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Uniqueness Theorems for Subharmonic and Holomorphic Functions of Several Variables on a Domain

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Abstract. We establish a general uniqueness theorem for subharmonic functions of several variables on a domain. A corollary from this uniqueness theorem for holomorphic functions is formulated in terms of the zero subset of holomorphic functions and restrictions on the growth of functions near the boundary of domain.

Key Words and Phrases: holomorphic function, zero set, uniqueness set, subharmonic function, Riesz measure, Jensen measure, potential, balayage.

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1. Introduction

1.1. Definitions and notations

We use an information and definitions from [1, 2, 3, 4, 5]. As usual, $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all natural, real and complex numbers, resp. We set

$$\mathbb{R}^+ := \{x \in \mathbb{R} \colon x \geqslant 0\},\ \mathbb{R}_{-\infty} := \{-\infty\} \cup \mathbb{R},\ \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\},\ \mathbb{R}_{\pm\infty} := \mathbb{R}_{-\infty} \cup \mathbb{R}_{+\infty}, \tag{1}$$

where the usual order relation $\leqslant$ on $\mathbb{R}$ is complemented by the inequalities $-\infty \leqslant x \leqslant +\infty$ for all $x \in \mathbb{R}_{\pm\infty}$. Let $f \colon X \to Y$ be a function. Given $S \subset X$, we denote by $f\big|_S$ the restriction of f to S. For $Y \subset \mathbb{R}_{\pm\infty}$, $g \colon X \to \mathbb{R}_{\pm\infty}$ and $S \subset X$, we write "$f \leqslant g$ *on* S" if $f(x) \leqslant g(x)$ for all $x \in S$.

Let $m \in \mathbb{N}$. Denote by $\mathbb{R}^m$ the *m-dimensional Euclidian real space.* Then $\mathbb{R}^m_\infty := \mathbb{R}^m \cup \{\infty\}$ is the *Alexandroff* (⇔one-point) *compactification of* $\mathbb{R}^m$. Given a subset S of $\mathbb{R}^m$ (or $\mathbb{R}^m_\infty$), the closure $\operatorname{clos} S$, the interior $\operatorname{int} S$ and the boundary ∂S will always be taken relative to $\mathbb{R}^m_\infty$. Let $S_0 \subset S \subset \mathbb{R}^m_\infty$. If the closure $\operatorname{clos} S_0$ is a compact subset of S in the topology induced on S from $\mathbb{R}^m_\infty$, then the set S_0 is a *relatively compact subset* of S, and we write $S_0 \Subset S$.

Let $n \in \mathbb{N}$. Denote by $\mathbb{C}^n$ the *n-dimensional Euclidian complex space.* Then $\mathbb{C}^n_\infty := \mathbb{C}^n \cup \{\infty\}$ is the *Alexandroff* (⇔one-point) *compactification of* $\mathbb{C}^n$. If necessary, we identify

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$\mathbb{C}^n$ (or $\mathbb{C}^n_\infty$) with $\mathbb{R}^{2n}$ (or $\mathbb{R}^{2n}_\infty$). Given a subset S of $\mathbb{C}^n$ (or $\mathbb{C}^n_\infty$), its closure $\operatorname{clos} S$ and its boundary ∂S will always be taken relative to $\mathbb{C}^n_\infty$.

Let A, B be sets, and $A \subset B$. The set A is a *non-trivial subset* of the set B if the subset $A \subset B$ is non-empty ($A \neq \varnothing$) and *proper* ($A \neq B$).

We always understand the *"positivity"* or *"positive"* as $\geqslant 0$, where the symbol 0 denotes the number zero, the zero function, the zero measure, etc. So, a function $f\colon X \to R \overset{(1)}{\subset} \mathbb{R}_{\pm\infty}$ is positive on X if $f(x) \geqslant 0$ for all $x \in X$. In such case we write "$f \geqslant 0$ *on* X".

