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Construction of Continuous Frames in Hilbert spaces

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Abstract. Extending the concept of frame to continuous frame, in this manuscript we will show that under certain conditions on the measure space Ω and the dimension of the underlying Hilbert space $\mathcal{H}$, we can construct continuous frames. Also, some examples are given.

Key Words and Phrases: frame, Bessel sequence, continuous frame, measure space, wavelet frame, short-time Fourier transform, Gabor frame.

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1. Introduction

A discrete frame is a countable family of elements in a separable Hilbert space which allows stable but not necessarily unique decomposition of arbitrary elements into expansion of the frame elements. The concept of generalization of frames was proposed by G. Kaiser [13] and independently by Ali, Antoine and Gazeau [2] to a family indexed by some locally compact space endowed with a Radon measure. These frames are known as continuous frames. Gabardo and Han in [11] called these frames *Frames associated with measurable spaces*, Askari-Hemmat, Dehghan and Radjabalipour in [3] called them *generalized frames* and in mathematical physics they are referred to as *Coherent states* [2].

For more studies on continuous frames and its applications, the interested reader can refer to [1, 2, 3, 4, 8, 9, 11]. In this paper, we focus on positive measures and separable complex Hilbert spaces.

Wavelet and Gabor frames are used very often in signal processing algorithms. Both systems are derived from a continuous transform, which can be seen as a continuous frame [1, 10, 12].

2. Continuous frames

Throughout this paper, $\mathcal{H}$ is a separable Hilbert space and (Ω, μ) is a measure space with positive measure μ.

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Definition 1. *Let $\mathcal{H}$ be a complex Hilbert space and (Ω, μ) be a measure space with positive measure μ. The mapping $F : \Omega \to \mathcal{H}$ is called a continuous frame with respect to (Ω, μ), if*

1. *F is weakly-measurable, i.e., for all $f \in \mathcal{H}$, the function $\omega \to \langle f, F(\omega)\rangle$ is a measurable function on Ω;*

2. *there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{1}$$

The constants A and B are called continuous frame bounds. F is called a tight continuous frame if $A = B$ and Parseval if $A = B = 1$. The mapping F is called Bessel *if the second inequality in (1) holds. In this case, B is called the* Bessel constant.

Example 1. *Let D be a bounded, Lebesgue measurable subset of $\mathbb{R}^n$ and let $\mathcal{H} = L^2(D)$ and $\Omega = \mathbb{R}^n$. Define $F(\omega)(t) = e^{2\pi i\langle \omega, t\rangle}\chi_D$ for each $\omega \in \Omega$. Clearly, for each $f \in \mathcal{H}$,*

$$\langle f, F(\omega)\rangle = \int_\Omega f(t)^{-2\pi i\langle \omega, t\rangle}\chi_D dt = \hat{f}(\omega),$$

for $\omega \in \Omega$, where $\hat{f}$ denotes the Fourier transform of f. By using Plancherel's identity, we have

$$\int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) = \int_\Omega |\hat{f}(\omega)|^2 d\mu(\omega) = \|\hat{f}\|_2^2 = \|f\|_2^2, \quad (f \in \mathcal{H}),$$

which shows that F is a Parseval continuous frame with respect to (Ω, μ) for $\mathcal{H}$.

It is obvious that, if μ is counting measure and $\Omega = \mathbb{N}$, then F is a discrete frame. In this sense, continuous frames are the more general setting.

The first inequality in (1) shows that F is complete, i.e.,

$$\overline{\text{span}}\{F(\omega)\}_{\omega\in\Omega} = \mathcal{H}.$$

Like orthonormal bases, we have the following proposition.

