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Determination of the Type and Parameters of a Beam End Fastening

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Abstract. A homogeneous Euler-Bernoulli beam is considered. Left end of the beam is rigid clamped. Right end of the beam can be fastened by following types: (i) rigid clamping, (ii) free support, (iii) free end, (iv) floating fixing, (v) five types of elastic fixing, (vi) a concentrated inertial element at the end, (vii) elastic fixing with a concentrated inertial element at the end. The aim of this paper is to determine the type of these seven forms of the end fastening and their parameters (relative stiffness coefficients of springs, mass and moment of inertia) by natural frequencies of the beam flexural vibrations. It is proved that the types (i)–(vii) and corresponding parameters of the boundary conditions on the right end of the beam is uniquely determined by five natural frequencies of flexural vibrations. It is shown that the four natural frequencies for the unique identification of the types of boundary conditions (i)–(vii) and corresponding parameters are not enough. The special cases of the boundary conditions identification are considered. It is shown that the types of boundary conditions (i)–(vi) with two unknown parameters are uniquely determined by three eigenvalues. Two eigenvalues are not enough for the unique determination. The types of boundary conditions (i)–(vii) with three unknown parameters are uniquely determined by four eigenvalues. Three eigenvalues are not enough for this. Corresponding examples are given.

Key Words and Phrases: inverse eigenvalue problem, eigenvalues, natural frequencies, beam, concentrated inertia element, spring stiffness coefficients, boundary conditions.

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1. Introduction

Beams are important elements of the mechanical systems of cars, tractors, ships, planes, etc. Their vibrations often cause drumming, leading to discomfort for crew members and passengers. This is due to the fact that the frequency spectra of beam vibrations are sometimes in a range hazardous to human health. To change the beam vibration frequencies, it is not always reasonable to change the beam length or attach concentrated masses. Therefore, to produce comfort conditions for passengers, it is required to determine the types of beam fastening that provide the necessary (safe) range of beam vibration frequencies. This refers not only to the fundamental vibration mode but also overtones. This problem is related to issues of noise suppression [19, 26, 40], acoustic

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diagnostics [1, 13, 30, 37], the theory of inverse problems of mathematical physics [33], the direct and inverse spectral problems [10, 14, 24, 28, 29, 34], boundary-value problems for differential equations of fourth order [2, 3, 21], identification of cracks and point masses [9, 17, 20, 22, 23, 25, 27, 31, 32, 35], and problems of the theory of oscillations [15, 16, 18, 36, 38, 39].

The aim of the present work is to determine the fastening parameters of a beam from the eigenfrequencies of its flexural vibrations.

Similarly formulated problems also arise in the spectral theory of differential operators, where it is required to establish the coefficients of a differential equation and the boundary conditions using a set of eigenvalues (for more details, see [10, 14, 24, 28, 29, 34]). However, as data for finding the boundary conditions, it was not one spectrum (as in this paper) but several spectra or also other additional spectral data (for example, the spectral function, the Weyl function or so-called weighting numbers) that were used in those papers. Moreover, the main aim there was to determine the coefficients in the equation and not in the boundary conditions. The aim of this paper is, in the case of a known differential equation, to establish some of the boundary conditions of the eigenvalue problem from its spectrum.

The problems of diagnosing the fastening of strings, membranes, rods, beams, pipes and plates have been studied previously in [5, 4, 6, 7, 8, 11].

2. Formulation of the inverse problem

The small free vibrations of an incompressible beam is described by the following equation [39, p. 152]:

$$EI\,\frac{\partial^4 u(x,t)}{\partial x^4} + \rho\,F\,\frac{\partial^2 u(x,t)}{\partial t^2} = 0,$$

where $\alpha = EI$ is the flexural rigidity, ρ is the density and F is the cross-section area of the beam. The problem of the small free vibrations of a beam with a rigidly clamped left-hand end reduces, after making the substitution $u(x,t) = y(x)\,\cos(\omega\,t)$ (see, for example, [18] or [36]), to the following eigenvalue problem:

$$y^{(4)} = \lambda^4\,y, \qquad U_1(y,\lambda) = y(0) = 0, \quad U_2(y,\lambda) = y'(0) = 0, \quad U_3(y,\lambda) = 0, \quad U_4(y,\lambda) = 0. \tag{1}$$