The class of all Borel real measures on local compact space X is denoted by $\mathcal{M}(X)$, $\mathcal{M}_{\mathrm{c}}(X)$ is the subclass of all Borel measures μ on X with compact support $\operatorname{supp}\mu \subset X$, and $\mathcal{M}^+(X) \subset \mathcal{M}(X)$ is the subclass of all Borel positive measures on X, $\mathcal{M}^+_{\mathrm{c}}(X) := \mathcal{M}^+(X) \cap \mathcal{M}_{\mathrm{c}}(X)$. Given $\mu \in \mathcal{M}(X)$ and $S \subset X$, we denote by $\mu\big|_S$ the restriction of ν to S. For $\nu \in \mathcal{M}(X)$, we write "$\nu \geqslant \mu$ *on* S" if $\big(\nu\big|_S - \mu\big|_S\big) \in \mathcal{M}^+(S)$.

Let $\mathcal{O}$ be a non-trivial open subset of $\mathbb{R}^m_\infty$. We denote by $\operatorname{sbh}(\mathcal{O})$ the class of all subharmonic functions $u\colon \mathcal{O} \to \mathbb{R}_{-\infty}$ on $\mathcal{O}$ for $m \geqslant 2$, and all (local) convex functions $u\colon \mathcal{O} \to \mathbb{R}_{-\infty}$ on $\mathcal{O}$ for $m = 1$. The class $\operatorname{sbh}(\mathcal{O})$ contains the function $\boldsymbol{-\infty}\colon x \mapsto -\infty$, $x \in \mathcal{O}$ (identical to $-\infty$); $\operatorname{sbh}^+(\mathcal{O}) := \{u \in \operatorname{sbh}(\mathcal{O})\colon u \geqslant 0 \text{ on } \mathcal{O}\}$. We set $\operatorname{sbh}_*(\mathcal{O}) := \operatorname{sbh}(\mathcal{O}) \setminus \{\boldsymbol{-\infty}\}$. For $u \in \operatorname{sbh}_*(\mathcal{O})$, the *Riesz measure of* u is the Borel positive measure

$$\nu_u := c_m \, \Delta u \in \mathcal{M}^+(\mathcal{O}), \quad c_m := \frac{\Gamma(m/2)}{2\pi^{m/2}\max\{1, (m-2)\}}\,, \tag{2}$$

where Δ is the *Laplace operator* acting in the sense of distribution theory, and Γ is the gamma function. Such measures ν_u are Radon measures, i. e. $\nu_u(S) < +\infty$ for each subset $S \Subset \mathcal{O}$. By definition, $\nu_{\boldsymbol{-\infty}}(S) := +\infty$ for all $S \subset \mathcal{O}$.

Let $\mathcal{O}$ be a non-trivial open subset of $\mathbb{C}^n_\infty$. We denote by $\operatorname{Hol}(\mathcal{O})$ and $\operatorname{sbh}(\mathcal{O})$ the class of holomorphic and subharmonic functions on $\mathcal{O}$, resp. For $u \in \operatorname{sbh}_*(\mathcal{O})$, the *Riesz measure of* u is the Borel (and Radon) positive measure

$$\nu_u := c_{2n} \, \Delta u \in \mathcal{M}^+(\mathcal{O}), \quad c_{2n} = \frac{(n-1)!}{2\pi^n \max\{1, 2n-2\}}\,.$$

1.2. Main Theorem and Corollary

Definition 1. *Let D be a non-trivial open connected subset of $\mathbb{R}^m_\infty$, i. e. D is a non-trivial domain in $\mathbb{R}^m_\infty$. Let K be a non-trivial compact subset of D, i. e. $\varnothing \neq K = \operatorname{clos} K \subset D$. A function $v \in \operatorname{sbh}^+(D \setminus K)$ is called a test function for D outside of K if*

$$\lim_{D \ni x' \to x} v(x') = 0 \quad \text{for each } x \in \partial D \quad \text{and} \quad \sup_{x \in D \setminus K} v(x) < +\infty. \tag{3}$$

The class of all such test functions for D outside of K is denoted by $\operatorname{sbh}^+_0(D \setminus K)$.