Proposition 1. *Let $F : \Omega \to \mathcal{H}$ be a continuous Bessel function, $\Lambda \subseteq \Omega$ and $f \in \mathcal{H}$. Then $E_\Lambda := \{\omega : \omega \in \Lambda, \langle f, F(\omega)\rangle \neq 0\}$ is σ-finite.*

Proof. For $n \in \mathbb{N}$ and $f \in \mathcal{H}$, let

$$K_n = \{\omega : \omega \in \Lambda, |\langle f, F(\omega)\rangle| \geq \frac{1}{n}\}.$$

Then $E_\Lambda = \bigcup_n K_n$. So

$$\begin{aligned}\frac{1}{n^2}\mu(K_n) &\leq \int_{K_n} |\langle f, F(\omega)\rangle|^2\, d\mu(\omega)\\ &\leq \int_{E_\Lambda} |\langle f, F(\omega)\rangle|^2\, d\mu(\omega)\\ &= \int_{\Omega} |\langle f, F(\omega)\rangle|^2\, d\mu(\omega)\\ &\leq B\|f\|^2.\end{aligned}$$

Therefore $\mu(K_n) < \infty$. Hence E_Λ is σ-finite. ◄

Let F be a continuous frame with respect to (Ω, μ). Then the mapping

$$\Psi : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

defined by

$$\Psi(f, g) = \int_\Omega \langle f, F(\omega)\rangle \langle F(\omega), g\rangle\, d\mu(\omega), \quad (f, g \in \mathcal{H})$$

is well defined, sesquilinear and bounded. By Cauchy-Schwarz's inequality, we get

$$\begin{aligned}|\Psi(f,g)| &\leq \int_\Omega |\langle f, F(\omega)\rangle \langle F(\omega), g\rangle|\, d\mu(\omega)\\ &\leq \left(\int_\Omega |\langle f, F(\omega)\rangle|^2\, d\mu(\omega)\right)^{\frac{1}{2}} \left(\int_\Omega |\langle F(\omega), g\rangle|^2\, d\mu(\omega)\right)^{\frac{1}{2}}\\ &\leq B\|f\|\|g\|.\end{aligned}$$

Hence $\|\Psi\| \leq B$. It follows that there exists a unique bounded operator (Riesz Representation Theorem) $S_F : \mathcal{H} \to \mathcal{H}$ such that

$$\Psi(f, g) = \langle S_F f, g\rangle, \quad (f, g \in \mathcal{H})$$

and, moreover, $\|\Psi\| = \|S_F\|$.

Since $\langle S_F f, f\rangle = \int_\Omega |\langle f, F(\omega)\rangle|^2\, d\mu(\omega)$, then S_F is positive and $AI \leq S_F \leq BI$. Hence S_F is invertible, positive and $\frac{1}{B}I \leq S_F^{-1} \leq \frac{1}{A}I$. We call S_F the continuous frame operator of F and use the notation

$$S_F f = \int_\Omega \langle f, F(\omega)\rangle F(\omega)\, d\mu(\omega) \quad (f \in \mathcal{H}),$$

which is valid in the weak sense. Thus, every $f \in \mathcal{H}$ has the representations

$$f = S_F^{-1} S_F f = \int_\Omega \langle f, F(\omega)\rangle S_F^{-1} F(\omega)\, d\mu(\omega),$$

$$f = S_F S_F^{-1} f = \int_\Omega \langle f, S_F^{-1} F(\omega)\rangle F(\omega)\, d\mu(\omega).$$

Theorem 1. *[14] Let (Ω, μ) be a measure space and F be a Bessel mapping from Ω to $\mathcal{H}$. Then the operator $T_F : L^2(\Omega, \mu) \to \mathcal{H}$ weakly defined by*

$$\langle T_F \varphi, h\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), h\rangle \, d\mu(\omega), \quad (h \in \mathcal{H})$$

is well defined, linear, bounded and its adjoint is given by

$$T_F^* : \mathcal{H} \to L^2(\Omega, \mu), \quad (T_F^* h)(\omega) = \langle h, F(\omega)\rangle, \quad (\omega \in \Omega).$$

The operator T_F is called the *pre-frame operator or synthesis operator* and T_F^* is called the *analysis operator* of F.

The converse of Theorem 1 holds when μ is a σ-finite measure [14].