Here α, ρ and F are constants, $\lambda^4 = \rho\,F\,\omega^2/\alpha$:

$$\begin{array}{l} U_3(y) = a_{11}\,y'''(1) + \left(a_{15} + a_{16}\,\lambda^4\right)\,y(1) = 0, \\ U_4(y) = a_{22}\,y''(1) + \left(a_{23} + a_{24}\,\lambda^4\right)\,y'(1) = 0, \end{array} \qquad (a_{ij} \in \mathbb{R}) \tag{2}$$

are linear forms which characterize the fixing at the point $x = 1$ (rigid clamping, free support, free end, floating fixing, elastic fixing, concentrated inertial element at the end, elastic fixing with concentrated inertial element at the end).

We shall denote the matrix, consisting of the coefficients a_{ij} of the forms $U_1(y)$ and $U_2(y)$, by A and its minors by M_{ij} :

$$A = \left\| \begin{array}{cccccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{array} \right\|, \qquad M_{ij} = \left| \begin{array}{cc} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{array} \right|,$$

where $a_{12} = a_{13} = a_{14} = a_{21} = a_{25} = a_{26} = 0$.

The search for the forms $U_1(y,\lambda)$, $U_2(y,\lambda)$ is equivalent to finding the linear envelope $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$, constructed in the vectors $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}, a_{i6})^{\mathrm{T}}$ $(i = 1, 2)$.

The different cases for the clamping of one end of a rod [39, p. 153–155], [18] are presented below:

(i) rigid clamping

$$A = \left\| \begin{matrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{matrix} \right\|,$$

(ii) free support

$$A = \left\| \begin{matrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{matrix} \right\|,$$

(iii) free end

$$A = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{matrix} \right\|,$$

(iv) floating fixing

$$A = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{matrix} \right\|,$$

(v) five types of elastic fixing

$$A = \left\| \begin{matrix} 1 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{matrix} \right\|, \quad \left\| \begin{matrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & c_2 & 0 & 0 & 0 \end{matrix} \right\|, \quad \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{matrix} \right\|,$$

$$\left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & c_2 & 0 & 0 & 0 \end{matrix} \right\|, \quad \left\| \begin{matrix} 1 & 0 & 0 & 0 & c_1 & 0 \\ 0 & 1 & c_2 & 0 & 0 & 0 \end{matrix} \right\|.$$

(vi) a concentrated inertial element at the end

$$A = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & -m\,\lambda^4 \\ 0 & 1 & 0 & -I_1\,\lambda^4 & 0 & 0 \end{matrix} \right\|,$$

(vii) elastic fixing with a concentrated inertial element at the end (see Figure 1)

$$A = \left\| \begin{matrix} 1 & 0 & 0 & 0 & c_1 & -m\,\lambda^4 \\ 0 & 1 & c_2 & -I_1\,\lambda^4 & 0 & 0 \end{matrix} \right\|,$$

Hence, in terms of eigenvalue problem (1), the inverse problem which has been constructed above can be formulated as follows: the coefficients a_{ij} of the forms $U_1(y,\lambda)$, $U_2(y,\lambda)$ of problem (1) are unknown, the rank of the matrix A, which is made up of these coefficients, is equal to two, the eigenvalues λ_k of problem (1) are known and it is required to find the linear envelope $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ of the vectors $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3}, a_{i4}, a_{i5}, a_{i6})^{\mathrm{T}}$ $(i = 1, 2)$.