Our main result for subharmonic functions is the following

Theorem 1 (see [6, Corollary 1.1] for the case $m=2$)**.** *Let D be a non-trivial domain in $\mathbb{R}^m_\infty$, K a compact subset of D with non-empty interior* $\operatorname{int} K \neq \varnothing$. *Let $M \in \operatorname{sbh}_*(D)$ be a function with the Riesz measure $\nu_M \in \mathcal{M}^+(D)$, $v \in \operatorname{sbh}_0^+(D \setminus K)$ a test function for D outside of K. Assume that*

$$\int_{D\setminus K} v \,\mathrm{d}\nu_M < +\infty. \tag{4}$$

If $u \in \operatorname{sbh}(D)$ is a function with the Riesz measure $\nu_u \in \mathcal{M}^+(D)$ such that

$$\nu_u \geqslant \nu \in \mathcal{M}^+(D \setminus K) \text{ on } D \setminus K, \tag{5a}$$

$$\int_{D\setminus K} v \,\mathrm{d}\nu = +\infty, \tag{5b}$$

$$u \leqslant M + \mathrm{const} \text{ on } D, \tag{5c}$$

where const *is a constant, then $u = -\infty$.*

We denote by σ_{2n-2} the $(2n-2)$-*dimensional surface* ($\Leftrightarrow$Hausdorff) *measure* on $\mathbb{C}^n$ and its restrictions to subsets of $\mathbb{C}^n$. So, if $n=1$, i.e. $2n-2=0$, then $\sigma_0(S) = \sum_{z\in S} 1$ for each $S \subset \mathbb{C}$, i.e. $\sigma_0(S)$ is equal to the number of points in the set $S \subset \mathbb{C}$.

Below we identify $\mathbb{C}^n$ (or $\mathbb{C}^n_\infty$) with $\mathbb{R}^m$ (or $\mathbb{R}^m_\infty$) where $m = 2n$.

Our main result for holomorphic functions is the following

Corollary 1. *Let all conditions of Theorem* 1 *be fulfilled including* (4)*. Let $f \in \operatorname{Hol}(D)$ be a holomorphic function on D and* $\operatorname{Zero}_f := \{z \in D \colon f(z) = 0\}$. *If*

$$\mathsf{Z} \subset (D \setminus K) \cap \operatorname{Zero}_f, \tag{6a}$$

$$\int_{\mathsf{Z}} v \,\mathrm{d}\sigma_{2n-2} = +\infty, \tag{6b}$$

$$|f| \leqslant \mathrm{const}\, e^M \text{ on } D, \tag{6c}$$

where const *is a constant, then $f \equiv 0$ on D, i.e.* $\operatorname{Zero}_f = D$.

Proof. Under the conditions of Corollary 1, suppose that $f \neq 0$. Then we have $\log|f| \in \operatorname{sbh}_*(D)$ with the Riesz measure $\nu_{\log|f|} \in \mathcal{M}^+(D)$. Let $n_f \colon D \to \{0\} \cup \mathbb{N}$ be the *multiplicity function of* f [7, 4]. It is known that $\operatorname{supp} n_f = \operatorname{Zero}_f$. By the classical Poincaré–Lelong formula [8] we have $n_f \,\mathrm{d}\sigma_{2n-2} = \mathrm{d}\nu_{\log|f|}$ on D. Hence, if the condition (6a) is fulfilled, then we get

$$\int_{\mathsf{Z}} v \,\mathrm{d}\sigma_{2n-2} \overset{(6a)}{\leqslant} \int v \, n_f \,\mathrm{d}\sigma_{2n-2} \leqslant \int v \,\mathrm{d}\nu_{\log|f|}. \tag{7}$$

If the condition (6c) is also fulfilled, then for $u := \log|f|$ we have (5c) together with (5a) for $\nu = \nu_u = \nu_{\log|f|}$. Since $u \neq -\infty$, by Theorem 1 we obtain the negation of the equality (5b). Therefore we get

$$\int_{\mathsf{Z}} v \,\mathrm{d}\sigma_{2n-2} \overset{(7)}{\leqslant} \int v \,\mathrm{d}\nu_{\log|f|} = \int v \,\mathrm{d}\nu_u < +\infty$$

which contradicts (6b). Corollary 1 is proved. ◄

2. Main results

2.1. Gluing Theorem for $m \in \mathbb{N}$

The next result shows how two subharmonic functions can be glued together.

