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# On Approximation by Stancu Type Jakimovski-Leviatan-Durrmeyer Operators

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**Abstract.** In this paper we introduce and study the Stancu type generalization of the Jakimovski-Leviatan-Durrmeyer operators and examine their approximation properties. We investigate the convergence of these operators with the help of Korovkin's approximation theorem. Also, we study local approximation properties and some direct theorems for these operators.

**Key Words and Phrases**: Szász operators, Favard-Szász operators, Appell polynomials, Jakimovski-Leviatan-Durrmeyer operators, modulus of continuity, positive linear operators, Korovkin type approximation theorem, local approximation, weighted space.

**2010 Mathematics Subject Classifications**: 40A30, 41A10, 41A25, 41A36

## 1. Introduction and preliminaries

The approximation process given by Korovkin gave a new shoot up to the approximation theory. It arises naturally in many situations connected with measure theory, functional analysis, partial differential equations, harmonic analysis, probability theory, etc. One of the most useful operators of the type are the Favard-Szász operators defined as follows:

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right),$$

where $x \geq 0$ and $f \in C[0,\infty)$, provided the sum converges. Several authors have investigated many interesting properties of these operators in [1, 3, 4, 5, 6, 11, 12, 15, 16]. Later Jakimovski and Leviatan generalized these operators in [8] using Appell polynomials defined as follows:

Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 \neq 0)$ be an analytic function defined in the disk $|z| < R$, $(R > 0)$ with $g(1) \neq 0$. The Appell polynomials are generated by the functions of the type

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{1}$$

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with the condition that $p_k(x) \geq 0$ for every $x \in [0,\infty)$. Jakimovski and Leviatan introduced the following positive linear operators:

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right).$$

They investigated approximation properties of these operaors in [8]. Ismail [7] generalized these operators using Sheffer polynomials and studied some approximation properties. Büyükyazici et al. [2] obtained Chlodowsky type generalization of these operators and investigated many poroperties by using Appell polynomials. Mursaleen et al. obtained another Chlodowsky type generalization and examined several approximation properties of these operators in [14]. Karaisa gave Durrmeyer type generalization of these operators and investigated the approximation properties in [9]. The Durrmeyer type generalization of these operators is defined as follows:

$$S_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f(t)dt, \quad x \geq 0, \tag{2}$$

where $B(n+1,k)$ is the beta function defined by

$$B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y > 0.$$

## 2. Construction of Operators

We introduce the Stancu type generalization of the Jakimovski-Leviatan-Durrmeyer operators as follows:

$$L_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f\left(\frac{nt+\alpha}{n+\beta}\right) dt, \quad x \geq 0, \tag{3}$$

where $B(x,y)$ is the beta function defined above and $\alpha, \beta$ are such that $0 \leq \alpha \leq \beta$. Taking $\alpha = 0,\ \beta = 0$ in (3), we get the Jakimovski-Leviatan-Durrmeyer operators (2). To examine the approximation results for the newly constructed operators, we need the following lemmas.

**Lemma 1.** *By the Appell polynomials* (1), *we easily get*

*(1)* $\sum\limits_{k=0}^{\infty} p_k(nx) = e^{nx} g(1)$,

*(2)* $\sum\limits_{k=0}^{\infty} k p_k(nx)) = e^{nx}[nxg(1) + g'(1)]$,

*(3)* $\sum\limits_{k=0}^{\infty} k^2 p_k(nx)) = e^{nx}[n^2x^2 g(1) + nx(2g'(1) + g(1)) + g''(1) + g'(1)]$,

*(4)* $\sum\limits_{k=0}^{\infty} k^3 p_k(nx)) = e^{nx}[n^3x^3g(1) + n^2x^2(3g'(1) + 4g(1)) + nx(3g''(1) + 8g'(1)) + g(1)) + g'''(1) + 4g''(1) + g'(1)],$

*(5)* $\sum\limits_{k=0}^{\infty} k^4 p_k(nx)) = e^{nx}[n^4x^4g(1) + n^3x^3(4g'(1) + 10g(1)) + n^2x^2(6g''(1) + 30g'(1) + 14g(1)) + nx(4g'''(1) + 30g''(1) + 28g'(1)) + g(1)) + g''''(1) + 1og''' + 14g''(1) + g'(1)].$

