Azerbaijan Journal of Mathematics
V. 7, No 1, 2017, January
ISSN 2218-6816

Potential Estimates and a Priori Estimates for Elliptic Equations of Cordes Type

F. Mamedov, S. Monsurrò*, M. Transirico

Abstract. We prove some a priori estimates for the solutions of some classes of Dirichlet problems associated to certain non divergence structure elliptic equations. This is done by means of a potential type estimate obtained for the solutions of the same kind of problems, but with more regular data.

Key Words and Phrases: elliptic equations, a priori bounds, potential estimates.

2010 Mathematics Subject Classifications: 35J25, 35B45, 31B10

1. Introduction

Let Ω be a sufficiently regular bounded open subset of $\mathbb{R}^n$, $n > 2$. We are interested in the study of some aspects related to the strong solvability of the Dirichlet problem

$$\begin{cases} Lu = -f, \quad f \in L^p(\Omega), \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

in the space $W^{2,p}(\Omega)$, $p > 1$, where L is the second order linear differential operator in non divergence form

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

The coefficients a_{ij} are supposed to be bounded measurable functions satisfying the uniform ellipticity condition.

As shown by a classical example due to C. Pucci (quoted by Talenti in [1]), for $n > 2$, the boundedness and ellipticity of the a_{ij} are not enough to derive the strong solvability of problem (1). Pucci's example suggests that two directions can be followed in order to overcome this difficulty. The first one is to impose conditions on the coefficient matrix stronger than the uniform ellipticity, while the second one is to assume suitable regularity of the a_{ij} (see e.g. [2] for a wider survey on this subject).

*Corresponding author.

Following the first order of ideas, Talenti considered, in [1], coefficients that do not scatter too much, i.e. satisfying the so called Cordes condition

$$\frac{\sum_{i,j=1}^{n} a_{ij}^2(x)}{\left(\sum_{i=1}^{n} a_{ii}(x)\right)^2} \leq \frac{1}{n-1} - \delta, \quad \delta > 0. \tag{2}$$

As proved by Talenti, (2) entails the uniform ellipticity of the operator and gives existence and uniqueness results for problem (1), for $p = 2$, together with the estimate

$$\|u\|_{W^{2,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{3}$$

For sake of completeness, in the same framework, we recall the work [3], where also lower order terms were considered, and [4, 5, 6] where unbounded domains in weighted and non-weighted cases were treated. Moreover, quasilinear elliptic equations of Cordes type have been studied, for example, in [7, 8]. Let us also mention that, in [9], S. Campanato extended (3) to values of p sufficiently close to 2.

Concerning the second order of ideas, it is well known (see, for instance, [10]) that if Ω has the $C^{1,1}$-regularity property and the coefficients $a_{ij} \in C^0(\overline{\Omega})$, $i, j = 1, \dots, n$, then one has the unique strong solvability of (1). Moreover, one also has the bound

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \tag{4}$$

where the constant C is independent of u, but depends on the required regularity of the coefficients. By classical embedding theorems, one gets then

$$\|u\|_{L^{\frac{np}{n-2p}}(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \tag{5}$$

for $p < \frac{n}{2}$.

In this direction, we remind the very relevant contribution to the theory given in [11, 12] where the authors obtain (4) assuming that the coefficients a_{ij} are in the class VMO, and also the more recent works on unbounded domains, always in the case of VMO-coefficients, [13, 14, 15, 16] in non-weighted and weighted contexts. Moreover, in this framework we also mention [17, 18], where the a_{ij} satisfy different hypotheses and Ω is unbounded.

Here, we assume that Ω has the $C^{1,1}$-regularity property and that the coefficients $a_{ij} \in C^0(\overline{\Omega})$, $i, j = 1, \dots, n$, are symmetric bounded measurable functions satisfying the Cordes condition. We also require that the datum f belongs to certain spaces generalizing the notion of Sobolev spaces to the fractional case. Roughly speaking, we consider as f restrictions to Ω of functions that are Riesz potentials of kernels in suitable Lebesgue spaces (see Section 2 for details). Our assumptions allow us to improve both the results obtained under the Cordes condition and those achieved by means of the continuity of the coefficients. Indeed, we get two kinds of estimates for the solution of (1). On one hand, in Theorem 1, we estimate, for $n/\sqrt{n-1} < p < n/2$, the $L^{\frac{np}{n-2p}}$-norm of the solution u by means of the L^p-norm of the Riesz derivative of f. On the other hand, in Theorem 2, we obtain (5) showing that actually, for the above range of exponent p, the constant C is independent of the regularity of the coefficients. This seems to open the future perspective to drop the hypothesis on the continuity of the coefficients.

Our main results can be achieved by means of a potential type estimate previously obtained for the solutions of the same kind of problems, but with more regular data (see Section 4).