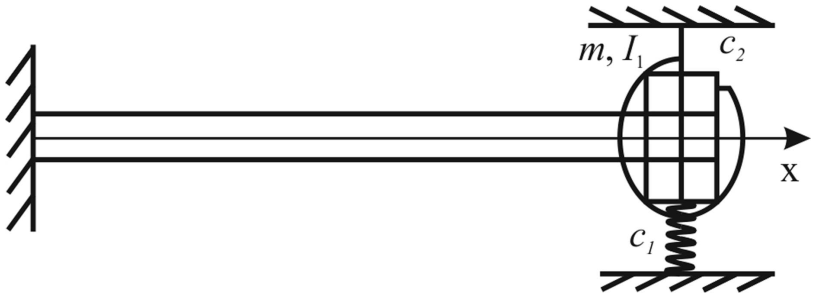

Figure 1: Boundary conditions (vii).

3. The uniqueness of the reconstructing boundary conditions (i)–(vii) from all eigenvalues

Together with problem (1), we consider the following eigenvalue problem

$$y^{(4)} = \lambda^4 y, \qquad U_1(y,\lambda) = y(0) = 0, \quad U_2(y,\lambda) = y'(0) = 0, \quad \widetilde{U}_3(y,\lambda) = 0, \quad \widetilde{U}_4(y,\lambda) = 0, \tag{3}$$

where $\widetilde{U}_i(y,\lambda) = \sum_{j=1}^{4} b_{i-2j}\, y^{(j-1)}(0) \ \ (i = 3,\, 4)$.

We denote the matrix composed of the coefficients b_{ij} of the forms $\widetilde{U}_1(y,\lambda)$ and $\widetilde{U}_2(y,\lambda)$ by B and its minors by $\widetilde{M}_{ij}$:

$$B = \left\| \begin{matrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \end{matrix} \right\|, \qquad \widetilde{M}_{ij} \equiv \left| \begin{matrix} b_{1i} & b_{1j} \\ b_{2i} & b_{2j} \end{matrix} \right|,$$

where $b_{12} = b_{13} = b_{14} = b_{21} = b_{25} = b_{26} = 0$.

The linear envelope of the vectors $\mathbf{b}_i = (b_{i1},\, b_{i2},\, b_{i3},\, b_{i4}\, b_{i5},\, b_{i6})^{\mathrm{T}}$ $(i = 1,\, 2)$ is denoted by $\langle \mathbf{b}_1,\, \mathbf{b}_2 \rangle$.

Theorem 1. *Suppose the following conditions are satisfied*

$$\operatorname{rank} A = \operatorname{rank} B = 2. \tag{4}$$

If the eigenvalues $\{\lambda_k\}$ of problem (1) and the eigenvalues $\{\widetilde{\lambda}_k\}$ of problem (3) are identical, with their multiplicities taken into account, then the linear envelopes $\langle \mathbf{a}_1,\, \mathbf{a}_2 \rangle$ and $\langle \mathbf{b}_1,\, \mathbf{b}_2 \rangle$ are also identical.

This theorem can be proved by methods of [4].

4. The uniqueness of the reconstructing boundary conditions (i)–(vii) from five eigenvalues

Theorem 1 uses all natural frequencies to determine the boundary conditions (i)–(vii).

Let us show that five natural frequencies are sufficient to identify the boundary conditions (i)–(vii).

First, we show this for the following boundary conditions (case (vii)):

$$U_3(y,\lambda) = y'''(1) + (a_1 - a_2\lambda^4)y(1) = 0, \quad U_4(y,\lambda) = y''(1) + (a_3 - a_4\lambda^4)y'(1) = 0. \quad (5)$$

We note that the functions

$$\begin{array}{ll} y_1(x,\lambda) = (\cos\lambda x + \cosh\lambda x)/2, & y_2(x,\lambda) = (\sin\lambda x + \sinh\lambda x)/(2\lambda), \\ y_3(x,\lambda) = (-\cos\lambda x + \cosh\lambda x)/(2\lambda^2), & y_4(x,\lambda) = (-\sin\lambda x + \sinh\lambda x)/(2\lambda^3), \end{array} \quad (6)$$

are linearly independent solutions of the equation $y^{(4)} = \lambda^4 y$ which satisfy the conditions

$$y_j^{(r-1)}(0,\lambda) = \begin{cases} 0 & j \neq r, \\ 1 & j = r, \end{cases} \qquad j,\, r = 1,\, 2,\, 3,\, 4 \quad (7)$$

(in other words, the solutions $y_1(x,\lambda)$ $(j = 1,2,3,4)$ form a fundamental Cauchy system and are expressed in terms of Krylov functions [39, p. 194]).