Theorem 2 (see [4, Corollary 2.4.5], and also [2, Theorem 2.4.5] for $m = 2$)**.** *Let $\mathcal{O}, O_0$ be open sets in $\mathbb{R}^m_\infty$, and $\mathcal{O} \subset \mathcal{O}_0$. Let $v_0 \in \operatorname{sbh}(\mathcal{O}_0)$, and $v \in \operatorname{sbh}(\mathcal{O})$. If*

$$\limsup_{\mathcal{O} \ni x' \to x} v(x') \leqslant v_0(x) \quad \text{for all points } x \in \mathcal{O}_0 \cap \partial\mathcal{O}, \tag{8}$$

then the function

$$\widetilde{v} := \begin{cases} \max\{v, v_0\} & \text{on } \mathcal{O}, \\ v_0 & \text{on } \mathcal{O}_0 \setminus \mathcal{O}, \end{cases} \tag{9}$$

belongs to the class $\operatorname{sbh}(\mathcal{O}_0)$.

Remark 1. A similar gluing theorem is true also for classes of plurisubharmonic functions [4, Corollary 2.9.5].

2.2. Jensen measures and potentials

Let $m \in \mathbb{N}$. Given $t \in \mathbb{R}_* := \mathbb{R} \setminus \{0\}$, we set

$$h_m(t) := \begin{cases} |t| & \text{for } m = 1, \\ \log |t| & \text{for } m = 2, \\ -\dfrac{1}{|t|^{m-2}} & \text{for } m \geqslant 3. \end{cases}$$

For simplicity, we consider only domains D in $\mathbb{R}^m \subset \mathbb{R}^m_\infty$, i. e. $\infty \notin D$.

Definition 2 ([9]–[13])**.** *Let $D \subset \mathbb{R}^m$ be a subdomain, $x_0 \in D$. A measure $\mu \in \mathcal{M}^+_{\mathrm{c}}(D)$ is called the Jensen measure for* $\operatorname{sbh}(D)$ *at $x_0 \in D$ if*

$$u(x_0) \leqslant \int u \, \mathrm{d}\mu \quad \text{for all } u \in \operatorname{sbh}(D).$$

By $J_{x_0}(D)$ we denote the class of all Jensen measures for D at x_0. Each Jensen measure $\mu \in J_{x_0}(D)$ is a probability measure, i. e. $\mu(D) = 1$.

For $\mu \in J_{x_0}(D)$ we shall say that the function

$$V_\mu(x) := \int h_m(x - y) \, \mathrm{d}\mu(y) - h_m(x - x_0), \quad x \in \mathbb{R}^m_\infty \setminus \{x_0\}, \tag{10}$$

is a potential of the Jensen measure $\mu \in J_0(D)$.

A function $V \in \operatorname{sbh}^+\big(\mathbb{R}^m_\infty \setminus \{x_0\}\big)$ is called the Jensen potential inside of D with pole at $x_0 \in D$ if the following two conditions hold:

(i) *there is a compact subset* $K_V \subset D$ *such that* $V \equiv 0$ *on* $\mathbb{R}^m_\infty \setminus K_V$ (finiteness),

(ii) $\limsup\limits_{x_0 \neq x \to x_0} \dfrac{V(x)}{\bigl|h_m(x-x_0)\bigr|} \leqslant 1$ (semi-normalization at x_0).

By $PJ_{x_0}(D)$ *we denote the class of all Jensen potentials inside of* D *with pole at* $x_0 \in D$.