2. A class of suitable functional spaces

In this section we introduce a new functional space where the datum f will be taken (see Definition 1). Roughly speaking, denoting by Ω an open subset of $\mathbb{R}^n$, $n > 2$, we consider as datum f restrictions to Ω of functions that are Riesz potentials of kernels in suitable Lebesgue spaces.

To be more precise, let us start recalling some definitions and properties of Riesz potentials and hypersingular integrals. Note that here we focus just on some specific aspects of the above mentioned topics, required for our needs. Wider and deeper surveys can be found, for instance, in [19, 20, 21, 22].

Let $0 < \alpha < n$ and $1 \le \nu < +\infty$. It is known that if $g \in L^\nu(\mathbb{R}^n)$, then the integral

$$(I^\alpha g)(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}}\, dy, \quad x \in \mathbb{R}^n, \tag{6}$$

converges absolutely for almost every x.

The function $I^\alpha g$ in (6) defines, up to a positive multiplicative constant depending on α, the Riesz potential of g.

For $0 < \alpha < n$ and $1 \le \nu < +\infty$, it is therefore possible to consider the space

$$I^\alpha(L^\nu) = \{f : f = I^\alpha(g),\ g \in L^\nu(\mathbb{R}^n)\},$$

that is a Banach space with respect to the norm

$$\|f\|_{I^\alpha(L^\nu)} = \|g\|_{L^\nu(\mathbb{R}^n)}.$$

If, in addition, the condition $1 < \nu < n/\alpha$ is satisfied, then one has

$$I^\alpha(L^\nu) \subseteq L^q(\mathbb{R}^n), \quad \text{with } q = \frac{n\nu}{n-\alpha\nu}, \tag{7}$$

where, as showed in [20], the inclusion is strict; moreover, one also has

$$\|I^\alpha g\|_{L^q(\mathbb{R}^n)} \le c\, \|g\|_{L^\nu(\mathbb{R}^n)}. \tag{8}$$

Now, let $0 < \alpha < n$, $1 < \nu < n/\alpha$ and $q = \frac{n\nu}{n-\alpha\nu}$. In view of a characterization lemma concerning the space of hypersingular integrals in terms of Riesz potentials (see [20]), it is possible to define the space $L^\alpha_{\nu,q}(\mathbb{R}^n)$ of hypersingular integrals as

$$L^\alpha_{\nu,q}(\mathbb{R}^n) = \{f \in L^q(\mathbb{R}^n) : f = I^\alpha(g) \text{ with } g \text{ in } L^\nu(\mathbb{R}^n)\}. \tag{9}$$

The function g in (9) is the Riesz derivative of f, and is denoted by $\mathbb{D}^\alpha f$.

The space $L^\alpha_{\nu,q}(\mathbb{R}^n)$ generalizes the notion of Sobolev space to the fractional case and is a Banach space endowed with the norm

$$\|f\|_{L^\alpha_{\nu,q}(\mathbb{R}^n)} = \|f\|_{L^q(\mathbb{R}^n)} + \|\mathbb{D}^\alpha f\|_{L^\nu(\mathbb{R}^n)}.$$

It is known that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^\alpha_{\nu,q}(\mathbb{R}^n)$, where, as usual, $C_0^\infty(\mathbb{R}^n)$ stands for the class of all C^∞ functions on $\mathbb{R}^n$ with compact support.

Definition 1. *Let Ω be an open subset of $\mathbb{R}^n$, $n > 2$. For $0 < \alpha < n$, $1 < \nu < n/\alpha$ and $q = \frac{n\nu}{n-\alpha\nu}$, we define $L^\alpha_{\nu,q}(\Omega)$ as the set of restrictions to Ω of functions in $L^\alpha_{\nu,q}(\mathbb{R}^n)$. More precisely, $f \in L^\alpha_{\nu,q}(\Omega)$ if there exists a function $F \in L^\alpha_{\nu,q}(\mathbb{R}^n)$ such that $F = f$ in Ω. This is a Banach space endowed with the norm*

$$\|f\|_{L^\alpha_{\nu,q}(\Omega)} = \inf\{\|F\|_{L^\alpha_{\nu,q}(\mathbb{R}^n)} \mid F_{|\Omega} = f\}.$$

3. Preliminary lemmas

Let Ω be an open subset of $\mathbb{R}^n$, $n > 2$. We consider in Ω the operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{10}$$

where the coefficients $a_{ij}(x)$ are symmetric bounded measurable functions satisfying Cordes condition

$$\frac{\sum_{i,j=1}^n a_{ij}^2(x)}{\left(\sum_{i=1}^n a_{ii}(x)\right)^2} \le \frac{1}{n-1} - \delta, \quad \delta > 0, \quad \text{a.e. in } \Omega. \tag{11}$$

Let us explicitly remark that the Cordes condition entails the uniform ellipticity of the operator L (see, e.g., [1]). This, together with the boundedness of the a_{ij}, gives the existence of a constant $\mu \in (0,1)$ such that

$$\mu|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \mu^{-1}|\xi|^2, \tag{12}$$

for almost every $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$.