The function

$$\Delta(\lambda) \equiv \det(\|U_i(y_j)\|_{i,j=1,2,3,4}) \equiv \begin{vmatrix} U_3(y_3,\lambda) & U_3(y_4,\lambda) \\ U_4(y_3,\lambda) & U_4(y_4,\lambda) \end{vmatrix},$$

(conditions (7) have been taken into account here) is the characteristic determinant of the boundary-value problem (1).

Applying Laplace's theorem for evaluating determinants and using trigonometric formulae and equalities (6), we obtain

$$\begin{array}{l} \Delta(\lambda) \equiv -f_0(\lambda) + a_1 f_1(\lambda) + a_2 f_2(\lambda) + a_3 f_3(\lambda) + a_4 f_4(\lambda) + (a_1 a_4 + a_2 a_3) f_5(\lambda) + \\ \qquad + a_1 a_3 f_6(\lambda) + a_2 a_4 f_7(\lambda), \end{array} \quad (8)$$

where

$$\begin{array}{ll} f_0(\lambda) = (1 + \cos\lambda\cosh\lambda)/2; & f_1(\lambda) = (-\cos\lambda\sinh\lambda + \sin\lambda\cosh\lambda)/(2\lambda^3); \\ f_2(\lambda) = -\lambda^4 f_1(\lambda); & f_3(\lambda) = -(\sin\lambda\cosh\lambda + \cos\lambda\sinh\lambda)/(2\lambda); \\ f_4(\lambda) = -\lambda^4 f_3(\lambda); & f_5(\lambda) = (\cos\lambda\cosh\lambda - 1)/2; \\ f_6(\lambda) = -f_5(\lambda)/\lambda^4; & f_7(\lambda) = -\lambda^4 f_5(\lambda). \end{array}$$

Then, the inverse problem, i.e., the problem of identifying the boundary conditions (5) from the natural frequencies, can be formulated in terms of function (8) as follows: the roots λ_k of characteristic determinant (8) are known. It is necessary to identify the coefficients a_i $(i = 1,2,3,4)$.

We substitute the values λ_k, $(j = 1,2,3,4,5)$ which are the first five eigenvalues of problem (1), into (8). We obtain a system of five algebraic equations in four unknowns

$$\begin{array}{l} a_3 f_3(\lambda_k) + a_4 f_4(\lambda_k) + (a_1 a_4 + a_2 a_3) f_5(\lambda_k) + a_1 a_3 f_6(\lambda_k) + a_2 a_4 f_7(\lambda_k) = \\ = f_0(\lambda_k) - a_1 f_1(\lambda_k) - a_2 f_2(\lambda_k), \qquad k = 1,2,...,5. \end{array} \quad (9)$$

If the determinant

$$\Delta_0 = \det(\| \; f_3(\lambda_j) \quad f_4(\lambda_j) \quad f_5(\lambda_j) \quad f_6(\lambda_j) \quad f_7(\lambda_j) \; \|_{j=1,2,3,4,5}), \quad (10)$$

of system (9) w.r.t. unknowns a_3, a_4, $(a_1\,a_4 + a_2\,a_3)$, $a_1\,a_3$, $a_2\,a_4$ is not equal to zero, then system of equations (9) has the unique solution determined, for example, by Cramer's formulae:

$$a_3 = \frac{\Delta_1}{\Delta_0}, \quad a_4 = \frac{\Delta_2}{\Delta_0}, \tag{11}$$

$$a_1\,a_4 + a_2\,a_3 = \frac{\Delta_3}{\Delta_0}, \quad a_1\,a_3 = \frac{\Delta_4}{\Delta_0}, \quad a_2\,a_4 = \frac{\Delta_5}{\Delta_0}. \tag{12}$$

Here the determinants Δ_i $(i = 1, 2, 3, 4, 5)$ are obtained from the determinant Δ_0 by replacing its ith column by the column of the right hand sides in the system of equations (9).