**Lemma 2.** *For $n > 3$, we have the following:*

*(1)* $L_n(e_0; x) = 1,$

*(2)* $L_n(e_1; x) = \frac{1}{n+\beta}(nx + (A_0 + \alpha)),$

*(3)* $L_n(e_2; x) = \frac{n^2}{n(n+1)(n+\beta)^2}[n^2x^2 + nx(2A_0+1) + (A_0+B_0)] + \frac{2n\alpha}{(n+1)(n+\beta)^2}[nx + (A_0+1)] + \frac{\alpha^2}{(n+\beta)^2}.$

*Proof.* (1) $L_n(e_0; x) = \frac{e^{-nx}}{g(1)} \sum\limits_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_0^\infty \frac{t^{k-1}}{(1+t)^{n+k+1}} dt = \frac{e^{-nx}}{g(1)} \sum\limits_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} B(k, n+1) = 1.$

(2)
$$\begin{aligned} L_n(e_1; x) &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_0^\infty \frac{t^{k-1}}{(1+t)^{n+k+1}} \left(\frac{nt+\alpha}{(n+\beta)}\right) dt \\ &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \frac{1}{n+\beta} [nB(k+1,n) + \alpha B(k, n+1)] \\ &= \frac{1}{n+\beta}(nx + (A_0 + \alpha)). \end{aligned}$$

(3)
$$\begin{aligned} L_n(e_2; x) &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_0^\infty \frac{t^{k-1}}{(1+t)^{n+k+1}} \left(\frac{nt+\alpha}{n+\beta}\right)^2 dt \\ &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \left[\frac{n^2}{(n+\beta)^2} B(k+2, n-1) + \frac{2n\alpha}{(n+\beta)^2} B(k+1, n) \right. \\ &\quad \left. + \frac{\alpha^2}{(n+\beta)^2} B(k, n+1)\right] \\ &= \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \left[\frac{n^2}{(n+\beta)^2} \frac{(k+2)(k+1)}{n(n+1)}) + \frac{2n\alpha}{(n+\beta)^2} \frac{(k+1)}{(n+1)} + \frac{\alpha^2}{(n+\beta)^2}\right] \\ &= \frac{n^2}{n(n+1)(n+\beta)^2} \left[n^2x^2 + nx(2A_0+1) + (A_0+B_0)\right] + \frac{2n\alpha}{n(n+1)(n+\beta)^2} \\ &\quad \times [nx + (A_0+1)] + \frac{\alpha^2}{(n+\beta)^2}. \end{aligned}$$

Hence the lemma is proved. ◀

**Theorem 1.** *For $f \in C[0,\infty)$, the operators $L_n$ converge uniformly to $f$ on the compact domain $[0,a], a > 0$ as $n \to \infty$.*

*Proof.* By the Lemma 2, we have

$$\lim_{n\to\infty} L_n(e_0;x) = 1,$$

$$\lim_{n\to\infty} L_n(e_1;x) = x,$$

$$\lim_{n\to\infty} L_n(e_2;x) = x^2.$$

Thus the operators $L_n$ converge unifomly to $f$, where $f$ is one of the test functions 1, $t$, $t^2$ on the compact interval $[0,a]$ . So by the Korovkin approximation theorem ([10], [13]), the result is true for every continuous function $f$ defined on the compact interval $[0,a]$. Hence the theorem is proved. ◄

## 3. Weighted Approximation

In this section, we investigate some approximation properties of the operators $L_n$ in the weighted space of continuous functions. We do this for the following class of continuous functions defined on $[0,\infty)$.

Let $B_{x^2}[0,\infty)$ be the linear space of all functions $h$ satisfying the condition $|h(x)| \leq K_h(1+x^2)$, where $K_h$ is a constant connected with $h$. We denote the subspace of all continuous functions of $B_{x^2}[0,\infty)$ by $C_{x^2}[0,\infty)$. Also, we denote by $C^*_{x^2}[0,\infty)$, the subclass of $C_{x^2}[0,\infty)$ of those functions $h$ for which $\lim\limits_{x\to\infty}\frac{h(x)}{1+x^2}$ is finite. It is obvious that $C^*_{x^2}[0,\infty) \subset C_{x^2}[0,\infty) \subset B_{x^2}[0,\infty)$. The norm for the space $C^*_{x^2}[0,\infty)$ is defined by