Let us now prove the following two lemmas that are essential tools in the proofs of our main results:

Lemma 1. *Let L be the operator defined in (10) satisfying (11). If $n > 5$ and $0 < s < \sqrt{n-1} - 2$, then for sufficiently small $\varepsilon > 0$ and for any fixed $y \in \mathbb{R}^n$ one has*

$$L\left[\left(|x-y|^2 + \varepsilon^2\right)^{-\frac{s}{2}}\right] \le$$
$$-n\mu s(s+2)\left(\tfrac{1}{s+2} - \tfrac{1}{\sqrt{n-1}}\right)\left(|x-y|^2 + \varepsilon^2\right)^{-\frac{s+2}{2}} \le 0$$

for a.e. $x \in \Omega$, and therefore the function $\left(|x-y|^2 + \varepsilon^2\right)^{-\frac{s}{2}}$ is L-superharmonic.

Proof. Fix $y \in \mathbb{R}^n$, set $R^\varepsilon(x) = \left(|x-y|^2+\varepsilon^2\right)^{\frac{1}{2}}$ and $G^\varepsilon(x) = R^\varepsilon(x)^{-s}$.

Easy calculations give

$$G^\varepsilon_{x_i} = -sR^\varepsilon(x)^{-s-2}(x_i - y_i)$$

and

$$G^\varepsilon_{x_i x_j} = sR^\varepsilon(x)^{-s-2}\left[(s+2)\gamma_i\gamma_j\frac{|x-y|^2}{R^\varepsilon(x)^2} - \delta_{ij}\right],$$

where $\gamma = (x-y)/|x-y|$ and δ_{ij} denotes the Kronecker delta.

It follows

$$\begin{aligned} LG^\varepsilon = s\, R^\varepsilon(x)^{-s-2}\Big[(s+2)\tfrac{|x-y|^2}{R^\varepsilon(x)^2}\textstyle\sum_{i,j=1}^n a_{ij}(x)\gamma_i\gamma_j - \sum_{i=1}^n a_{ii}(x)\Big] = \\ s(s+2)\, R^\varepsilon(x)^{-s-2}\Big(\sum_{i=1}^n a_{ii}(x)\Big)\Big[\frac{|x-y|^2}{R^\varepsilon(x)^2}\cdot\frac{\sum_{i,j=1}^n a_{ij}(x)\gamma_i\gamma_j}{\sum_{i=1}^n a_{ii}(x)} - \frac{1}{s+2}\Big]. \end{aligned} \tag{13}$$

Finally, by Hölder inequality for sums, (11) and (12) we obtain

$$\begin{aligned} LG^\varepsilon \le\; & s(s+2)\, R^\varepsilon(x)^{-s-2}\Big(\textstyle\sum_{i=1}^n a_{ii}(x)\Big)\Big[\tfrac{|x-y|^2}{R^\varepsilon(x)^2}\cdot\tfrac{\left(\sum_{i,j=1}^n a_{ij}^2(x)\right)^{1/2}}{\sum_{i=1}^n a_{ii}(x)} - \tfrac{1}{s+2}\Big] \\ & \le s(s+2)\, R^\varepsilon(x)^{-s-2}\Big(\textstyle\sum_{i=1}^n a_{ii}(x)\Big)\Big(\sqrt{\tfrac{1}{n-1}-\delta} - \frac{1}{s+2}\Big) \\ & \le -n\mu s(s+2)\,\left(|x-y|^2+\varepsilon^2\right)^{-\frac{s+2}{2}}\left(\frac{1}{s+2} - \frac{1}{\sqrt{n-1}}\right) \le 0. \end{aligned}$$

◄

Lemma 2. *Let L be the operator defined in* (10) *satisfying* (11). *If $n > 5$, $0 < s < \sqrt{n-1} - 2$ and g is a non negative function in $L^\nu(\mathbb{R}^n)$, by some fixed $1 < \nu < n/(n-s)$, then for sufficiently small $\varepsilon > 0$ the potential*

$$W^\varepsilon(x) = \int_{\mathbb{R}^n}\frac{g(y)}{\left(|x-y|^2+\varepsilon^2\right)^{\frac{s}{2}}}\,dy, \quad x \in \Omega, \tag{14}$$

is such that

$$LW^\varepsilon(x) \le -n\mu s(s+2)\left(\tfrac{1}{s+2} - \tfrac{1}{\sqrt{n-1}}\right)\int_{\mathbb{R}^n}\frac{g(y)}{\left(|x-y|^2+\varepsilon^2\right)^{\frac{s+2}{2}}}\,dy \le 0, \tag{15}$$

for a.e. $x \in \Omega$, and therefore W^ε is L-superharmonic.