Substituting (11) into (12), we obtain the following system of algebraic equations w.r.t. unknown coefficients a_i $(i = 1, 2, 3, 4)$:

$$\begin{array}{c} \Delta_1^1\,(a_1)^2 - \left(\Delta_0^1 + \Delta_1^4\right)\,a_1 - \Delta_2^4\,a_2 + \Delta_2^1\,a_1\,a_2 + \Delta_0^4 = 0, \\ \Delta_2^2\,(a_2)^2 - \left(\Delta_0^2 + \Delta_2^5\right)\,a_2 - \Delta_1^5\,a_1 + \Delta_1^2\,a_1\,a_2 + \Delta_0^5 = 0, \\ \Delta_1^2\,(a_1)^2 + \Delta_2^1\,(a_2)^2 - \left(\Delta_0^2 + \Delta_1^3\right)\,a_1 - \\ - \left(\Delta_0^1 + \Delta_2^3\right)a_2 + \left(\Delta_1^1 + \Delta_2^2\right)\,a_1\,a_2 + \Delta_0^3 = 0, \end{array} \tag{13}$$

where

$$\begin{array}{ll} \Delta_i^1 = \det\left(\|\ f_i(\lambda_j)\quad f_4(\lambda_j)\quad f_5(\lambda_j)\quad f_6(\lambda_j)\quad f_7(\lambda_j)\ \|_{j=1,2,3,4,5}\right), & i = 0, 1, 2, \\ \Delta_i^2 = \det\left(\|\ f_3(\lambda_j)\quad f_i(\lambda_j)\quad f_5(\lambda_j)\quad f_6(\lambda_j)\quad f_7(\lambda_j)\ \|_{j=1,2,3,4,5}\right), & i = 0, 1, 2, \\ \Delta_i^3 = \det\left(\|\ f_3(\lambda_j)\quad f_4(\lambda_j)\quad f_i(\lambda_j)\quad f_6(\lambda_j)\quad f_7(\lambda_j)\ \|_{j=1,2,3,4,5}\right), & i = 0, 1, 2, \\ \Delta_i^4 = \det\left(\|\ f_3(\lambda_j)\quad f_4(\lambda_j)\quad f_5(\lambda_j)\quad f_i(\lambda_j)\quad f_7(\lambda_j)\ \|_{j=1,2,3,4,5}\right), & i = 0, 1, 2, \\ \Delta_i^5 = \det\left(\|\ f_3(\lambda_j)\quad f_4(\lambda_j)\quad f_5(\lambda_j)\quad f_6(\lambda_j)\quad f_i(\lambda_j)\ \|_{j=1,2,3,4,5}\right), & i = 0, 1, 2. \end{array} \tag{14}$$

$(\Delta_j = \Delta_0^j - a_1\Delta_1^j - a_2\Delta_2^j,\ \ j = 1, 2, 3, 4, 5)$.

This proves

Theorem 2. *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be eigenvalues of boundary problem (1), where $U_3(y, \lambda)$ and $U_4(y, \lambda)$ have the form (5). If the determinant (10) is not equal to zero and the system (13) has the unique solution, then the coefficients a_i, $i = 1, 2, 3, 4$ are uniquely determined by the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$.*

The proven theorem justify the possibility of applying numerical experiment to the problem of determining the boundary conditions. Below, we present this experiment.

Example 1. *Let $\lambda_1 = 3,046515$, $\lambda_2 = 4.782842$, $\lambda_3 = 7.815980$, $\lambda_4 = 10.95883$, $\lambda_5 = 14.10555$ be the first five eigenvalues of Problem (1), where $U_3(y, \lambda)$ and $U_4(y, \lambda)$ have the form (5). The determinant (10) is not equal to zero ($\Delta_0 = 3,605140 \cdot 10^{15}$). From (14) it follows that*

$$\Delta_0^1 = -1.389961 \cdot 10^{13}, \Delta_0^2 = -3.074774 \cdot 10^{11}, \Delta_0^3 = 5.380211 \cdot 10^{15},$$