$$\|h\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|\ h(x)\ |}{1+x^2}.$$

**Lemma 3.** *Let $\rho(x) = 1 + x^2$ be a weight function. Then*

$$\|L_n(\rho;x)\|_{x^2} \leq K,$$

*where $K$ is a positive constant greater than unity.*

*Proof.* For $n > 1$, using the Lemma 2, we get $L_n(\rho;x) =$

$$1+\tfrac{n^4}{n(n+1)(n+\beta)^2}x^2+\left(\tfrac{n^3(2A_0+1)}{n(n+1)(n+\beta)^2}+\tfrac{2\alpha n^2}{(n+1)(n+\beta)^2}\right)x+\tfrac{n^2((1+2\alpha)A_0+B_0+2\alpha)}{n(n+1)(n+\beta)^2}+\tfrac{\alpha^2}{(n+\beta)^2},$$

therefore

$$\|L_n(\rho;x)\|_{x^2} = \sup_{x\geq 0}\left\{\frac{1}{1+x^2}+\frac{n^4}{n(n+1)(n+\beta)^2}\frac{x^2}{1+x^2}+\frac{n^3(2A_0+1)}{n(n+1)(n+\beta)^2}\frac{x}{1+x^2}\right.$$

$$\left.+\frac{2\alpha n^2}{(n+1)(n+\beta)^2}\frac{x}{1+x^2}+\frac{n^2((1+2\alpha)A_0+B_0+2\alpha)}{n(n+1)(n+\beta)^2}\frac{1}{1+x^2}+\frac{\alpha^2}{(n+\beta)^2}\frac{1}{1+x^2}\right\}$$

$$\leq 1 + \frac{n^4}{n(n+1)(n+\beta)^2} + \frac{n^3(2A_0+1)}{n(n+1)(n+\beta)^2} + \frac{2\alpha n^2}{(n+1)(n+\beta)^2}$$
$$+ \frac{n^2((1+2\alpha)A_0 + B_0 + 2\alpha)}{n(n+1)(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}.$$

Now since $\lim\limits_{n\to\infty} \frac{n^4}{n(n+1)(n+\beta)^2} = 1$, $\lim\limits_{n\to\infty} \frac{n^3}{n(n+1)(n+\beta)^2} = 0$, $\lim\limits_{n\to\infty} \frac{n^2}{(n+1)(n+\beta)^2} = 0$, $\lim\limits_{n\to\infty} \frac{n^2}{n(n+1)(n+\beta)^2} = 0$ and $\lim\limits_{n\to\infty} \frac{1}{(n+\beta)^2} = 0$, there exists a positive constant $K > 1$ such that

$$\|L_n(\rho; x)\|_{x^2} \leq K.$$

This completes the proof. ◄

By the Lemma 3, it is easily seen that the operators $L_n$ defined by (3) act from the space $C_{x^2}[0,\infty)$ to the space $B_{x^2}[0,\infty)$.

**Theorem 2.** *Let $L_n$ be the sequence of positive linear operators defined by (3) and $\rho(x) = 1 + x^2$ be the weight function. Then, for each $f \in C^*_{x^2}[0,\infty)$,*

$$\lim_{n\to\infty} \|L_n(f;x) - f(x)\|_{x^2} = 0.$$

*Proof.* In view of the Korovkin Theorem it is sufficient to prove that

$$\lim_{n\to\infty} \|L_n(t^i;x) - x^i\|_{x^2} = 0, \quad i = 0,1,2.$$

By Lemma 2 (1), it is obvious that

$$\lim_{n\to\infty} \|L_n(1;x) - 1\|_{x^2} = 0.$$

By Lemma 2 (2), $\|L_n(e_1;x) - e_1(x)\|_{x^2} = \sup\limits_{x\geq 0} \left| \frac{n}{n+\beta}\frac{x}{1+x^2} + \frac{A_0+\alpha}{n+\beta}\frac{1}{1+x^2} - \frac{x}{1+x^2} \right|$

$$= \sup_{x\geq 0} \left| \left(\frac{n}{n+\beta} - 1\right)\frac{x}{1+x^2} + \frac{A_0+\alpha}{n+\beta}\frac{1}{1+x^2} \right|$$
$$= \frac{\beta}{n+\beta} + \frac{A_0+\alpha}{n+\beta},$$

therefore,

$$\lim_{n\to\infty} \|L_n(e_1;x) - e_1(x)\|_{x^2} = 0.$$

By Lemma 2 (3), $\|L_n(e_2;x) - e_2(x)\|_{x^2} = \sup\limits_{x\geq 0} \left| \left(\frac{n^4}{n(n+1)(n+\beta)^2} - 1\right)\frac{x^2}{(1+x^2)} + \left(\frac{n^3(2A_0+1)}{n(n+1)(n+\beta)^2}\right.\right.$

