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Paley-Wiener Type Perturbations of Frames and the Deviation from Perfect Reconstruction

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Abstract. Frame theory is an efficient tool to obtain expansions of elements in separable Hilbert spaces that are similar to the ones obtained via orthonormal bases, however, with considerably more flexibility. In this paper we give a survey of known results about frame expansions and perturbation theory, combined with an extension to approximately dual frames. We will show, e.g., that perturbation of a pair of dual frames in the Paley-Wiener sense leads to a deviation from perfect reconstruction that can be controlled in terms of the frame bounds of the involved sequences. The paper contains an Appendix, which motivates the analysis of frames via classical results.

Key Words and Phrases: frames, dual frames, approximately dual frames.

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1. Introduction

Frame theory is a tool to obtain decompositions of elements in a separable Hilbert space $\mathcal{H}$ in terms of "convenient building blocks." To be more precise, a sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\,||f||^2, \,\forall f \in \mathcal{H}. \tag{1}$$

Given any frame $\{f_k\}_{k=1}^{\infty}$, there exists a so-called dual frame, i.e., a frame $\{g_k\}_{k=1}^{\infty}$ such that each $f \in \mathcal{H}$ has a representation

$$f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k. \tag{2}$$

Perturbation theory is one of the well established research areas within frame theory. A typical question is as follows: assuming that $\{f_k\}_{k=1}^{\infty}$ is a frame and that a sequence $\{\widetilde{f}_k\}_{k=1}^{\infty}$ is "close" to $\{f_k\}_{k=1}^{\infty}$, can we conclude that $\{\widetilde{f}_k\}_{k=1}^{\infty}$ is a frame as well? In this

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paper we give a survey of some key results in perturbation theory, combined with new results. In contrast to most results in the literature, we will not focus on the perturbation of a frame itself, but rather on its effect on the reconstruction formula (2). Thus, a typical question will be as follows: assuming that two sequences $\{\widetilde{f}_k\}_{k=1}^\infty, \{\widetilde{g}_k\}_{k=1}^\infty$ are "close" to the dual frames $\{f_k\}_{k=1}^\infty, \{g_k\}_{k=1}^\infty$, how is the identity (2) affected when the elements f_k, g_k are replaced by $\widetilde{f}_k, \widetilde{g}_k$?

In the rest of this section we give a more detailed description of the necessary background in frame theory. The classical perturbation results are summarized in Section 2, and the new results concerning the associated deviation from perfect reconstruction are presented in Section 3. For convenience of the reader without experience in frame theory we have included an Appendix containing some motivation for the study of frames; the information here is taken from the recent book [4]. The Appendix can be read independently of Sections 2–3.

A sequence $\{f_k\}_{k=1}^\infty$ in $\mathcal{H}$ is called a Bessel sequence if at least the upper condition in (1) holds. If $\{f_k\}_{k=1}^\infty$ is a frame, the numbers A, B in (1) are called frame bounds. It is well known that if $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are Bessel sequences and (2) holds, then $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are automatically frames; in this case $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are said to be dual frames. The property (2) clearly resembles the well known expansion in terms of an orthonormal basis, although the sequences $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ might not be identical. The key difference between a frame and a basis is that a frame $\{f_k\}_{k=1}^\infty$ might be redundant: in fact, if $\{f_k\}_{k=1}^\infty$ is a frame but not a basis, then the dual frame $\{g_k\}_{k=1}^\infty$ is not unique. In concrete applications (typically taking place in concrete Hilbert spaces like $L^2(\mathbb{R})$ and dealing with explicitly given frames) this flexibility is used to construct dual frames with particularly attractive properties. We also note a well-known relation to Riesz bases: any Riesz basis is indeed a frame. On the other hand, a frame is a Riesz basis if and only if

$$[\{c_k\}_{k=1}^\infty \in \ell^2(\mathbb{N}), \ \sum_{k=1}^\infty c_k f_k = 0] \Rightarrow c_k = 0, \forall k \in \mathbb{N}.$$

Given any Bessel sequence $\{f_k\}_{k=1}^\infty$ one can define a bounded operator $T : \ell^2(\mathbb{N}) \to \mathcal{H}$ by $T\{c_k\}_{k=1}^\infty := \sum c_k f_k$; the operator T is called the synthesis operator or preframe operator. It is easy to see that the adjoint operator is given by $T^* : \mathcal{H} \to \ell^2(\mathbb{N})$, $T^* f = \{\langle f, f_k\rangle\}_{k=1}^\infty$. Denoting the preframe operators for two Bessel sequences $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ by T and U, respectively, it is clear that $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are dual frames if and only if

$$TU^* = I. \tag{3}$$

In the rest of this paper the formula (2) associated with a pair of dual frames $\{f_k\}_{k=1}^\infty$, $\{g_k\}_{k=1}^\infty$ will play the key role. In signal processing terms the result is phrased as "dual pairs of frames leads to *perfect reconstruction.*" For more general information about frames we refer to the monograph [4].