$$\Delta_0^4 = 2.115762 \cdot 10^{17}, \Delta_0^5 = -6.655016 \cdot 10^{11},$$

$$\Delta_1^1 = -1.213977 \cdot 10^{14}, \Delta_1^2 = -1.255696 \cdot 10^{9}, \Delta_1^3 = -2.330674 \cdot 10^{14},$$

$$\Delta_1^4 = -6.299423 \cdot 10^{15}, \Delta_1^5 = -2.689428 \cdot 10^{9},$$

$$\Delta_2^1 = -5.411625 \cdot 10^{15}, \Delta_2^2 = -1.263336 \cdot 10^{14}, \Delta_2^3 = -8.125798 \cdot 10^{15},$$

$$\Delta_2^4 = 1.056752 \cdot 10^{17}, \Delta_2^5 = -2.526925 \cdot 10^{14}.$$

Substituting these values into (13), we obtain a system of algebraic equations which has the following unique solution: $a_1 = 0.050, a_2 = 2.000, a_3 = 3.000, a_4 = 0.070$.

Theorem 3. *Let* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ *be the first five eigenvalues of the boundary problem (1), where* $U_3(y)$ *and* $U_4(y)$ *have the general form (2). If* $\operatorname{rank} A = 2$, *then the boundary conditions (2) are uniquely determined by the eigenvalues* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$.

Proof. Cases (iii), (v), and (vi) of boundary conditions (2) are the special cases of (vii). So for these cases Theorem 3 is proved. Cases (i) clamping, (ii) free support, and (iv) floating fixing are not special cases of (vii). So we need to exclude the existance of the eigenvalue problem with boundary conditions (vii), which is different from cases (i), (ii), and (iv), but has the same first five eigenvalues as in (i), (ii), and (iv).

Case (i): rigid clamping. For the rigid clamping case, the first five eigenvalues of problem (1) are

$$\begin{gathered} \lambda_1 = 4.730041, \quad \lambda_2 = 7.853205, \quad \lambda_3 = 10.99561, \\ \lambda_4 = 14.137167, \quad \lambda_5 = 17.27876. \end{gathered} \tag{15}$$

They are the roots of the function $f_5(\lambda)$. Hence determinant (10) is equal to zero and we can not use the methods of the proof of Theorem 2.

We use other method of finding a_i $(i = 1, 2, 3, 4)$. The eigenvalues (15) are roots of the functions $f_5(\lambda)$, $f_6(\lambda)$, and $f_7(\lambda)$. So substituting the known eigenvalues (15) into (9), we obtain a system of linear algebraic equations for unknown coefficients a_i $(i = 1, 2, 3, 4)$:

$$a_1\, f_1(\lambda_k) + a_2\, f_2(\lambda_k) + a_3\, f_3(\lambda_k) + a_4\, f_4(\lambda_k) = f_0(\lambda_k), \qquad k = 1, 2, ..., 5. \tag{16}$$

The determinant of first four equations of (16) w.r.t. unknowns a_1, a_2, a_3, a_4 is $-2.332 \cdot 10^{14} \neq 0$. Hence, the system of first four equations of (16) has the following unique solution:

$$a_1 = -8.73, \quad a_2 = 0.274 \cdot 10^{-2}, \quad a_3 = -0.294, \quad a_4 = 0.744 \cdot 10^{-5}. \tag{17}$$

The determinant of four equations of (16) as $k = 1, 2, 3, 5$ w.r.t. unknowns a_1, a_2, a_3, a_4 is $1.76 \cdot 10^{16} \neq 0$. Hence, the system of four equations for $k = 1, 2, 3, 5$ of (16) has the following unique solution:

$$a_1 = -8.31, \quad a_2 = 0.217 \cdot 10^{-2}, \quad a_3 = -0.263, a_4 = 0.464 \cdot 10^{-5}. \tag{18}$$

Solutions (17) and (18) do not coincide. So system (16) has no solution. Consequently, type (i) is the only one of (i)–(vii) which provides that the values (15) are the first five eigenvalues of problem (1).