Proof. It is immediate to verify that $0 < s < n-2$ and that the function W^ε has second weak derivatives. Moreover,

$$LW^\varepsilon(x) = \int_{\mathbb{R}^n} g(y)L\left[\left(|x-y|^2+\varepsilon^2\right)^{-\frac{s}{2}}\right]\,dy, \text{ for a.e. } x \in \Omega.$$

This, together with Lemma 1 and the hypothesis on the sign of the function g, gives (15).◄

4. Main results

In this section we prove our main results. To this aim, some preliminary facts are needed.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 5$, with the $C^{1,1}$-regularity property and $0 < s < \sqrt{n-1} - 2$.

Given a non negative smooth function g and $\varepsilon > 0$, we set

$$\overline{F}_\varepsilon(x) = \int_{\mathbb{R}^n} \frac{g(y)}{(|x-y|^2 + \varepsilon^2)^{\frac{s+2}{2}}} dy, \;\; x \in \mathbb{R}^n. \tag{16}$$

Then, we consider the classical solution u_ε of the problem

$$\begin{cases} Lu_\varepsilon = -\overline{F}_\varepsilon \text{ in } \Omega, \\ u_\varepsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{17}$$

where L is the operator defined in (10) satisfying (11) with $a_{ij} \in C^0(\overline{\Omega})$, $i, j = 1, \ldots, n$.

Let us prove the following lemma.

Lemma 3. *Let u_ε be the solution of problem* (17)*. Then there exists a positive constant $C = C(n, \mu, s)$ such that*

$$0 \leq u_\varepsilon(x) \leq C \int_{\mathbb{R}^n} \frac{\overline{F}(y)}{|x-y|^{n-2}}\, dy \quad \textit{in } \Omega, \tag{18}$$

where

$$\overline{F}(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{s+2}} dy, \;\; x \in \mathbb{R}^n. \tag{19}$$

Proof. Let $W_\varepsilon(x)$ be defined in (14). By Lemma 2 we get that there exists a positive constant C_1 such that

$$LW_\varepsilon \leq -C_1 \int_{\mathbb{R}^n} \frac{g(y)}{(|x-y|^2 + \varepsilon^2)^{\frac{s+2}{2}}}\, dy = -C_1\, \overline{F}_\varepsilon, \tag{20}$$

with $C_1 = C_1(n, \mu, s)$. Thus

$$L\Big(u_\varepsilon - \frac{W_\varepsilon}{C_1}\Big) \geq 0 \quad \text{in} \quad \Omega.$$

Moreover, it is easy to check that

$$u_\varepsilon - \frac{W_\varepsilon}{C_1} \leq 0 \quad \text{on } \partial\Omega.$$

Thus, using the maximum principle in its classical forms (see e.g. [10], Theorems 3.1 and 3.3) we obtain

$$0 \leq u_\varepsilon(x) \leq \frac{W_\varepsilon(x)}{C_1} \leq \frac{1}{C_1} \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^s}\, dy \quad \text{in } \Omega.$$

Hence by the composition formula for fractional integrals (see e.g. [19]) we conclude that

$$u_\varepsilon(x) \le C \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \left(\int_{\mathbb{R}^n} \frac{g(z)}{|y-z|^{s+2}}\, dz \right) dy \quad \text{in } \Omega,$$

with $C = C(n, \mu, s)$. The thesis is then obtained in view of the definition of $\overline{F}$. ◄

We want to exploit the potential type estimate just proved to get some a priori estimates for strong solutions of the following Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ \\ Lu = -f, \quad f \in L^p(\Omega), \end{cases} \tag{21}$$

where the coefficients $a_{ij} \in C^0(\overline{\Omega})$, $i, j = 1, \dots, n$, and $p > 1$.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 5$, with the $C^{1,1}$-regularity property and L be the operator defined in (10) satisfying (11). Let $\dfrac{n}{\sqrt{n-1}} < p < \dfrac{n}{2}$, $1 < \nu < \dfrac{n}{n - \sqrt{n-1} + \frac{n}{p}}$. If u is a solution of problem (21) with datum $f \in L_{\nu,p}^{\frac{n}{\nu} - \frac{n}{p}}(\Omega)$, then there exists a positive constant $C = C(n, \mu, p, \nu)$ such that*

$$\|u\|_{L^{\frac{np}{n-2p}}(\Omega)} \le C \|\mathbb{D}^{\frac{n}{\nu} - \frac{n}{p}} F\|_{L^\nu(\mathbb{R}^n)}, \tag{22}$$

with F as in Definition 1.