We present interrelations between Jensen measures and potentials. The first is

Proposition 1 ([13, Proposition 1.4, Duality Theorem])**.** *The map*

$$\mathcal{P}\colon J_{x_0}(D) \to PJ_{x_0}(D), \quad \mathcal{P}(\mu) \overset{(10)}{:=} V_\mu, \quad \mu \in J_{z_0}(D), \tag{11}$$

is the bijection from $J_{x_0}(D)$ *to* $PJ_{x_0}(D)$ *such that* $\mathcal{P}\bigl(t\mu_1+(1-t)\mu_2\bigr) = t\mathcal{P}(\mu_1)+(1-t)\mathcal{P}(\mu_2)$ *for all* $t \in [0,1]$ *and for all* $\mu_1, \mu_2 \in J_{x_0}(D)$*. Besides,*

$$\mathcal{P}^{-1}(V) \overset{(2)}{=} c_m\, \Delta V\,\Bigl|_{D\setminus\{x_0\}} + \left(1 - \limsup_{x_0 \neq x \to x_0} \frac{V(x)}{|h_m(x-x_0)|}\right) \cdot \delta_{x_0}\,, \quad V \in PJ_{x_0}(D), \tag{12}$$

where δ_{x_0} *is the Dirac measure at the point* x_0*, i. e.* $\operatorname{supp}\delta_{x_0} = \{x_0\}$ *and* $\delta_{x_0}\bigl(\{x_0\}\bigr) = 1$*.*

The second is a generalized Poisson – Jensen formula.

Proposition 2 ([13, Proposition 1.2])**.** *Let* $\mu \in J_{x_0}(D)$*. For each function* $u \in \operatorname{sbh}(D)$ *with* $u(x_0) \neq -\infty$ *and the Riesz measure* $\nu_u \in \mathcal{M}^+(D)$ *we have the equality*

$$u(x_0) + \int_{D\setminus\{x_0\}} V_\mu \, d\nu_u = \int_D u\, d\mu. \tag{13}$$

Given $x \in \mathbb{R}^m$ and $r \overset{(1)}{\in} \mathbb{R}^+$, we set $B(x,r) := \{x' \in \mathbb{R}^m \colon |x'-x| < r\}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$; $B_*(x,r) := B(x,r) \setminus \{x\}$; $\overline{B}(x,r) := \operatorname{clos} B(x,r)$.

By $\operatorname{har}(\mathcal{O})$ denote the class of all harmonic functions on an open subset $\mathcal{O} \subset \mathbb{R}^m$.

Corollary 2. *Let* D *be a domain in* $\mathbb{R}^m$*,* $x_0 \in D$*,* $r_0 > 0$*, and* $B(x_0, r_0) \Subset D$*;* $b \in \mathbb{R}^+$*. If the functions* $u \in \operatorname{sbh}_*(D)$ *with the Riesz measure* ν_u *and* $M \in \operatorname{sbh}_*(D)$ *with the Riesz measure* ν_M *satisfy the inequality*

$$u \leqslant M + \mathrm{const} \quad \text{on } D, \tag{14}$$

then there is a constant $C \in \mathbb{R}$ *such that*

$$\int_{D\setminus\overline{B}(x_0,r_0)} V \,\mathrm{d}\nu_u \leqslant \int_{D\setminus\overline{B}(x_0,r_0)} V\,\mathrm{d}\nu_M + C \tag{15}$$

for all functions $V \in PJ_{x_0}(D)$ *satisfying the following three conditions:*

$$\begin{gathered} V\,\Bigl|_{B_*(x_0,r_0)} \in \operatorname{har}\bigl(B_*(x_0,r_0)\bigr), \\ \limsup_{x_0 \neq x \to x_0} \frac{V(x)}{\bigl|h_m(x-x_0)\bigr|} \overset{\text{(ii)}}{=} 1 \quad \text{(normalization at } x_0), \\ \sup_{x \in \partial B(x_0,r_0)} V(x) \leqslant b. \end{gathered} \tag{16}$$

