overline{z}) = \mathrm{Re} \sum_{m=o}^{\infty} c_m \varphi^m(z) J_m^* \left(\varphi(z)\overline{\varphi(z)}\right). \tag{32}$$

6.1. Solution in a disk

In this section, we find a solution of equation (29) in the disk $K := \{w : \ |w| < 1\}$, provided that

$$U(w, \overline{w})\big|_C = \mathrm{Re} f(t), \quad t \in C = \partial K, \tag{33}$$

where $f(t)$ is a function expanded into a uniformly convergent series, that is,

$$f(t) = \sum_{m=0}^{\infty} d_m t^m. \tag{34}$$

Note that, if the series whose terms are all functions analytic in the closed domain $\overline{G}$ converges uniformly on the boundary $L = \partial G$, then it converges uniformly in $\overline{G}$, and the limit function is continuous on L and analytic in G (see [10, p. 192]). One of the sufficient conditions for a function to be expanded into a uniformly convergent series whose terms are analytic functions is a condition of being Hölder continuous (see [10, p. 275]).

Thus, from the uniform convergence of series (34) it follows that this series converges uniformly in $\overline{K}$, and the limit function is analytic in K and continuous on C.

Substituting expression (34) into condition (33) and using relation (30) and the equality $w\overline{w} = 1$, $w \in C$, we obtain

$$\sum_{m=0}^{\infty} c_m \mathrm{e}^{im\psi} J_m^*(1) = \sum_{m=0}^{\infty} d_m \mathrm{e}^{im\psi}.$$

Hence, we find coefficients $c_m = \dfrac{d_m}{J_m^*(1)}$ and write the solution to the problem as follows:

$$U(w, \overline{w}) = \mathrm{Re} \sum_{m=0}^{\infty} \frac{d_m}{J_m^*(1)} w^m J_m^*(w\overline{w}). \tag{35}$$

Thus, the solution to our problem is found in the form of a series in terms of the functions $\{w^m J_m^*(w\overline{w})\}$. At the same time, because of the boundedness of $J_m^*\left(|w|^2\right)$ and the uniform convergence of series (34) in $\overline{K}$, series (35) also converges uniformly in $\overline{K}$.

If we have an explicit expression for the boundary function $\mathrm{Re}f(t) = u(t)$, then, using the Schwartz formula and the expansion of the corresponding analytic function into a series, we obtain

$$f(w) = \frac{1}{2\pi i} \int\limits_{|t|=1} u(t) \frac{t-w}{t+w} \frac{dt}{t} = \sum_{m=0}^{\infty} d_m w^m,$$

where $d_0 = \frac{1}{2\pi i} \int\limits_{|t|=1} u(t)\frac{dt}{t}$; $d_k = \frac{1}{2\pi i} \int\limits_{|t|=1} u(t)\frac{dt}{t^{k+1}}$, $k \geq 1$. Substituting the expressions for the coefficients of this series into (35), we obtain a solution to the corresponding boundary value problem for the Helmholtz equation.

6.2. The construction of a solution to the boundary value problem in an arbitrary domain

Let us find a solution to equation (31) in a domain D, provided that

$$U(z, \overline{z})\big|_L = \mathrm{Re}f(t), \quad t \in L, \tag{36}$$

where $f(t)$ is a function expanded into a uniformly convergent series in terms of the first system in (4), that is,

$$f(t) = \sum_{n=0}^{\infty} a_n g_n(t). \tag{37}$$

The uniform convergence of series (37) in $\overline{D}$, the analytic property of the function $f(z)$ in D, as well as the continuity of this function on the boundary L follow from the conditions on the function $f(t)$.

Substituting the expression for mapping (1) into the expression for the solution (30) to the problem in the disk, we obtain the set of solutions to equation (31) in D in the form of series (32). We determine the coefficients of this series from condition (36). Substituting formulas (32) and (26) into the boundary condition and using the relations $g_n(z) = \varphi^n(z)$, $\varphi(t)\overline{\varphi(t)} = 1$, $\varphi(t) = \mathrm{e}^{i\psi}$, $0 \leq \psi < 2\pi$, we obtain the following equality:

$$\sum_{n=0}^{\infty} c_n \mathrm{e}^{in\psi} J_n^*(1) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{in\psi}.$$

Hence, finding that $c_n = \dfrac{a_n}{J_n^*(1)}$, we write the solution to our problem as follows:

$$U(z,\overline{z}) = \mathrm{Re}\sum_{n=1}^{\infty} \frac{a_n}{J_n^*(1)} g_n(z) J_n^*\left(\varphi(z)\overline{\varphi(z)}\right). \tag{38}$$

Substituting the values $z = t \in L$ into series (38), we immediately obtain the boundary condition (36). The uniform convergence of series (38) in the domain D follows from the boundedness of $J_n^*(\varphi(z)\overline{\varphi(z)})$ whenever $0 \le |\varphi(z)| \le 1$, $z \in D$, and the uniform convergence of series (37).

Example 4. *If $f(t) = \mathrm{Re}\, g_n(t)$ and, respectively, $f(z) = g_n(z)$, then the solution of equation (31) satisfying the condition $U(z,\overline{z})|_L = \mathrm{Re}\, g_n(t)$ has the form*

$$U(z,\overline{z}) = \frac{1}{J_n^*(1)} \mathrm{Re}\, g_n(z) J_n^*\left(\varphi(z)\overline{\varphi(z)}\right).$$

But if $f(t) = 1$, then the solution to this equation is as follows:

$$U(z,\overline{z}) = \frac{1}{J_0^*(1)} J_0^*\left(\varphi(z)\overline{\varphi(z)}\right),$$

where $J_0^*\left(\varphi(z)\overline{\varphi(z)}\right) = J_0^*\left(|\varphi(z)|^2\right)$ is the Bessel function of zero order.

Conclusions. In this article, applying conformal mappings $w = \varphi(z)$ of simply connected domains onto a unit disk such that $\varphi(0) = 0$, $\varphi'(0) = 1$, we construct bases for the spaces of analytic functions and find corresponding solutions to the boundary problems for the Helmholtz equation. This approach can be extended to the case of mappings satisfying the conditions $\varphi(z_0) = 0$, $\varphi'(z_0) = 1$. In this case we must use the expansions of corresponding functions into the Maclaurin and the Laurent series in a neighborhood of z_0.

Similar bases can be used for constructing solutions to the boundary value problems of the second or the third kind for the Helmholtz equation.

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