Proof. We prove Theorem 1 in two steps. In the first one we prove a potential type estimate for the solutions of some auxiliary problems. In the second one we exploit this estimate to obtain (22).

Step 1. Observe that since $f \in L_{\nu,p}^{\frac{n}{\nu} - \frac{n}{p}}(\Omega)$, by Definition 1 there exists a function $F \in L^\alpha_{\nu,q}(\mathbb{R}^n)$ such that $F = f$ in Ω. Therefore, in view of (9) we can find a $g \in L^\nu(\mathbb{R}^n)$ such that

$$F(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n - \left(\frac{n}{\nu} - \frac{n}{p}\right)}}\, dy, \quad x \in \mathbb{R}^n. \tag{23}$$

Let g^+ and g^- be the positive and negative parts of the function g, respectively, and put

$$F^+(x) = \int_{\mathbb{R}^n} \frac{g^+(y)}{|x-y|^{n - \left(\frac{n}{\nu} - \frac{n}{p}\right)}}\, dy, \; F^-(x) = \int_{\mathbb{R}^n} \frac{g^-(y)}{|x-y|^{n - \left(\frac{n}{\nu} - \frac{n}{p}\right)}}\, dy, \tag{24}$$

$x \in \mathbb{R}^n$. One clearly has $F = F^+ - F^-$.

Classical results give the existence of sequences of smooth and non negative functions g_h^+ and g_h^- such that

$$g_h^\pm \to g^\pm \text{ in } L^\nu(\mathbb{R}^n) \text{ as } h \to 0. \tag{25}$$

Let ε take its values in a sequence of positive reals numbers converging to zero. Set

$$F_{h,\varepsilon}^{\pm}(x) = \int_{\mathbb{R}^n} \frac{g_h^{\pm}(y)dy}{(|x-y|^2+\varepsilon^2)^{\frac{s+2}{2}}}, \quad x \in \mathbb{R}^n, \tag{26}$$

with $s = n - \left(\frac{n}{\nu} - \frac{n}{p}\right) - 2$, and let $u_{h,\varepsilon}^{+}$ and $u_{h,\varepsilon}^{-}$ be the classical solutions of the problems

$$\begin{cases} Lu_{h,\varepsilon}^{+} = -F_{h,\varepsilon}^{+} \text{ in } \Omega, \\ u_{h,\varepsilon}^{+} = 0 \text{ on } \partial\Omega, \end{cases} \tag{27}$$

and

$$\begin{cases} Lu_{h,\varepsilon}^{-} = -F_{h,\varepsilon}^{-} \text{ in } \Omega, \\ u_{h,\varepsilon}^{-} = 0 \text{ on } \partial\Omega, \end{cases} \tag{28}$$

respectively.

It is easy to check that $0 < s < \sqrt{n-1} - 2$ and that all the hypotheses of Lemma 3 are fulfilled. Hence by (18) we get

$$u_{h,\varepsilon}^{+}(x) \le C_1 \int_{\mathbb{R}^n} \frac{F_h^{+}(y)}{|x-y|^{n-2}}\, dy \quad \text{in } \Omega \tag{29}$$

and

$$u_{h,\varepsilon}^{-}(x) \le C_1 \int_{\mathbb{R}^n} \frac{F_h^{-}(y)}{|x-y|^{n-2}}\, dy \quad \text{in } \Omega, \tag{30}$$

with $C_1 = C_1(n, \mu, p, \nu)$ and

$$F_h^{\pm}(x) = \int_{\mathbb{R}^n} \frac{g_h^{\pm}(y)}{|x-y|^{s+2}} dy. \tag{31}$$

Now, set $F_{h,\varepsilon} = F_{h,\varepsilon}^{+} - F_{h,\varepsilon}^{-}$, and let $u_{h,\varepsilon}$ be the classical solution of the problem

$$\begin{cases} Lu_{h,\varepsilon} = -F_{h,\varepsilon} \text{ in } \Omega, \\ u_{h,\varepsilon} = 0 \text{ on } \partial\Omega. \end{cases} \tag{32}$$

In view of the uniqueness of the solutions of problems (27), (28) and (32) (see e.g. [10], Theorem 9.15), one has $u_{h,\varepsilon} = u_{h,\varepsilon}^{+} - u_{h,\varepsilon}^{-}$ and therefore $|u_{h,\varepsilon}| \le u_{h,\varepsilon}^{+} + u_{h,\varepsilon}^{-}$, since $u_{h,\varepsilon}^{+}$ and $u_{h,\varepsilon}^{-}$ are positive as a consequence of Lemma 3. Thus (29) and (30) give the potential type estimate

$$|u_{h,\varepsilon}(x)| \le C_1 \int_{\mathbb{R}^n} \frac{[F_h^{+}(y) + F_h^{-}(y)]}{|x-y|^{n-2}}\, dy \quad \text{in } \Omega. \tag{33}$$

We want to pass to the limit as $\varepsilon \to 0$ in (33). To this aim, observe that the smoothness of the coefficients of problem (32) guaranties that

$$\|u_{h,\varepsilon} - u_{h,\varepsilon'}\|_{W^{2,p}(\Omega)} \le C\|F_{h,\varepsilon} - F_{h,\varepsilon'}\|_{L^p(\Omega)}. \tag{34}$$

(see e.g. [10], Lemma 9.17).