Proof. The technique of balayage out from the ball $B(x_0,r_0)$ gives two functions $u_0\in \operatorname{sbh}_*(D)$ with the Riesz measure ν_{u_0} and $M_0\in \operatorname{sbh}_*(D)$ with the Riesz measure ν_{M_0} such that

$$u_0\,\Big|_{B(x_0,r_0)}\in \operatorname{har}\bigl(B(x_0,r_0)\bigr),\quad u_0=u \text{ on } D\setminus B(x_0,r_0), \tag{17u}$$

$$M_0\,\Big|_{B(x_0,r_0)}\in \operatorname{har}\bigl(B(x_0,r_0)\bigr),\quad M_0=M \text{ on } D\setminus B(x_0,r_0),$$

$$u_0 \overset{(14)}{\leqslant} M_0+C_0 \quad \text{on } D \quad \text{where } C_0\in\mathbb{R}^+ \text{ is a constant}, \tag{17b}$$

$$\operatorname{supp}\nu_{M_0}\subset D\setminus B(x_0,r_0),\quad \nu_{M_0}\bigl(\partial B(x_0,r_0)\bigr)=\nu_M\bigl(\overline{B}(x_0,r_0)\bigr) \tag{17n}$$

$$\nu_{M_0}\,\Big|_{D\setminus\overline{B}(x_0,r_0)}=\quad \nu_M\,\Big|_{D\setminus\overline{B}(x_0,r_0)}\ . \tag{17r}$$

By Proposition 1, the measure $\mu\overset{(12)}{:=}\mathcal{P}^{-1}(V)\overset{(11)}{\in} J_{x_0}(D)$ satisfies the following conditions: $\operatorname{supp}\mu\overset{(16)}{\subset} D\setminus B(x_0,r_0)$, $\mu(D)=1$. The inequality (17b) entails the inequality

$$\int u_0\,\mathrm{d}\mu\leqslant\int M_0\,\mathrm{d}\mu+C_0.$$

Hence by Proposition 2

$$\int_{D\setminus\overline{B}(x_0,r_0)} V\,\mathrm{d}\nu_u \overset{(17u)}{\leqslant}\int_D V\,\mathrm{d}\nu_{u_0}\overset{(13)}{\leqslant}\int_D V\,\mathrm{d}\nu_{M_0}+\bigl(C_0-u_0(x_0)+M_0(x_0)\bigr).$$

Put $C_1:=C_0-u_0(x_0)+M_0(x_0)\in\mathbb{R}$. We continue this inequality as

$$\begin{aligned}\int_{D\setminus\overline{B}(x_0,r_0)} V\,\mathrm{d}\nu_u &\overset{(17n)}{\leqslant}\int_{D\setminus\overline{B}(x_0,r_0)} V\,\mathrm{d}\nu_{M_0}+\int_{B(x_0,r_0)} V\,\mathrm{d}\nu_{M_0}+C_1\\ &\overset{(17r)}{=}\int_{D\setminus\overline{B}(x_0,r_0)} V\,\mathrm{d}\nu_{M}+\int_{\partial B(x_0,r_0)} V\,\mathrm{d}\nu_{M_0}+C_1\\ &\overset{(16)}{\leqslant}\int_{D\setminus\overline{B}(x_0,r_0)} V\,\mathrm{d}\nu_{M}+b\,\nu_{M_0}\bigl(\overline{B}(x_0,r_0)\bigr)+C_1\\ &\overset{(17n)}{=}\int_{D\setminus\overline{B}(x_0,r_0)} V\,\mathrm{d}\nu_{M}+b\,\nu_{M}\bigl(\overline{B}(x_0,r_0)\bigr)+C_1.\end{aligned}$$

We choose $C:=b\,\nu_M\bigl(\overline{B}(x_0,r_0)\bigr)+C_1$ and obtain the inequality (15). ◀