Case (iv): floating fixing. For the floating fixing case, the first five eigenvalues of problem (1) are

$$\begin{gathered}\lambda_1 = 2.365020, \quad \lambda_2 = 5.497804, \quad \lambda_3 = 8.639380, \\ \lambda_4 = 11.78097, \quad \lambda_5 = 18.06416.\end{gathered} \tag{19}$$

They are the roots of the function $f_3(\lambda)$. Therefore, determinant (10) is equal to zero and as above we can not use the methods of the proof of Theorem 2.

Solving the system of nonlinear equations

$$\begin{gathered}a_1\, f_1(\lambda_k) + a_2\, f_2(\lambda_k) + a_3\, f_3(\lambda_k) + a_4\, f_4(\lambda_k) + (a_1\, a_4 + a_2\, a_3)\, f_5(\lambda_k) + \\ + a_1\, a_3\, f_6(\lambda_k) + a_2\, a_4\, f_7(\lambda_k) = f_0(\lambda_k),\end{gathered} \tag{20}$$

we obtain that the solutions set of the system of equations (20) is empty.

Consequently, type (iv) is the only one of (i)–(vii) which provides that the values (15) are the first five eigenvalues of problem (1).

Case (ii): free support. For the free support case, the first five eigenvalues of problem (1) are

$$\begin{gathered}\lambda_1 = 3.926602, \quad \lambda_2 = 7.068583, \quad \lambda_3 = 10.21018, \\ \lambda_4 = 13.35177, \quad \lambda_5 = 16.49336.\end{gathered} \tag{21}$$

They are the roots of the function $f_1(\lambda)$. Determinant (10) is equal to $D_0 = 2.829 \cdot 10^{18} \neq 0$ and we can use the methods of the proof of Theorem 2. The result is a system of three equations (13) w.r.t. unknown a_2. The set of solutiuons of this system is empty. Consequently, type (ii) is the only one of (i)–(vii) which provides that the values (21) are the first five eigenvalues of problem (1).

Cases (iii), (v), (vi), and (vii) are boundary conditions of problem (1), where $U_3(y)$ and $U_4(y)$ have the form (5). From Theorem 2 it follows that if the determinant (10) is not equal to zero and the system (13) has the unique solution, then the coefficients a_i, $i = 1, 2, 3, 4$ are uniquely determined by the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$. This completes the proof of Theorem 3. ◄

The following question arises: taking into account that, from five eigenvalues, we obtain a unique solution, is it possible to obtain a unique solution to the problem of determining boundary conditions (2) by using four of eigenvalues of problem (5)? The answer is no.

Example 2. *Let* $\lambda_1 = 3.046515$*,* $\lambda_2 = 4.782842$*,* $\lambda_3 = 7.815980$*,* $\lambda_4 = 10.95883$ *be the first four eigenvalues of Problem (5) from Example 1. Two natural problems (1),(5) have these eigenvalues. For the first one, the coefficients* a_i *(*$i = 1, 2, 3, 4$*) of the form (5) are*

$$a_1 = 147.3199, \quad a_2 = 1.984184, \quad a_3 = 4.449441, \quad a_4 = 0.066191.$$

For the second problem, the coefficients a_i *(*$i = 1, 2, 3, 4$*) of the form (5) are:*

$$a_1 = 0.05000, \quad a_2 = 2.00000, \quad a_3 = 3.00000, \quad a_4 = 0.07000.$$

Note that the second problem is the problem of Example 1.

5. The special cases of the reconstructing boundary conditions from eigenvalues

1. Cases (i)-(vi). How many eigenvalues do we need to uniquelly reconstruct boundary conditions (i)-(vi) (without (vii))? The answer is three.