Concerning the $F^+_{h,\varepsilon}$ and $F^-_{h,\varepsilon}$, for any fixed h, by (31), (26) and Beppo Levi's theorem, we obtain

$$F^{\pm}_{h,\varepsilon}(x) \to F^{\pm}_h(x) \text{ a.e. in } \mathbb{R}^n \quad \text{as} \quad \varepsilon \to 0. \tag{35}$$

Moreover, in view of Lebesgue's dominated convergence theorem, these last convergences take place also in $L^p(\mathbb{R}^n)$, i.e.

$$F^{\pm}_{h,\varepsilon}(x) \to F^{\pm}_h(x) \text{ in } L^p(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \to 0. \tag{36}$$

Therefore, clearly

$$F_{h,\varepsilon}(x) \to F_h(x) \text{ in } L^p(\mathbb{R}^n) \quad \text{as} \quad \varepsilon \to 0. \tag{37}$$

Combining (34) and (37), we obtain that the sequence $u_{h,\varepsilon}$ is a Cauchy sequence in $W^{2,p}(\Omega)$. Thus there exists $u_h \in W^{2,p}(\Omega)$ such that $u_{h,\varepsilon} \to u_h$ in $W^{2,p}(\Omega)$ as $\varepsilon \to 0$. Hence, up to a subsequence, still denoted by $u_{h,\varepsilon}$, we have

$$u_{h,\varepsilon} \to u_h \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0. \tag{38}$$

Now, in view of (38), we can pass to the limit as $\varepsilon \to 0$ in (33), obtaining the claimed potential type estimate for the u_h:

$$|u_h(x)| \leq C_1 \int_{\mathbb{R}^n} \frac{[F^+_h(y) + F^-_h(y)]}{|x-y|^{n-2}}\, dy \quad \text{in } \Omega. \tag{39}$$

Step 2. By (39) and (8) we have

$$\begin{aligned} \|u_h\|_{L^{\frac{np}{n-2p}}(\Omega)} &\leq C_1 \Big(\int_\Omega \Big(\int_{\mathbb{R}^n} \frac{[F^+_h(y) + F^-_h(y)]}{|x-y|^{n-2}}\, dy \Big)^{\frac{np}{n-2p}} dx \Big)^{\frac{n-2p}{np}} \\ &\leq C_1 \|I^2(F^+_h + F^-_h)\|_{L^{\frac{np}{n-2p}}(\mathbb{R}^n)} \leq C_2 \|F^+_h + F^-_h\|_{L^p(\mathbb{R}^n)} \\ &\leq C_2 (\|F^+_h\|_{L^p(\mathbb{R}^n)} + \|F^-_h\|_{L^p(\mathbb{R}^n)}), \end{aligned} \tag{40}$$

with $C_2 = C_2(n, \mu, p, \nu)$.

We want to pass to the limit as $h \to 0$ in (40). Observe that, by uniqueness, u_h is the solution of the problem

$$\begin{cases} Lu_h = -F_h \text{ in } \Omega, \\ u_h = 0 \text{ on } \partial\Omega, \end{cases} \tag{41}$$

where $F_h = F^+_h - F^-_h$. Moreover, from (25), (31) and (8), it follows that

$$F^{\pm}_h(x) \to F^{\pm}(x) \text{ in } L^p(\mathbb{R}^n) \quad \text{as} \quad h \to 0, \tag{42}$$

and therefore

$$F_h(x) \to F(x) \text{ in } L^p(\mathbb{R}^n) \quad \text{as} \quad h \to 0. \tag{43}$$

Now, since u_h solves (41), we have that

$$\|u_h - u_{h'}\|_{W^{2,p}(\Omega)} \le C\|F_h - F_{h'}\|_{L^p(\Omega)}, \tag{44}$$

and so, in view of convergence (43) we obtain

$$u_h \to u \quad \text{in } W^{2,p}(\Omega), \text{ as } h \to 0, \tag{45}$$

where u is the solution of problem (21).