$$\left.\left. + \frac{2\alpha n^2}{(n+1)(n+\beta)^2}\right)\frac{x}{(1+x^2)} + \left(\frac{n^2(A_0+B_0)}{n(n+1)(n+\beta)^2} + \frac{2\alpha n^2(A_0+1)}{n(n+1)(n+\beta)^2}\right.\right.$$
$$\left.\left. + \frac{\alpha^2}{(n+\beta)^2}\right)\frac{1}{1+x^2}\right|$$

$$\leq \left\{ \left( \frac{n^4}{n(n+1)(n+\beta)^2} - 1 \right) + \left( \frac{n^3(2A_0+1)}{n(n+1)(n+\beta)^2} + \frac{2\alpha n^2}{(n+1)(n+\beta)^2} \right) + \left( \frac{n^2(A_0+B_0)}{n(n+1)(n+\beta)^2} + \frac{2\alpha n^2(A_0+1)}{n(n+1)(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2} \right) \right\},$$

therefore,

$$\lim_{n\to\infty} \|L_n(e_2;x) - e_2(x)\|_{x^2} = 0.$$

This proves the theorem. ◄

## 4. Rate of Convergence

In this section, we compute the rate of convergence of our operators in terms of the modulus of continuity which is defined as follows:

The modulus of continuity of $f$, denoted by $\omega_f(\delta)$, gives the maximum oscillation of $f$ in any interval of length not exceeding $\delta(>0)$ and it is given by the relation

$$\omega_f(\delta) = \sup_{|x-y|\leq\delta} |f(x) - f(y)|, \quad x, y \in [0, 1+\ell].$$

It is known that $\lim\limits_{\delta\to 0^+} \omega_f(\delta) = 0$ for $f \in C[0, 1+\ell]$ and for any $\delta > 0$ one has

$$|f(y) - f(x)| \leq \omega_f(\delta)\left( \frac{|y-x|}{\delta} + 1 \right).$$

We denote by $C_E[0,\infty)$, the set of all continuous functions $f$ on $[0,\infty)$ with the property that $|f(x)| \leq be^{ax}$ for all $x \geq 0$ and for some positive finite constants $a, b$.

**Theorem 3.** *Let $f \in C_E[0,\infty), x \geq 0$ and $n > 1$. Then we have*

$$\left|L_n(f;x) - f(x)\right| \leq 2\omega_f\left(\delta_{n,x}\right),$$

*where*

$$\delta_{n,x}^2 = \omega_{n,1}x^2 + \omega_{n,2}x + \omega_{n,3} + \frac{\alpha^2}{(n+\beta)^2},$$

*with*

$$\omega_{n,1} = \frac{n((n+1)\beta^2) - 2(n+\beta) - 2n\beta - 3n^2}{(n+1)^2(n+\beta)^2},$$

$$\omega_{n,2} = \frac{n^5 - n^4(2\alpha(1+\beta) + 2A_0(2+\beta)) - 2n^3((1+2\beta)(A_0+\alpha) - 2n^2\beta(1+A_0))}{n^2(n+1)^2(n+\beta)^2},$$

$$\omega_{n,3} = \frac{n^4(A_0+B_0)}{n^2(n+1)^2(n+\beta)^2} + \frac{n^3(A_0+1)}{n(n+1)(n+\beta)^2}.$$

*Proof.* By linearity and positivity of the operators $L_n$, we get

$$\left|L_n(f;x) - f(x)\right| \leq \omega\,(f,\delta)\left\{1 + \frac{1}{\delta}(L_n((t-x)^2;x))^{\frac{1}{2}}\right\}.$$

Making use of the Lemma 2 and linearity of the operators $L_n$, for $n > 1$, we have $L_n((t-x)^2;x) = \frac{1}{n^2(n+1)^2(n+\beta)^2}\Bigg((n^3(n+1)\beta^2 - 2(n+\beta) - 2n\beta - 3n^2)x^2$

$$+(n^5 - n^4(2\alpha(1+\beta) + 2A_0(2+\beta) - 2n^3(1+2\beta)(A_0+\alpha) - 2n^2\beta(A_0+1)))x$$
$$+n^4((A_0+B_0) + (n+1)(A_0+1)) + n^2\alpha^2(n+1)^2\Bigg).$$