Proposition 2. *Let (Ω, μ) be a measure space, where μ is a σ-finite measure and $F : \Omega \to \mathcal{H}$ be a measurable function. If the mapping $T_F : L^2(\Omega, \mu) \mapsto \mathcal{H}$ defined by*

$$\langle T_F \varphi, h\rangle = \int_\Omega \varphi(\omega)\langle F(\omega), h\rangle \, d\mu(\omega), \quad (\varphi \in L^2(\Omega, \mu), h \in \mathcal{H})$$

is a bounded operator, then F is Bessel.

The next theorem gives an equivalent characterization of continuous frame [14].

Theorem 2. *Let (Ω, μ) be a measure space where μ is a σ-finite measure. The mapping $F : \Omega \to \mathcal{H}$ is a continuous frame with respect to (Ω, μ) for $\mathcal{H}$ if and only if the operator T_F defined in Theorem 1 is a bounded and onto operator.*

As in discrete case, we have the following lemma.

Lemma 1. *[14] Let $F : \Omega \to \mathcal{H}$ be a Bessel function with respect to (Ω, μ). By the above notations, $S_F = T_F T_F^*$.*

The following proposition is a criterion for a continuous frame for a closed subspace of $\mathcal{H}$ to be a continuous frame. For discrete case see [4, Lemma 5.2.1].

Proposition 3. *[14] Suppose that F is a continuous frame with respect to (Ω, μ) for a closed subspace K of $\mathcal{H}$, where μ is a σ-finite measure. Then F is a continuous frame for $\mathcal{H}$ if and only if T^* is injective.*

It is well known that discrete Bessel sequences in a Hilbert space are norm bounded above: if

$$\sum_n |\langle f, f_n\rangle|^2 \leq B \parallel f \parallel^2,$$

for all $f \in \mathcal{H}$, then $\parallel f_n \parallel \leq \sqrt{B}$ for all n. For continuous Bessel mappings, the following example shows that, it is possible to make a continuous Bessel mapping which is unbounded.

Example 2. *Take an (essentially) unbounded (Lebesgue) measurable function $a : \mathbb{R} \to \mathbb{C}$ such that $a \in L^2(\mathbb{R}) \setminus L^\infty(\mathbb{R})$. It is easy to see that such functions indeed exist; consider, for example, the function*

$$b(x) := \frac{1}{\sqrt{|x|}}, 0 < |x| < 1, \quad b(x) = \frac{1}{|x|^2}, |x| \geq 1 \quad \textit{and} \quad b(x) = 0, x = 0.$$

This function is clearly in $L^1(\mathbb{R}) \setminus L^\infty(\mathbb{R})$ and furthermore, $b(x) \geq 0$ for all $x \in \mathbb{R}$. Now take $a(x) = \sqrt{b(x)}$. Choose a fixed vector $h \in \mathcal{H}, h \neq 0$. Then, the mapping

$$F : \mathbb{R} \to \mathcal{H}, \omega \mapsto F(\omega) = a(\omega)h,$$

is weakly (Lebesgue) measurable and a continuous Bessel mapping, since

$$\int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) \leq \|h\|^2 \|a\|^2 \|f\|^2,$$

for all $f \in \mathcal{H}$. But $\|F(\omega)\|$ is unbounded, since a is unbounded.

Also, it is possible to construct a continuous frame in case $\mu(\Omega) = \infty$, $\dim \mathcal{H} = \infty$ with $\int_\Omega \|F(\omega)\|^2 d\mu(\omega) < \infty$.

Example 3. *Let $\Omega = \mathbb{R}$ and $\mathcal{H}$ be an infinite dimensional Hilbert space. Take an essentially unbounded Lebesgue measurable function $a : \mathbb{R} \to \mathbb{C}$, such that $a \in L^2(\mathbb{R}) \setminus L^\infty(\mathbb{R})$. For $0 \neq h \in \mathcal{H}$, take $F : \mathbb{R} \to \mathcal{H}$ by $F(\omega) = a(\omega)h$. Then F is Bessel and $\int_\Omega \|F(\omega)\|^2 d\mu(\omega) < \infty$.*

The following example shows that even continuous frames need not necessarily be norm bounded.