Hence, by the embedding $W^{2,p}(\Omega) \subset L^{\frac{np}{n-2p}}(\Omega)$, convergences (45) and (43) we can pass to the limit as $h \to 0$ in (40), obtaining

$$\|u\|_{L^{\frac{np}{n-2p}}(\Omega)} \le C_2(\|F^+\|_{L^p(\mathbb{R}^n)} + \|F^-\|_{L^p(\mathbb{R}^n)}). \tag{46}$$

Finally, using (8) we conclude our proof to get

$$\begin{aligned} \|u\|_{L^{\frac{np}{n-2p}}(\Omega)} &\le C_3 \left(\|g^+\|_{L^\nu(\mathbb{R}^n)} + \|g^-\|_{L^\nu(\mathbb{R}^n)}\right) \\ &\le C_4\|g\|_{L^\nu(\mathbb{R}^n)} = C_4\|\mathbb{D}^\alpha f\|_{L^\nu(\mathbb{R}^n)}, \end{aligned} \tag{47}$$

with $C_3 = C_3(n, \mu, p, \nu)$ and $C_4 = C_4(n, \mu, p, \nu)$, where we have used

$$\|g\|_{L^\nu(\mathbb{R}^n)} \approx \|g^+\|_{L^\nu(\mathbb{R}^n)} + \|g^-\|_{L^\nu(\mathbb{R}^n)}, \tag{48}$$

since the functions g^+ and g^- have disjoint supports. ◄

Making some additional assumptions it is possible to achieve an estimate in terms of the L^p-norm of f.

Let Ω, n, p and ν be as in Theorem 1. We suppose that there exists an operator $P : L_{\nu,p}^{\frac{n}{\nu}-\frac{n}{p}}(\Omega) \to L_{\nu,p}^{\frac{n}{\nu}-\frac{n}{p}}(\mathbb{R}^n)$ such that

$$\|Pf\|_{L^p(\mathbb{R}^n)} \le c\|f\|_{L^p(\Omega)}, \quad f \in L_{\nu,p}^{\frac{n}{\nu}-\frac{n}{p}}(\Omega), \tag{49}$$

with $c = c(n, p, \nu, \Omega)$ positive constant.

If the datum f in (21) is in $L_{\nu,p}^{\frac{n}{\nu}-\frac{n}{p}}(\Omega)$, then we can find a $g \in L^\nu(\mathbb{R}^n)$ such that

$$Pf(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\left(\frac{n}{\nu}-\frac{n}{p}\right)}}\, dy, \quad x \in \mathbb{R}^n. \tag{50}$$

We set

$$Pf^+(x) = \int_{\mathbb{R}^n} \frac{g^+(y)}{|x-y|^{n-\left(\frac{n}{\nu}-\frac{n}{p}\right)}}\, dy, \quad Pf^-(x) = \int_{\mathbb{R}^n} \frac{g^-(y)}{|x-y|^{n-\left(\frac{n}{\nu}-\frac{n}{p}\right)}}\, dy, \tag{51}$$

$x \in \mathbb{R}^n$, where g^+ and g^- are the positive and negative parts of the function g, respectively. We explicitly observe that clearly one has $Pf = Pf^+ - Pf^-$.

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 5$, with the $C^{1,1}$-regularity property and L be the operator defined in* (10) *satisfying* (11)*. Let* $\dfrac{n}{\sqrt{n-1}} < p < \dfrac{n}{2}$, $1 < \nu < \dfrac{n}{n - \sqrt{n-1} + \frac{n}{p}}$, *and assume that* (49) *holds. If u is a solution of problem* (21) *with datum $f \in L^{\frac{n}{\nu}-\frac{n}{p}}_{\nu,p}(\Omega)$ such that*

$$\|Pf\|_{L^p(\mathbb{R}^n)} \approx \|Pf^+\|_{L^p(\mathbb{R}^n)} + \|Pf^-\|_{L^p(\mathbb{R}^n)}, \tag{52}$$

then there exists a positive constant $C = C(n, \mu, p, \nu)$ such that

$$\|u\|_{L^{\frac{np}{n-2p}}(\Omega)} \leq C\|f\|_{L^p(\Omega)}. \tag{53}$$

Proof. Following along the lines of the proof of (46) of Theorem 1, with corresponding modifications, we obtain

$$\|u\|_{L^{\frac{np}{n-2p}}(\Omega)} \leq C_1(\|Pf^+\|_{L^p(\mathbb{R}^n)} + \|Pf^-\|_{L^p(\mathbb{R}^n)}), \tag{54}$$

with $C_1 = C_1(n, \mu, p, \nu)$. Therefore by (52) and (49) one gets the thesis

$$\|u\|_{L^{\frac{np}{n-2p}}(\Omega)} \leq C_2\|Pf\|_{L^p(\mathbb{R}^n)} \leq C_3\|f\|_{L^p(\Omega)}, \tag{55}$$

with $C_2 = C_2(n, \mu, p, \nu)$ and $C_3 = C_3(n, \mu, p, \nu, \Omega)$. Let us explicitly remark that the constant C_3 does not depend on the regularity of the coefficients. ◀

References

[1] G. Talenti, *Sopra una classe di equazioni ellittiche a coefficienti misurabili*, Ann. Mat. Pura Appl., **69**, 1965, 285–304.