2.3. Continuation of test functions

Proposition 3. *Let $v\in \operatorname{sbh}_0^+(D\setminus K)$ be a test function for D outside of K, $r_0>0$, and $B(x_0,2r_0)\subset K$. Then there are subdomains $D_0\Subset D_1\Subset D$, a number $r_0>0$ and a function $\widetilde{v}\in \operatorname{sbh}^+\bigl(D\setminus\{x_0\}\bigr)$ such that*

$$\overline{B}(x_0,2r_0)\subset K\subset D_0, \tag{18a}$$

$$\widetilde{v}\,\Big|_{B_*(x_0,2r_0)} \in \operatorname{har}\bigl(B_*(x_0,2r_0)\bigr), \tag{18b}$$

$$\exists \lim_{x_0\neq x\to x_0} \frac{\widetilde{v}(x)}{\bigl|h_m(x-x_0)\bigr|} \in (0,+\infty), \tag{18c}$$

$$\widetilde{v} = v \;\; on\; D\setminus D_1. \tag{18d}$$

Proof. Obviously, there is a subdomain $D_0 \Subset D$ satisfying (18a). Besides, there is a subdomain $D_1 \Subset D$ such that this domain D_1 is regular for the Dirichlet problem and $D_0 \Subset D_1$. Let $g_{D_1}(\cdot, x_0)$ be the Green's function of D_1 with the pole x_0 where $g_{D_1}(x,x_0) \equiv 0$ for all $x \in D\setminus D_1$. Then $g_{D_1}(\cdot,x_0) \in \operatorname{sbh}\bigl(D\setminus\{x_0\}\bigr)$. Put

$$q := \sup_{x\in\partial D_0} v(x), \quad a := \inf_{x\in\partial D_0} g_{D_1}(x,x_0) > 0, \quad v_0 := \frac{q}{a}\, g_{D_1}(\cdot, x_0),$$

and $\mathcal{O} := D\setminus \operatorname{clos} D_0$, $\mathcal{O}_0 := D\setminus\{x_0\}$. Then the condition (8) is fulfilled. By Theorem 2, the function (9) is the required one according to the known properties of the Green's function. ◀

2.4. Proof of Theorem 1

Proof. Let $B(x_0, 2r_0) \subset K$, where $r_0 > 0$. Suppose that the inequalities (5a), (5c) are fulfilled for $u \neq -\infty$. We must prove that the integral in (5b) with $\nu := \nu_u$ is finite. Consider the function $\widetilde{v}$ from Proposition 3 where

$$0 < c \overset{(18c)}{:=} \lim_{x_0\neq x\to x_0} \frac{\widetilde{v}(x)}{\bigl|h_m(x-x_0)\bigr|} < +\infty.$$

Then the function $V := \frac{1}{c}\,\widetilde{v}$ satisfies the following conditions

$$V\,\Big|_{B_*(x_0,2r_0)} \overset{(18b)}{\in} \operatorname{har}\bigl(B_*(x_0,2r_0)\bigr), \tag{19b}$$

$$\lim_{x_0\neq x\to x_0} \frac{V(x)}{\bigl|h_m(x-x_0)\bigr|} \overset{(18c)}{=} 1, \tag{19c}$$

$$V \overset{(18d)}{=} \frac{1}{c}\, v \text{ on } D\setminus D_1. \tag{19d}$$

$$\lim_{D\ni x'\to x} V(x') \overset{(3)}{=} 0 \quad \text{for each } x\in\partial D, \tag{19e}$$

$$\sup_{x\in\partial B(x_0,r_0)} V(x) =: b \overset{(3)}{<} +\infty. \tag{19f}$$

We put $V(x) \equiv 0$ at $x \in \mathbb{R}^m_\infty \setminus D$. Then $V \overset{(19e)}{\in} \operatorname{sbh}^+\bigl(\mathbb{R}^m_\infty\setminus\{x_0\}\bigr)$. Consider the sequence of functions $V_n := \max\{0, V - 1/n\}$, $\;n\in\mathbb{N}$. If $n_0 \in \mathbb{N}$ is a sufficiently large number, then every function V_n, $n \geqslant n_0$, is a Jensen function inside of D with the pole at $x_0 \in D$ such that