The purpose of this paper is to discuss the deviation in the reconstruction formula that occurs when the frames $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are perturbed. The corresponding question for the frame property is well studied in the literature; see the short survey in Section 2. However, for concrete applications it is even more important to know the deviation from perfect reconstruction when dual frames $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ are replaced by perturbations $\{\widetilde{f}_k\}_{k=1}^\infty$ and $\{\widetilde{g}_k\}_{k=1}^\infty$. For example, it would be preferable to obtain an estimate of the form

$$\left|\left|f - \sum_{k=1}^\infty \langle f, \widetilde{g}_k\rangle \widetilde{f}_k\right|\right| \le \epsilon\, ||f||,\ \forall f \in \mathcal{H} \tag{4}$$

for a small value of $\epsilon > 0$. For the case where $\{\widetilde{f}_k\}_{k=1}^\infty$ and $\{\widetilde{g}_k\}_{k=1}^\infty$ are Bessel sequences, this idea has been formalized in the paper [6] by saying that $\{\widetilde{f}_k\}_{k=1}^\infty$ and $\{\widetilde{g}_k\}_{k=1}^\infty$ form *approximately dual frames* if (4) holds for some $\epsilon < 1$. In Section 3 we will show how Paley-Wiener type perturbations on a pair of dual frames affects the deviation from perfect reconstruction; in particular, the results will show that the deviation can be controlled if we are able to control the Bessel bounds for the involved sequences.

The current paper will keep the discussion on the level of a general Hilbert space. Applications to concrete settings appear in the papers [6], [9], and [1]. Note also that the paper [10] contains a discussion of wavelet frames and approximation on subspaces, however, not with the exact concept of approximately dual frames as defined in [6].

2. Some classical perturbation results for frames

A classical perturbation result, attributed to Neumann/Paley/Wiener, states that if $\{f_k\}_{k=1}^\infty$ is a basis for a Banach space X, then a sequence $\{\widetilde{f}_k\}_{k=1}^\infty$ in X is also a basis if there exists a constant $\lambda \in]0,1[$ such that

$$\left|\left|\sum_{k=1}^\infty c_k(f_k - \widetilde{f}_k)\right|\right| \le \lambda \left|\left|\sum_{k=1}^\infty c_k f_k\right|\right| \tag{5}$$

for all finite sequences of scalars $\{c_k\}_{k=1}^\infty$.

Note that (5) is a condition on the *perturbation operator*

$$K : \mathcal{D}(K) \to \mathcal{H},\ K\{c_k\}_{k=1}^\infty = \sum_{k=1}^\infty c_k(f_k - \widetilde{f}_k). \tag{6}$$

This is typical for the results considered in this paper; in fact, all results will be formulated in terms of conditions on operators of this type. We also note that the condition (5) with $\lambda < 1$ is not suitable for an immediate extension to the case where $\{f_k\}_{k=1}^\infty$ is a frame for a Hilbert space; indeed, if $\sum_{k=1}^\infty c_k f_k$ and $\sum_{k=1}^\infty c_k \widetilde{f}_k$ are both convergent, the condition implies that

$$\sum_{k=1}^\infty c_k f_k = 0 \Leftrightarrow \sum_{k=1}^\infty c_k \widetilde{f}_k = 0,$$

and hereby forces the perturbed family $\{\widetilde{f}_k\}_{k=1}^{\infty}$ to exhibit the same linear dependence as the given sequence $\{f_k\}_{k=1}^{\infty}$. A considerably more flexible result appeared in [2] under the name *A Paley-Wiener Theorem for frames:*

Theorem 1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with bounds A, B. Let $\{\widetilde{f}_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$ and assume that there exist constants $\lambda, \mu \geq 0$ such that $\lambda + \frac{\mu}{\sqrt{A}} < 1$ and*

$$\left|\left|\sum_{k=1}^{\infty} c_k(f_k - \widetilde{f}_k)\right|\right| \leq \lambda \left|\left|\sum_{k=1}^{\infty} c_k f_k\right|\right| + \mu \left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2} \tag{7}$$

for all finite scalar sequences $\{c_k\}_{k=1}^{\infty}$. Then $\{\widetilde{f}_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds

$$A\left(1 - \left(\lambda + \frac{\mu}{\sqrt{A}}\right)\right)^2, \quad B\left(1 + \lambda + \frac{\mu}{\sqrt{B}}\right)^2.$$