Now using this and taking $\delta_{n,x} = \delta$, we arrive at the required inequality. Hence the theorem is proved. ◄

## 5. Direct theorems

By $C_B[0,\infty)$ we denote the space of all bounded and continuous functions $f$ on $[0,\infty)$ with the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|.$$

For all $\delta > 0$, the Peetre's $K$-functional is defined by

$$K_2(f,\delta) = \inf_{h\in C_B^2[0,\infty)} \{\|f-h\| + \delta\|h''\|\},$$

where $C_B^2[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$. By [17, Theorem 2.4, p. 177], there exists a constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\delta),$$

where $\omega_2(h,\delta)$, the second order modulus of continuity of $h \in C_B[0,\infty)$, is defined as

$$\omega_2(h,\delta^{\frac{1}{2}}) = \sup_{0<p<\delta^{\frac{1}{2}}} \sup_{x\in[0,\infty)} |h(x+2p) - 2h(x+p) + h(x)|.$$

Also by $\omega(h,\delta)$ we denote the first order modulus of continuity of $h \in C_B[0,\infty)$. Next, for $f \in C_B[0,\infty)$, we define the following associated operators:

$$\tilde{L_n}(f;x) = L_n(f;x) - \frac{1}{n+\beta}\Bigg((n+\beta)x + (A_0+\alpha)\Bigg).$$

where $x \geq 0$.

**Lemma 4.** *Let $g \in C_B^2[0,\infty)$. Then for all $x \geq 0$ and $n > 1$, we get* $\left|\tilde{L_n}(g;x) - g(x)\right| \leq \phi_n^{\alpha,\beta}(x)\|g''\|$,

*where*

$$\phi_n^{\alpha,\beta}(x) = \frac{1}{n^2(n+1)^2(n+\beta)^2}\left(a_{n,x^2}x^2 + b_{n,x}x + c_n + \left(\frac{A_0+\alpha}{n+\beta}\right)^2\right),$$

$$a_{n,x^2} = n^3\left((n+1)\beta^2 - 2(n+\beta) - 1n\beta - 3n^2\right),$$

$$b_{n,x} = n^5 - n^4\left(2\alpha(1+\beta) + 2A_0(2+\beta) - 2n^3(1+2\beta)(A_0+\alpha) - 2n^2\beta(A_0+1)\right)$$

*and*

$$c_n = n^2\left(n^2(A_0+B_0) + (n+1)((A_0+1) + \alpha(n+1))\right).$$

*Proof.* It is obvious that $\tilde{L_n}(e_1(x) - x; x) = 0$. Let $g \in C_B^2[0,\infty)$. Then by Taylor's expansion of $g$, we have

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du,$$

where $t \in [0,\infty)$. Operating by $\tilde{L_n}$ on both sides of the above equality, we get $\tilde{L_n}(g;x) - g(x) = g'(x)\tilde{L_n}(t-x;x) + \tilde{L_n}\left(\int_x^t (t-u)g''(u)du; x\right)$

$$\begin{aligned} &= \tilde{L_n}\left(\int_x^t (t-u)g''(u)du; x\right) \\ &= L_n\left(\int_x^t (t-u)g''(u)du; x\right) - \int_x^{\frac{(n+\beta)x+(A_0+\alpha)}{n+\beta}} \left(\frac{(n+\beta)x+A_0}{n+\beta} - u\right) g''(u)du. \end{aligned}$$

Therefore

$$\begin{aligned} \left|L_n(g;x) - g(x)\right| &\leq L_n\left(\left|\int_x^t (t-u)g''(u)du\right|; x\right) + \\ &+ \left|\int_x^{\frac{(n+\beta)x+(A_0+\alpha)}{n+\beta}} \left(\frac{(n+\beta)x+A_0}{n+\beta} - u\right) g''(u)du\right|. \end{aligned} \tag{4}$$

Since $\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2\|g''\|$, we get

$$\left|\int_x^{\frac{(n+\beta)x+(A_0+\alpha)}{n+\beta}} \left(\frac{(n+\beta)x+A_0}{n+\beta} - u\right) g''(u)du\right| \leq \left(\frac{A_0+\alpha}{n+\beta}\right)^2 \|g''\|.$$

By (4), we have

$$\left|\tilde{L_n}(g;x) - g(x)\right| \leq \left\{L_n\left((t-x)^2; x\right) + \left(\frac{A_0+\alpha}{n+\beta}\right)^2\right\}\|g''\|.$$