$$V_n\,\Big|_{B_*(x_0,r_0)} \overset{(19b)}{\in} \operatorname{har}\bigl(B_*(x_0,r_0)\bigr), \quad \lim_{x_0\neq x\to x_0} \frac{V_n(x)}{\bigl|h_m(x-x_0)\bigr|} \overset{(19c)}{=} 1, \quad \sup_{x\in\partial B(x_0,r_0)} V_n(x) \overset{(19f)}{\leqslant} b,$$

$$\frac{1}{c}\,v(x)=V(x)\geqslant V_n(x)\nearrow\frac{1}{c}\,v(x),\quad \text{for all } x\overset{(19d)}{\in} D\setminus D_1 \text{ when } n\to\infty. \tag{20}$$

Hence, by Corollary 2 there is a constant $C\in\mathbb{R}^+$ such that

$$\int_{D\setminus\overline{B}(x_0,r_0)} V_n\,\mathrm{d}\nu_u \overset{(15)}{\leqslant} \int_{D\setminus\overline{B}(x_0,r_0)} V_n\,\mathrm{d}\nu_M + C \quad \text{for all } n\geqslant n_0.$$

Therefore, for all $n\geqslant n_0$,

$$\int_{D\setminus\overline{B}(x_0,r_0)} V_n\,\mathrm{d}\nu_u \overset{(15)}{\leqslant} \int_{D\setminus\overline{B}(x_0,r_0)} V_n\,\mathrm{d}\nu_M + C \overset{(20)}{\leqslant} \int_{D\setminus\overline{B}(x_0,r_0)} V\,\mathrm{d}\nu_M + C$$

$$\leqslant \int_{D\setminus D_1} V\,\mathrm{d}\nu_M + \int_{D_1\setminus\overline{B}(x_0,r_0)} V\,\mathrm{d}\nu_M + C \overset{(19f)}{\leqslant} \int_{D\setminus D_1} V\,\mathrm{d}\nu_M + b\,\nu_M\bigl(D_1\setminus\overline{B}(x_0,r_0)\bigr) + C,$$

where we used the maximum principle for $V\in \mathrm{sbh}\bigl(\mathbb{R}^m_\infty\setminus\{x_0\}\bigr)$ in $\mathbb{R}^m_\infty\setminus B(x_0,r_0)$. Further, we put $C_1:=b\,\nu_M\bigl(D_1\setminus\overline{B}(x_0,r_0)\bigr)+C\in\mathbb{R}$ and continue as

$$\int_{D\setminus\overline{B}(x_0,r_0)} V_n\,\mathrm{d}\nu_u \leqslant \int_{D\setminus D_1} V\,\mathrm{d}\nu_M + C_1 \overset{(20)}{=} \frac{1}{c}\int_{D\setminus D_1} v\,\mathrm{d}\nu_M + C_1 \leqslant \frac{1}{c}\int_{D\setminus K} v\,\mathrm{d}\nu_M + C_1.$$

So, when $n\to\infty$, we obtain in view of (20)

$$\frac{1}{c}\int_{D\setminus D_1} v\,\mathrm{d}\nu_u \overset{(20)}{=} \int_{D\setminus D_1} V\,\mathrm{d}\nu_u \leqslant C_2 := \frac{1}{c}\int_{D\setminus K} v\,\mathrm{d}\nu_M + C_1 \overset{(4)}{\in} \mathbb{R}.$$

Hence

$$\int_{D\setminus K} v\,\mathrm{d}\nu_u \leqslant \int_{D_1\setminus K} v\,\mathrm{d}\nu_u + c\,C_2 \leqslant \nu_u(D_1\setminus K)\sup_{x\in D\setminus K} v(x) + c\,C_2 \overset{(3)}{<} +\infty.$$

This completes the proof of Theorem 1. ◀

Remark 2. Our results show that the construction of test functions in the sense of Definition 2 is important. For $m=2=2n$ such constructions have been developed in [6]. We will consider the case $m\neq 2$ in one of our next works.

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