Theorem 1 has the following immediate consequence, which was also obtained independently by Favier and Zalik [8]:

Corollary 1. *Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$ with bounds A, B, and let $\{\widetilde{f}_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$. If there exists a constant $R < A$ such that*

$$\sum_{k=1}^{\infty} |\langle f, f_k - \widetilde{f}_k\rangle|^2 \leq R\ ||f||^2, \ \forall f \in \mathcal{H},$$

then $\{\widetilde{f}_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ with bounds

$$A\left(1 - \sqrt{\frac{R}{A}}\right)^2, \ B\left(1 + \sqrt{\frac{R}{B}}\right)^2.$$

A different type of perturbation result involving the same perturbation operator was obtained in [5]:

Proposition 1. *If $\{f_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$ and*

$$K : \ell^2(\mathbb{N}) \to \mathcal{H}, \ K\{c_k\}_{k=1}^{\infty} := \sum_{k=1}^{\infty} c_k(f_k - \widetilde{f}_k)$$

is well-defined and compact, then $\{\widetilde{f}_k\}_{k=1}^{\infty}$ is a frame for the Hilbert space $\overline{span}\{\widetilde{f}_k\}_{k=1}^{\infty}$.

Note that in Proposition 1 the space $\overline{\text{span}}\{\widetilde{f}_k\}_{k=1}^{\infty}$ might just be a subspace of $\mathcal{H}$.

3. Paley-Wiener type perturbation and deviation from perfect reconstruction

In this section we will reconsider the questions appearing in Section 2, but now with the purpose of analyzing the deviation form perfect reconstruction that occur when a pair of dual frames is perturbed. It turns out that the key role is played by the norm of the perturbation operator; for this reason we will consider the Paley-Wiener type perturbation condition (7), but without the λ–term.

In the entire section we will use the following setup and notation:

- Consider dual frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, with preframe operators

$$T, U : \ell^2(\mathbb{N}) \to \mathcal{H},\ T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k,\ U\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k g_k.$$

Then each $f \in \mathcal{H}$ has the representation

$$f = \sum_{k=1}^{\infty} \langle f, g_k \rangle f_k;$$

or, formulated in operator terms,

$$I = TU^*.$$

- We will consider approximately dual frames $\{\widetilde{f}_k\}_{k=1}^{\infty}$ and $\{\widetilde{g}_k\}_{k=1}^{\infty}$, with preframe operators

$$\widetilde{T}, \widetilde{U} : \ell^2(\mathbb{N}) \to \mathcal{H},\ \widetilde{T}\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k \widetilde{f}_k,\ \widetilde{U}\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k \widetilde{g}_k.$$

Note that we will always assure that $\{\widetilde{f}_k\}_{k=1}^{\infty}$ and $\{\widetilde{g}_k\}_{k=1}^{\infty}$ are Bessel sequences, so the operators $\widetilde{T}$ and $\widetilde{U}$ are actually well-defined. We will measure approximately duality in terms of the deviation

$$\left|\left| f - \sum_{k=1}^{\infty} \langle f, \widetilde{g}_k \rangle \widetilde{f}_k \right|\right| = \left|\left| (I - \widetilde{T}\widetilde{U}^*) f \right|\right|,\ f \in \mathcal{H},$$

or in terms of the operator norm

$$\left|\left| I - \widetilde{T}\widetilde{U}^* \right|\right|.$$

We first show that the operator norm $||I - \widetilde{T}\widetilde{U}^*||$ can be controlled in terms of the Bessel bounds for the sequences $\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty}, \{f_k - \widetilde{f}_k\}_{k=1}^{\infty}$, and $\{g_k - \widetilde{g}_k\}_{k=1}^{\infty}$. We will use the well-known result that if $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence with preframe operator T, then the *minimal* Bessel bound is given by $||T||^2$; in other words, any Bessel bound B satisfies $||T||^2 \leq B$.