Example 4. *Let $F : \mathbb{R} \to \mathcal{H}$ be a norm-unbounded continuous Bessel mapping with Bessel constant B_1 (like in Example 2) and $G : \mathbb{R} \to \mathcal{H}$ be a norm-bounded continuous frame with bounds $A_2 \leq B_2$ and also assume that $B_1 < A_2$. Then $F - G : \mathbb{R} \to \mathcal{H}$ is a norm-unbounded frame. It is clear that for any $f \in \mathcal{H}$*

$$\begin{aligned} \int_\Omega |\langle f, F(\omega) - G(\omega)\rangle|^2 d\mu(\omega) &\leq \int_\Omega |\langle f, F(\omega)\rangle|^2 d\mu(\omega) + \int_\Omega |\langle f, G(\omega)\rangle|^2 d\mu(\omega) \\ &\leq (B_1 + B_2)\|f\|^2. \end{aligned}$$

So $F - G$ is a continuous Bessel mapping with bound $B_1 + B_2$. For the lower bound, observe that

$$\begin{aligned} \left(\int_\Omega |\langle f, F(\omega) - G(\omega)\rangle|^2 \, d\mu(\omega)\right)^{\frac{1}{2}} &\geq \left(\int_\Omega |\langle f, G(\omega)\rangle|^2 \, d\mu(\omega)\right)^{\frac{1}{2}} - \left(\int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega)\right)^{\frac{1}{2}} \\ &\geq (\sqrt{A_2} - \sqrt{B_1})\|f\|, \end{aligned}$$

and the lower bound is established. The mapping $F - G$ is not norm bounded, since

$$\|F(\omega) - G(\omega)\| \geq \|F(\omega)\| - M,$$

where $\|G(\omega)\| \leq M$ a.e.

3. Construction of continuous frames

For any separable Hilbert space there exists a frame and, even more generally, any separable Banach space can be equipped with a Banach frame with respect to an appropriately chosen sequence space [5]. Concerning the existence of continuous frames, it is natural to ask: do there exist continuous frames for any Hilbert space and any measure space? The existence of continuous frame depends on the dimension of space and the measure of Ω which we derive at the following propositions. For the answer we consider four cases:

- $\mu(\Omega) = \infty$ and $dim\mathcal{H} = \infty$;
- $\mu(\Omega) < \infty$ and $dim\mathcal{H} < \infty$;
- $\mu(\Omega) = \infty$ and $dim\mathcal{H} < \infty$;
- $\mu(\Omega) < \infty$ and $dim\mathcal{H} = \infty$.

Proposition 4. *Let (Ω, μ) be a σ-finite measure space with infinite measure and $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Then there exists a continuous Parseval frame $F : \Omega \to \mathcal{H}$ with respect to (Ω, μ).*

Proof. Since Ω is σ-finite, it can be written as a disjoint union $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$ of countably many subsets $\Omega_k \subseteq \Omega$ such that $\mu(\Omega_k) < \infty$ for all $k \in \mathbb{N}$. Without loss of generality, assume that $\mu(\Omega_k) > 0$ for all k. Let $\{e_k\}_{k\in\mathbb{N}}$ be the orthonormal base of $\mathcal{H}$. Define the function $F : \Omega \to \mathcal{H}$ by

$$\omega \mapsto F(\omega) = \frac{1}{\sqrt{\mu(\Omega_k)}} e_k, \quad (\omega \in \Omega_k).$$

Then, for all $f \in \mathcal{H}$,

$$\int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) = \sum_{k\in\mathbb{N}} \int_{\Omega_k} |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) = \sum_{k\in\mathbb{N}} |\langle f, e_k\rangle|^2 = A\|f\|^2.$$

Thus, F is a continuous Parseval frame. ◀

It is possible to find a frame for any separable Hilbert space and by using the following proposition we can find a continuous frame for any separable Hilbert space.