Let $\lambda_1, \lambda_2, \lambda_3$ be the first five eigenvalues of boundary problem (1), where $U_3(y)$ and $U_4(y)$ have the general form (2). By definition, put

$$F_1 = \left\| \begin{matrix} f_0(\lambda_1) & f_1(\lambda_1) & f_3(\lambda_1) & f_6(\lambda_1) \\ f_0(\lambda_2) & f_1(\lambda_2) & f_3(\lambda_2) & f_6(\lambda_2) \\ f_0(\lambda_3) & f_1(\lambda_3) & f_3(\lambda_3) & f_6(\lambda_3) \end{matrix} \right\|, \quad F_1 = \left\| \begin{matrix} f_0(\lambda_1) & f_2(\lambda_1) & f_4(\lambda_1) & f_7(\lambda_1) \\ f_0(\lambda_2) & f_2(\lambda_2) & f_4(\lambda_2) & f_7(\lambda_2) \\ f_0(\lambda_3) & f_2(\lambda_3) & f_4(\lambda_3) & f_7(\lambda_3) \end{matrix} \right\|.$$

Theorem 4. *If* $\operatorname{rank} A = 2$, $\operatorname{rank} F_1 = \operatorname{rank} F_2 = 3$, *then boundary conditions (i)-(vi) are uniquely determined by three eigenvalues* $\lambda_1, \lambda_2, \lambda_3$.

Proof of Theorem 4 and a method of solving the problem are given in [12].

In the next two examples, we consider the cases in which the use of the third eigenvalue is essential for the unique reconstruction of boundary conditions (i)-(vi).

Example 3. *Suppose* $a_2 = a_4 = 0$; *and* $\lambda_1 = 4.639942$, $\lambda_2 = 7.855696$ *are the first two eigenvalues of Problem (5). Then there are two natural problems (1),(5) which have these eigenvalues. For the first problem* $a_1 = 10.000$, $a_3 = 0.170$, *and for the second problem* $a_1 = 1677.848$, $a_2 = 27.527$.

Example 4. *Suppose* $a_1 = a_3 = 0$; *and* $\lambda_1 = 1.671009$, $\lambda_2 = 2.764306$ *are the first two eigenvalues of Problem (5). Then there are two natural problems (1),(5) which have these eigenvalues. For the first problem* $a_2 = 0.0200$, $a_4 = 0.7000$, *and for the second problem* $a_2 = 0.1531$, $a_4 = 0.1137$.

2. Cases (i)–(vii) when one of the parameters a_i, $i = 1, 2, 3, 4$ is equal to zero. In case (vii), similar to the proof of Theorem 2, we can prove that four eigenvalues are sufficient to recover boundary conditions (vii) uniquely. So it follows from the proof of Theorem 3 that to recover boundary conditions (i)–(vii) uniquely we need the first four eigenvalues of problem (1),(5).

In the next two examples, we consider the cases in which the use of the fourth eigenvalue is essential for the unique reconstruction of boundary conditions (i)–(vii) when one of the parameters a_i, $i = 1, 2, 3, 4$ is equal to zero.

Example 5. *Suppose* $a_4 = 0$; *and* $\lambda_1 = 2.289615$, $\lambda_2 = 4.415309$, $\lambda_3 = 7.648880$ *are the first three eigenvalues of Problem (5). Then there are two natural problems (1),(5) which have these eigenvalues. For the first problem* $a_1 = 0.020$, $a_2 = 0.700$, $a_3 = 0.100$, *and for the second problem* $a_1 = 4.260$, $a_2 = 0.695$, $a_3 = 0.090$.

Example 6. *Suppose* $a_1 = 0$; *and* $\lambda_1 = 2.422582$, $\lambda_2 = 6.438223$, $\lambda_3 = 10.11708$ *are the first three eigenvalues of Problem (5). Then there are two natural problems (1),(5) which have these eigenvalues. For the first problem* $a_2 = 0.164$, $a_3 = 0.3$, $a_4 = 0.004$, *and for the second problem* $a_2 = 0.180$, $a_3 = 0.3$, $a_4 = 0.084$.

6. Conclusions

Three eigenvalues are sufficient for unique recovery of boundary conditions (i)–(vi) with two unknown coefficients. Four eigenvalues are sufficient for unique recovery of boundary conditions (i)-(vii) with three unknown coefficients. Five eigenvalues are sufficient for unique recovery of boundary conditions (i)-(vii) with four unknown coefficients. A smaller set of the eigenvalues of problem (1) is not enough for unique recovery of boundary conditions (i)-(vii) in these cases.

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