Theorem 2. *Assume that* $\{f_k\}_{k=1}^\infty$ *and* $\{g_k\}_{k=1}^\infty$ *are dual frames, with upper frame bounds* B_f, B_g*, respectively. Consider sequences* $\{\widetilde{f_k}\}_{k=1}^\infty$ *and* $\{\widetilde{g_k}\}_{k=1}^\infty$ *such that*

$$\left|\left|\sum_{k=1}^\infty c_k(f_k-\widetilde{f_k})\right|\right| \le \mu\left(\sum_{k=1}^\infty |c_k|^2\right)^{1/2}, \text{ and } \left|\left|\sum_{k=1}^\infty c_k(g_k-\widetilde{g_k})\right|\right| \le \mu\left(\sum_{k=1}^\infty |c_k|^2\right)^{1/2}$$

for some $\mu \ge 0$ *and all finite sequences* $\{c_k\}_{k=1}^\infty$*. Then*

$$||I-\widetilde{T}\widetilde{U}^*|| \le \mu\left(||T||+||\widetilde{U}||\right) \le \mu\left(\sqrt{B_f}+\mu+\sqrt{B_g}\right). \tag{8}$$

Proof. In terms of the preframe operators $T, \widetilde{T}, U$, and $\widetilde{U}$,

$$I-\widetilde{T}\widetilde{U}^* = TU^*-\widetilde{T}\widetilde{U}^* = T(U-\widetilde{U})^*+(T-\widetilde{T})\widetilde{U}^*;$$

thus

$$\left|\left|I-\widetilde{T}\widetilde{U}^*\right|\right| \le ||T||\,||U-\widetilde{U}||+||T-\widetilde{T}||\,||\widetilde{U}||.$$

This immediately gives the first estimate in (8). The second estimate now follows from $||\widetilde{U}|| \le ||\widetilde{U}-U||+||U||$. ◀

The estimate (8) shows that if we fix the dual frames $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$, the deviation from perfect reconstruction can be controlled in terms of the parameter μ. On the other hand, the estimate also indicates that we can not control the deviation from perfect reconstruction that occur by replacing $\{f_k\}_{k=1}^\infty$ by $\{\widetilde{f_k}\}_{k=1}^\infty$, uniformly over all dual frames $\{g_k\}_{k=1}^\infty$; in other words, we need to be able to control uniformly the Bessel bound on $\{g_k\}_{k=1}^\infty$ as well. The next example confirms this.

Example 1. *Let* $\{e_k\}_{k=1}^\infty$ *be an ONB for* $\mathcal{H}$*, and let, for some fixed* $N \in \mathbb{N}$*,*

$$\{f_k\}_{k=1}^\infty := \{0, e_1, e_2, \dots\},\ \{g_k\}_{k=1}^\infty := \{Ne_1, e_1, e_2, \dots\}.$$

Then $\{g_k\}_{k=1}^\infty$ *is a non-canonical dual frame of the frame* $\{f_k\}_{k=1}^\infty$*. Given* $\epsilon > 0$*, let*

$$\{\widetilde{f_k}\}_{k=1}^\infty = \{\epsilon e_1, e_1, e_2, \dots\},\ \{\widetilde{g_k}\}_{k=1}^\infty = \{g_k\}_{k=1}^\infty.$$

Then

$$\left|\left|\sum_{k=1}^\infty c_k(f_k-\widetilde{f_k})\right|\right| \le \epsilon\left(\sum_{k=1}^\infty |c_k|^2\right)^{1/2}, \text{ and } \left|\left|\sum_{k=1}^\infty c_k(g_k-\widetilde{g_k})\right|\right| = 0$$

for all finite sequences $\{c_k\}_{k=1}^\infty$*. However,*

$$||I-\widetilde{T}\widetilde{U}^*|| \ge ||(I-\widetilde{T}\widetilde{U}^*)e_1|| = ||e_1-(\epsilon Ne_1+e_1)|| = N\epsilon.$$

Thus, the deviation from perfect reconstruction can only be controlled with knowledge of the Bessel bound for $\{g_k\}_{k=1}^\infty$*: we are not able to obtain a uniform estimate on* $||I-\widetilde{T}\widetilde{U}^*||$ *that holds for all dual frames* $\{g_k\}_{k=1}^\infty$ *of* $\{f_k\}_{k=1}^\infty$*.* ◀

Recall that we considered compact perturbations in Proposition 1. We will now show that compact perturbation on a pair of dual frames also implies that $I - \widetilde{T}\widetilde{U}^*$ is compact.