Proposition 5. *Let (Ω, μ) be a σ-finite measure space with infinite measure and $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Then there exists a continuous frame $F : \Omega \to \mathcal{H}$ with respect to (Ω, μ).*

Proof. Since Ω is σ-finite, it can be written as a disjoint union $\Omega = \bigcup \Omega_k$ of countably many subsets $\Omega_k \subseteq \Omega$ such that $\mu(\Omega_k) < \infty$ for all $k \in \mathbb{N}$. Without loss of generality, assume that $\mu(\Omega_k) > 0$ for all k. Let $\{f_k\}_{k\in\mathbb{N}}$ be a frame for $\mathcal{H}$ with bounds A and B. Define the function $F : \Omega \to \mathcal{H}$ by

$$\omega \mapsto F(\omega) = \frac{1}{\sqrt{\mu(\Omega_k)}} f_k, \quad (\omega \in \Omega_k).$$

Then, for all $f \in \mathcal{H}$,

$$\int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) = \sum_{k\in\mathbb{N}} |\langle f, f_k\rangle|^2.$$

So

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) \leq B\|f\|^2 \quad (f \in \mathcal{H}).$$

Thus, F is a continuous frame with frame bounds A and B. ◀

Proposition 6. *Let (Ω, μ) be a measure space with finite measure and $\mathcal{H}$ be a finite dimensional Hilbert space. Then there exists a continuous frame (Parseval frame) $F : \Omega \to \mathcal{H}$ with respect to (Ω, μ).*

Proof. Let $dim\mathcal{H} = n$ and $\{f_k\}_{k=1}^N$ be a frame for $\mathcal{H}$ and $\Omega = \bigcup_{k=1}^N \Omega_k$, where $\Omega_k \subseteq \Omega$, $1 \leq k \leq N$, $0 < \mu(\Omega_k) < \infty$ and Ω_k's are mutually disjoint. Then $F(\omega) = \frac{1}{\sqrt{\mu(\Omega_k)}} f_k, \omega \in \Omega_k$ is a continuous frame for $\mathcal{H}$. If we choose the orthonormal base instance frame for $\mathcal{H}$, then F is a continuous Parseval frame. ◀

Proposition 7. *Let (Ω, μ) be a σ-finite measure space with infinite measure and $\mathcal{H}$ be a finite dimensional Hilbert space. Then there exists a continuous frame $F : \Omega \to \mathcal{H}$ with respect to (Ω, μ).*

Proof. Since Ω is σ-finite, it can be written as a disjoint union $\Omega = \bigcup \Omega_k$ of countably many subsets $\Omega_k \subseteq \Omega$ such that $\mu(\Omega_k) < \infty$ for all k. Without loss of generality, assume that $\mu(\Omega_k) > 0$ for all $k \in \mathbb{N}$. Let $\{f_k\}_{k=1}^\infty$ be a frame for $\mathcal{H}$ with bounds A and B. Define the function $F : \Omega \to \mathcal{H}$ by

$$\omega \mapsto F(\omega) = \frac{1}{\sqrt{\mu(\Omega_k)}} f_k, \quad (\omega \in \Omega_k).$$

Then, for all $f \in \mathcal{H}$,

$$\int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) = \sum_{k\in\mathbb{N}} |\langle f, f_k\rangle|^2.$$

So

$$A\|f\|^2 \leq \int_\Omega |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) \leq B\|f\|^2 \quad (f \in \mathcal{H}).$$

Thus, F is a continuous frame with frame bounds A and B. ◀

In the case $\mu(\Omega) < \infty$ and $dim\mathcal{H} = \infty$, we have only **Bessel mapping**.

Proposition 8. *Let (Ω, μ) be a σ-finite measure space with finite measure and $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Then there exists a continuous Bessel mapping $F : \Omega \to \mathcal{H}$ with respect to (Ω, μ).*

Proof. Let $\{f_k\}_{k=1}^{\infty}\infty$ be a Bessel sequence for $\mathcal{H}$ with bounds A, B and $\Omega = \bigcup_{k=1}^{N} \Omega_k$, where $\Omega_k \subseteq \Omega$, $1 \le k \le N$, $0 < \mu(\Omega_k) < \infty$. Let $F(\omega) = \frac{1}{\sqrt{\mu(\Omega_k)}} f_k, \omega \in \Omega_k$. Then for all $f \in \mathcal{H}$,

$$\int_{\Omega} |\langle f, F(\omega)\rangle|^2 \, d\mu(\omega) = \sum_{k=1}^{N} |\langle f, f_k\rangle|^2 \le B\|f\|^2.$$