[2] A. Maugeri, D.K. Palagachev, L.G. Softova, *Elliptic and parabolic equations with discontinuous coefficients*, Mathematical Research, **109**, Wiley-VCH Verlag Berlin GmbH, Berlin, 2000.

[3] M. Chicco, *Equazioni ellittiche del secondo ordine di tipo Cordes con termini di ordine inferiore*, Ann. Mat. Pura Appl., **85**, 1970, 347–356.

[4] L. Caso, R. D'Ambrosio, M. Transirico, *Weighted a priori bounds for elliptic operators of Cordes type*, J. Inequal. Appl., **2015:138**, 2015, 12 pages.

[5] L. Caso, R. D'Ambrosio, M. Transirico, *Well-posedness in weighted Sobolev spaces for elliptic equations of Cordes type*, Collect. Math., **67**, 2016, 539–554.

[6] M. Transirico, M. Troisi, *Equazioni ellittiche del secondo ordine di tipo Cordes in aperti non limitati di* $\mathbb{R}^n$, Boll. Un. Mat. Ital., **3**, 1989, 169–184 .

[7] E.I. Abduragimov, *The Dirichlet problem for a certain class of second order quasilinear elliptic equations of Cordes type (Russian)*, Functional analysis, theory of functions and their applications, **1**, Dagestan. Gos. Univ., Makhachkala, 1974, 6–13 .

[8] A.F. Guliyev, A.S. Hassanpour, *On uniqueness of strong solution of Dirichlet problem for second order quasilinear elliptic equations with Cordes condition*, Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **24**, 2004, 85–92.

[9] S. Campanato,*Un risultato relativo ad equazioni ellittiche del secondo ordine di tipo non variazionale*, Ann. Scuola Norm. Sup. Pisa, **21**, 1967, 701–707.

[10] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[11] F. Chiarenza, M. Frasca, P. Longo, *Interior* $W^{2,p}$ *estimates for non divergence elliptic equations with discontinuous coefficients*, Ricerche Mat., **40**, 1991, 149–168.

[12] F. Chiarenza, M. Frasca, P. Longo, $W^{2,p}$*-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc., **336**, 1993, 841–853.

[13] L. Caso, P. Cavaliere, M. Transirico, *An existence result for elliptic equations with VMO - coefficients*, J. Math. Anal. Appl., **325**, 2007, 1095–1102.

[14] P. Cavaliere, M. Transirico, *The Dirichlet problem for elliptic equations in unbounded domains of the plane*, J. Funct. Space Appl., **6**, 2008, 47-58.

[15] S. Boccia, S. Monsurrò, M. Transirico, *Solvability of the Dirichlet problem for elliptic equations in weighted Sobolev spaces on unbounded domains*, Bound. Value Probl., **2008:901503**, 2008, 13 pages.

[16] S. Boccia, M. Salvato, M. Transirico, *Existence and uniqueness results for Dirichlet problem in weighted Sobolev speces on unbounded domains*, Methods Appl. Anal., **18**, 2011, 203–213.

[17] S. Monsurrò, M Transirico, *A priori bounds in* L^p *for solutions of elliptic equations in divergence form*, Bull. Sci. Math., **7**, 2013, 16 pages.

[18] S. Monsurrò, M Transirico, *A* $W^{2,p}$*-estimate for a class of elliptic operators*, Int. J. Pure Appl. Math., **83**, 2013, 489-499.

[19] N.S. Landkof, *Foundations of modern potential theory*, Die Grundlehren der mathematischen Wissenschaften, Band 180, Springer-Verlag, New York-Heidelberg, 1972.

[20] S.G. Samko, *Hypersingular integrals and their applications*, Analytical Methods and Special Functions, **5**, Taylor & Francis, Ltd., London, 2002.

[21] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional integrals and derivatives. Theory and applications*, Translated from the 1987 Russian original, Gordon and Breach Science Publishers, Yverdon, 1993.

[22] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. **30**, Princeton University Press, Princeton, N.J., 1970.

Farman Mamedov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az1141, 9, B.Vahabzade Str., Baku, Azerbaijan
E-mail: farman-m@mail.ru

Sara Monsurrò
Dipartimento di Matematica, Università di Salerno, via Giovanni Paolo II, 132, I - 84084, Fisciano (SA), Italy
Member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM)
E-mail: smonsurro@unisa.it

Maria Transirico
Dipartimento di Matematica, Università di Salerno, via Giovanni Paolo II, 132, I - 84084, Fisciano (SA), Italy.
Member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM)
E-mail: mtransirico@unisa.it

Received 03 May 2016
Accepted 20 June 2016