Proposition 2. *If* $\{f_k\}_{k=1}^\infty$ *and* $\{g_k\}_{k=1}^\infty$ *are dual frames and the operators*

$$K, \widetilde{K} : \ell^2(\mathbb{N}) \to \mathcal{H},\ K\{c_k\}_{k=1}^\infty := \sum_{k=1}^\infty c_k(f_k - \widetilde{f}_k), \quad \widetilde{K}\{c_k\}_{k=1}^\infty := \sum_{k=1}^\infty c_k(g_k - \widetilde{g}_k)$$

are compact, then the operator

$$I - \widetilde{T}\widetilde{U}^*$$

is compact.

Proof. As in the proof of Theorem 2,

$$I - \widetilde{T}\widetilde{U}^* = T(U - \widetilde{U})^* + (T - \widetilde{T})\widetilde{U}^* = T\widetilde{K}^* + K\widetilde{U}^*.$$

It now follows from standard results in functional analysis that $I - \widetilde{T}\widetilde{U}^*$ is compact. ◄

Appendix

The purpose of this section is to give a short motivation for the study of frame theory.

Frame theory aims at obtaining expansions of elements in Hilbert spaces in terms of superpositions of "elementary building blocks." In many natural cases these building blocks do not form bases, as the following elementary example from [4] illustrates.

Example 2. *Consider the orthonormal basis* $\{e_k\}_{k\in\mathbb{Z}} := \{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ *for* $L^2(0,1)$. *We will now consider these functions on an open subinterval* $I \subset]0,1[$ *with* $|I| < 1$. *We can identify* $L^2(I)$ *with the subspace of* $L^2(0,1)$ *consisting of the functions which are zero on* $]0,1[\setminus I$. *Hereby a function* $f \in L^2(I)$ *is identified with a function (which we still denote* f*) in* $L^2(0,1)$, *and which has the expansion*

$$f = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k \ \text{in } L^2(0,1). \tag{9}$$

Since

$$\begin{aligned} \Big\| f - \sum_{|k|\le n} \langle f, e_k\rangle e_k \Big\|_{L^2(I)} &= \left(\int_I \Big| f(x) - \sum_{k=-n}^{n} \langle f, e_k\rangle e^{2\pi ikx} \Big|^2 dx \right)^{1/2} \\ &\le \left(\int_0^1 \Big| f(x) - \sum_{k=-n}^{n} \langle f, e_k\rangle e^{2\pi ikx} \Big|^2 dx \right)^{1/2} \\ &\to 0 \ \text{as } n \to \infty, \end{aligned}$$

we also have

$$f = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k \quad in \quad L^2(I). \tag{10}$$

That is, the functions $\{e_k\}_{k\in\mathbb{Z}}$ *also have the expansion property in* $L^2(I)$*. However, they are not a basis for* $L^2(I)$*. To see this, define the function*

$$\widetilde{f}(x) = \begin{cases} f(x) & if\ x \in I, \\ 1 & if\ x \notin I. \end{cases}$$

Then $\widetilde{f} \in L^2(0,1)$ *and we have the representation*

$$\widetilde{f} = \sum_{k\in\mathbb{Z}} \langle \widetilde{f}, e_k\rangle e_k \quad in \quad L^2(0,1). \tag{11}$$

By restricting to I*, the expansion* (11) *is also valid in* $L^2(I)$*; since* $f = \widetilde{f}$ *on* I*, this shows that*

$$f = \sum_{k\in\mathbb{Z}} \langle \widetilde{f}, e_k\rangle e_k \quad in \quad L^2(I). \tag{12}$$

Thus, (10) *and* (12) *are both expansions of* f *in* $L^2(I)$*, and they are non-identical; the argument is that since* $f \neq \widetilde{f}$ *in* $L^2(0,1)$*, the expansions* (9) *and* (11) *show that*

$$\{\langle f, e_k\rangle\}_{k\in\mathbb{Z}} \neq \{\langle \widetilde{f}, e_k\rangle\}_{k\in\mathbb{Z}}.$$

The conclusion is that the restriction of the functions $\{e_k\}_{k\in\mathbb{Z}}$ *to* I *is not a basis for* $L^2(I)$*, but the expansion property is preserved. In frame terminology, the sequence* $\{e_k\}_{k\in\mathbb{Z}}$ *is a tight frame for* $L^2(I)$*, meaning that we can take* $A = B$ *in the frame definition.* ◀

Another motivation for frame theory is that it gives much more flexibility than the theory for orthonormal bases. Indeed, the ONB conditions are very strong, and in many cases ONB's satisfying additional constraints can not be constructed; in some of these cases, a construction can be obtained by replacing the ONB condition with the frame condition. We will now give some examples of this, formulated in the concrete setting of Gabor systems and wavelet systems on $L^2(\mathbb{R})$. First, let us define the following unitary operators on $L^2(\mathbb{R})$:

(i) For $a \in \mathbb{R}$, the operator T_a, called translation by a, is defined by

$$T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (T_a f)(x) := f(x-a), \ x \in \mathbb{R}. \tag{13}$$

(ii) For $b \in \mathbb{R}$, the operator E_b, called modulation by b, is defined by

$$E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (E_b f)(x) := e^{2\pi i b x} f(x), \ x \in \mathbb{R}. \tag{14}$$

(iii) The dyadic scaling operator is

$$D : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (Df)(x) := 2^{1/2} f(2x). \tag{15}$$

A Gabor system in $L^2(\mathbb{R})$ is a collection of functions of the form

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx} g(x - na)\}_{m,n\in\mathbb{Z}},$$

where $g \in L^2(\mathbb{R})$ is a fixed function and $a, b > 0$. It is easy to construct an ONB for $L^2(\mathbb{R})$ of this form: in fact, $\{E_mT_n\chi_{[0,1]}\}_{m,n\in\mathbb{Z}}$ is an ONB for $L^2(\mathbb{R})$. However, for the sake of applications it is important to consider Gabor systems for which g is a continuous function with compact support. It turns out that this additional constraint can not be combined with the ONB condition, but frames satisfying this condition can be constructed:

Proposition 3. *Let g be a continuous function with compact support. Then the following hold:*

(i) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can not be an orthonormal basis for $L^2(\mathbb{R})$.

(ii) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can not be a Riesz basis for $L^2(\mathbb{R})$.

(iii) $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ can be a frame for $L^2(\mathbb{R})$ if $0 < ab < 1$.

A more precise version of Proposition 3 (iii) says that for any $a, b > 0$ with $ab < 1$, there exists a continuous function g with compact support such that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. For more information on Gabor systems and their applications we refer to the monographs [4, 13] and the research papers in the books [11, 12].

Another important class of frames is formed by wavelet systems. A wavelet system in $L^2(\mathbb{R})$ has the form $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}} = \{2^{j/2}\psi(2^jx - k)\}_{j,k\in\mathbb{Z}}$. Classical multiresolution analysis explains how to construct ONB's for $L^2(\mathbb{R})$ having the wavelet structure [7], but also in this case there are restrictions on the additional properties one can obtain:

Proposition 4. *Let $\psi \in L^2(\mathbb{R})$. Assume that ψ decays exponentially and that $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis. Then ψ can not be infinitely often differentiable with bounded derivatives.*

For a proof we refer to [7]. On the other hand, it is known that the properties in Proposition 4 can be obtained if we allow $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ to be a frame instead of an ONB; concrete cases are well-known, see, e.g., Example 15.2.7 in [4].

On a more concrete level, wavelet expansions are used for compression purposes. In order to explain this, assume that $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ and $\{D^jT_k\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames. Then the frame decomposition takes the form

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, D^jT_k\psi\rangle D^jT_k\widetilde{\psi}, \ \forall f \in L^2(\mathbb{R}). \tag{16}$$

It is known how to construct "efficient wavelet representation" in several concrete cases, e.g., within image analysis. The idea is that concrete images usually are known to belong to certain subspaces of $L^2(\mathbb{R})$; constructing the function ψ such that it has a high number of vanishing moments (for the exact definition of vanishing moments we refer to [7], or [3, 16] for more elementary introductions) it is possible to obtain that a large number of the coefficients $\{\langle f, D^jT_k\psi\rangle\}_{j,k\in\mathbb{Z}}$ almost vanishes and can be replaced by zero without affecting the quality of the image. The technology for doing this is already implemented in consumer electronics like digital cameras and MP3 players, and it is contained in the JPEG2000 standard for image processing. Even though the expansion (16) can be realized with an orthonormal basis $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ (corresponding to the case where $\psi = \widetilde{\psi}$) these applications are actually based on the more general case where $\{D^jT_k\psi\}_{j,k\in\mathbb{Z}}$ is a Riesz basis, i.e., the "intermediate step" between ONB's and frames.

We note that, despite the immediate differences between Gabor analysis and wavelet analysis, there exists a class of systems, the so-called generalized shift-invariant systems, that contains both systems as special cases. For an introduction to these systems we refer to the original papers [15] by Ron & Shen and [14] by Hernandez, Labate & Weiss. The analysis for approximately dual frames for GSI-systems have been developed in the recent paper [1].

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