Thus, F is a continuous Bessel mapping with bound B. ◄

4. Gabor and wavelet systems are continuous frames

Well known examples for frames are wavelet and Gabor systems. The corresponding continuous wavelet and STFT transforms give rise to continuous frames. We make use of the following unitary operators on $L^2(\mathbb{R})$:

- Translation: $T_x f(t) := f(t - x)$, for $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$;
- Modulation: $M_y f(t) := e^{2\pi i y \cdot t} f(t)$, for $f \in L^2(\mathbb{R})$ and $y \in \mathbb{R}$;
- Dilation: $D_z f(t) := \frac{1}{|z|^{\frac{1}{2}}} f(\frac{t}{z})$, for $f \in L^2(\mathbb{R})$ and $z \neq 0$.

Definition 2. *Let $\psi \in L^2(\mathbb{R})$ be admissible, i.e.,*

$$C_\psi := \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} \, d\gamma < +\infty.$$

For $a, b \in \mathbb{R}$ with $a \neq 0$, let

$$\psi^{a,b}(x) := (T_b D_a \psi)(x) = \frac{1}{|a|^{\frac{1}{2}}} \psi(\frac{x - b}{a}), \quad (x \in \mathbb{R}).$$

Then the continuous wavelet transform W_ψ is defined by

$$W_\psi(f)(a, b) := \langle f, \psi^{a,b}\rangle = \int_{-\infty}^{+\infty} f(x) \frac{1}{|a|^{\frac{1}{2}}} \overline{\psi(\frac{x - b}{a})} \, dx, \quad (f \in L^2(\mathbb{R})).$$

For an admissible function ψ in L^2, the system $\{\psi^{a,b}\}_{a \neq 0, b \in \mathbb{R}}$ is a continuous tight frame for $L^2(R)$ with respect to $\Omega = \mathbb{R} \setminus \{0\} \times \mathbb{R}$ equipped with the measure $\frac{dadb}{a^2}$ and for all $f \in L^2(\mathbb{R})$

$$f = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_\psi(f)(a, b) \psi^{a,b} \frac{dadb}{a^2},$$

where the integral is understood in weak sense. This system constitutes a continuous tight frame with frame bound $\frac{1}{C_\psi}$. If ψ is suitably normed so that $C_\psi = 1$, then the frame bound is 1, i.e. we have a continuous Parseval frame. For details, see the Proposition 11.1.1 and Corollary 11.1.2 of [6].

Definition 3. *Fix a function $g \in L^2(\mathbb{R}) \setminus \{0\}$. The short-time Fourier transform (STFT) of a function $f \in L^2(\mathbb{R})$ with respect to the window function g is given by*

$$\Psi_g(f)(y,\gamma) = \int_{-\infty}^{+\infty} f(x)\overline{g(x-y)}e^{-2\pi i x\gamma}dx, \qquad (y,\gamma \in \mathbb{R}).$$

Note that in terms of modulation operators and translation operators, $\Psi_g(f)(y,\gamma) = \langle f, M_\gamma T_y g\rangle$.

Let $g \in L^2(\mathbb{R}) \setminus \{0\}$. Then $\{M_b T_a g\}_{a,b\in\mathbb{R}}$ is a continuous frame for $L^2(\mathbb{R})$ with respect to $\Omega = \mathbb{R}^2$ equipped with the Lebesgue measure. Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$. Then

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Psi_{g_1}(f_1)(a,b)\overline{\Psi_{g_2}(f_2)(a,b)}dbda = \langle f_1, f_2\rangle\langle g_2, g_1\rangle.$$

So this system represents a continuous tight frame with bound $\|g\|^2$. For details see the Proposition 8.1.2 of [6].

Another example of continuous frames, called wave packets, can be constructed by combinations of modulations, translations and dilations to interpolate the time-frequency properties of analysis of Gabor and wavelet frames. The interested reader can refer to [7].

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