Now using the expression for $L_n\left((t-x)^2;x\right)$ from the Theorem 3, we get $\left|\tilde{L_n}(g;x)-g(x)\right|\leq$ $\frac{1}{n^2(n+1)^2(n+\beta)^2}\Big\{n^3((n+1)\beta^2-2(n+\beta)-1n\beta-3n^2)x^2$

$$+(n^5-n^4\left(2\alpha(1+\beta)+2A_0(2+\beta)-2n^3(1+2\beta)(A_0+\alpha)-2n^2\beta(A_0+1)\right)x$$

$$+\left(n^2\left(n^2(A_0+B_0\right)+(n+1)((A_0+1)+\alpha(n+1))\right)+\left(\frac{A_0+\alpha}{n+\beta}\right)^2\Big\}\|g''\|$$

$$=\quad \frac{1}{n^2(n+1)^2(n+\beta)^2}\left(a_{n,x^2}x^2+b_{n,x}x+c_n+\left(\frac{A_0+\alpha}{n+\beta}\right)^2\right)\|g''\|.$$

where $a_{n,x^2}$, $b_{n,x}$ are respectively the coefficients of $x^2$, $x$ in the above expression and $c_n=n^2\left(n^2(A_0+B_0\right)+(n+1)((A_0+1)+\alpha(n+1))$. More succinctly, it can be writen as

$$\left|\tilde{L_n}(g;x)-g(x)\right|\leq\phi_n^{\alpha,\beta}(x)\|g''\|,$$

where

$$\phi_n^{\alpha,\beta}(x)=\frac{1}{n^2(n+1)^2(n+\beta)^2}\left(a_{n,x}x^2+b_{n,x}x+c_n+\left(\frac{A_0+\alpha}{n+\beta}\right)^2\right)$$

and hence the lemma is proved. ◀

**Theorem 4.** *Let $f\in C_B[0,\infty)$. Then for every $x\geq 0$, there exists a constant $K>0$ such that*

$$\left|L_n(f;x)-f(x)\right|\leq K\omega_2\left(f,\sqrt{\phi_n^{\alpha,\beta}(x)}\right)+\omega\left(f,\frac{A_0+\alpha}{n+\beta}\right),$$

*where $\phi_n^{\alpha,\beta}(x)$ is as in Lemma 4 and $\omega(f,.)$, $\omega_2(f,.)$ are respectively the first order modulus of continuity and the second order modulus of continuity of $f$.*

*Proof.* For $f\in C_B[0,\infty)$, $g\in C_B^2[0,\infty)$, by the definition of the operators $\tilde{L}_n$, we have

$$\left|L_n(f;x)-f(x)\right|\leq\left|\tilde{L}_n(f-g;x)\right|+\left|(f-g)(x)\right|+$$
$$+\left|\tilde{L}_n(g;x)-g(x)\right|+\left|f\left(\frac{(n+\beta)x+(A_0+\alpha)}{n+\beta}\right)-f(x)\right|.$$

But

$$\left|\tilde{L_n}(f;x)\right|\leq\|f\|L_n(1;x)+2\|f\|=3\|f\|,$$

so we have

$$\left|L_n(f;x)-f(x)\right|\leq 4\|f-g\|+\left|\tilde{L_n}(g;x)-g(x)\right|+\omega\left(f,\frac{A_0+\alpha}{n+\beta}\right).$$

Using Lemma 4, we easily get

$$\left|L_n(f;x)-f(x)\right|\leq 4\left(\|f-g\|+\phi_n^{\alpha,\beta}(x)g''\right)+\omega\left(f,\frac{A_0+\alpha}{n+\beta}\right).$$

On taking the infimum over all $g \in C_B^2[0,\infty)$ on the right hand side of above inequality, we arrive at the desired result. ◄

## 6. Conclusions

In this paper, we introduced the operators

$$L_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(nx)}{B(n+1,k)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} f\left(\frac{nt+\alpha}{n+\beta}\right) dt, \quad x \geq 0,$$

which are Stancu type generalization of the Jakimovski-Leviatan-Durrmeyer operators which are more general than the operators defined by Jakimovski and Leviatan as well as Durrmeyer type operators. We have calculated the moments for these operators and examined their approximation properties. We investigated the convergence of these operators with the help of Korovkin's approximation theorem in the weighted function space of continuous functions $C^*_{x^2}[0,\infty)$. We also studied local approximation properties and some direct theorems for these operators and computed the rate of convergence by means of the modulus of continuity.

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