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# A Boundary Value Problem with Retarded Argument and Discontinuous Coefficient in the Differential Equation

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**Abstract.** This paper studies a boundary value problem with a retarded argument and a discontinuous coefficient in the differential equation. The equivalent integral representation for the solution of the boundary value problem is constructed. The simplicity of the eigenvalues is proved. The asymptotic behaviors of the eigenvalues and eigenfunctions are investigated.

**Key Words and Phrases**: asymptotics of eigenvalues, boundary value problems, retarded argument.

**2010 Mathematics Subject Classifications**: 34B09, 34L05, 34L20

## 1. Introduction

In this paper, we consider the boundary value problem generated by the differential equation

$$y''(x) + \lambda^2\rho(x)y(x) + q(x)y\left(x - \Delta(x)\right) = 0, \quad 0 \le x \le \pi, \tag{1}$$

with the boundary conditions

$$y\left(0\right)\cos\alpha + y'\left(0\right)\sin\alpha = 0, \tag{2}$$

$$y\left(\pi\right)\cos\beta + y'\left(\pi\right)\sin\beta = 0, \tag{3}$$

$$y\left(x - \Delta\left(x\right)\right) \equiv y\left(0\right)\phi\left(x - \Delta\left(x\right)\right), \tag{4}$$

where the real valued functions $q(x)$ and $\Delta(x)$ are continuous on the interval $[0, \pi]$, $\lambda$ is a real parameter, $\alpha$ and $\beta$ are arbitrary real numbers, $\phi(x)$ is a continuous initial function on the initial set

$$E_0 = \{x - \Delta(x) : x - \Delta(x) < 0, x > 0\} \quad \text{with} \quad \phi(0) = 1 \quad \text{and}$$

$$\rho(x) = \begin{cases} 1, & 0 \le x < a, \\ \alpha^2, & a < x \le \pi, \end{cases}$$

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with $0 < \alpha \neq 1$.

There has been a growing interest in the theory of boundary value problems for differential equations with retarded arguments. For a comprehensive treatment and for references to the extensive literature on the subject one may refer to [6, 7, 8, 10, 14]. Equations with a retarded argument have a wide area of applications in the theory of mathematical control, theory of optimal control, mathematical biology, mathematical economics, etc. (see [2, 9, 13]).

Let us recall that by a differential equation with a retarded argument we mean a differential equation where unknown functions and their derivatives have different arguments. Boundary value problems with retarded argument in differential equation and discontinuities inside the interval have been studied in [4, 5]. In [3] a boundary value problem with a retarded argument in the Sturm-Liouville equation and spectral parameter in the boundary conditions was examined and an efficient numerical procedure for solving boundary value problem for a differential equation with retarded argument was given in [1]. For discontinuous boundary value problems with retarded argument many results are restricted to the ones that have discontinuities inside the interval (see [5]).

The structure of this paper is as follows. Following this introduction we provide the properties of the eigenvalues of the boundary value problem (1)-(4) and then we obtain the asymptotic formulas for the eigenfunctions.

Let $\varphi(x,\lambda)$ be a solution of equation (1) satisfying the conditions

$$\varphi(0,\lambda) = \sin\alpha, \quad \varphi'(0,\lambda) = -\cos\alpha. \tag{5}$$

The initial conditions (4), (5) determine a unique solution of equation (1) on the interval $[0,\pi]$ (see [14]).

**Lemma 1.** *The solution $\varphi(x,\lambda)$ of the boundary value problem (1)-(4) admits the equivalent integral representation:*

$$\varphi(x,\lambda) = \left[\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^+(x) + \frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^-(x)\right]\sin\alpha -$$

$$-\left[\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\sin\lambda\mu^+(x) + \frac{1}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\sin\lambda\mu^-(x)\right]\frac{\cos\alpha}{\lambda} +$$

$$+\frac{1}{4\lambda}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\sin\lambda\left(\mu^+(t)-\mu^+(x)\right)dt +$$

$$+\frac{1}{4\lambda}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\sin\lambda\left(\mu^-(t)-\mu^+(x)\right)dt +$$

$$+\frac{1}{4\lambda}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\sin\lambda\left(\mu^+(t)-\mu^-(x)\right)dt+$$
$$+\frac{1}{4\lambda}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\sin\lambda\left(\mu^-(t)-\mu^-(x)\right)dt, \quad (6)$$

*where* $\mu^{\pm}(x)=\pm x\sqrt{\rho(x)}+a\left(1\mp\sqrt{\rho(x)}\right)$.

*Proof.* To prove (6) first let us seek the solution of equation

$$\varphi''(x,\lambda)+\lambda^2\rho(x)\varphi(x,\lambda)=-q(x)\varphi\left(x-\Delta(x),\lambda\right),$$

in the form of

$$\varphi(x,\lambda)=c_1e_0(x,\lambda)+c_2e_0(x,-\lambda),$$

where $e_0(x,\lambda)$ and $e_0(x,-\lambda)$ are linearly independent and

$$e_0(x,\lambda)=\begin{cases} e^{i\lambda x} & ,0\le x<a,\\ \frac{1}{2}\left(1+\frac{1}{\alpha}\right)e^{i\lambda(\alpha x+a(1-\alpha))}+\frac{1}{2}\left(1-\frac{1}{\alpha}\right)e^{i\lambda(-\alpha x+a(1+\alpha))} & ,a<x\le\pi.\end{cases}$$

Applying the method of variation of parameters, we have

$$\begin{aligned}\varphi(x,\lambda) &= \frac{1}{2i\lambda}\int_0^x\left[e_0(t,\lambda)e_0(x,-\lambda)-e_0(t,-\lambda)e_0(x,\lambda)\right]q(t)\varphi(t-\Delta(t),\lambda)dt+\\ &+ \tilde{c}_1e_0\left(x,\lambda\right)+\tilde{c}_2e_0\left(x,-\lambda\right).\end{aligned}$$

After some making necessary calculations and taking (5) into consideration, we obtain (6). ◀

The function $\varphi(x,\lambda)$ is a nontrivial solution of equation (1) satisfying the initial conditions at the left-hand endpoint. Substituting $\varphi(x,\lambda)$ into (3) we obtain the characteristic equation

$$F(\lambda)\equiv\varphi(\pi,\lambda)\cos\beta+\varphi'(\pi,\lambda)\sin\beta=0. \quad (7)$$

By Theorem 1.1 (see [14]), the set of eigenvalues of the boundary value problem (1)-(4) coincides with the set of real roots of equation (7).

## 2. Properties of the Eigenvalues

In this section, we prove the simplicity of eigenvalues and then we obtain the asymptotics of eigenvalues of the boundary value problem (1)-(4).

Let us denote the solution of the equation $y''=-\lambda^2\rho(x)y$ satisfying the conditions

$$\varphi_0(0,\lambda)=\sin\alpha,\quad \varphi_0'(0,\lambda)=-\cos\alpha$$

by $\varphi_0(x,\lambda)$.

$\varphi_0(x,\lambda)$ has the following form

$$\varphi_0(x,\lambda)=\frac{\sin\alpha}{2}\left[\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^+(x)+\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\cos\lambda\mu^-(x)\right]$$

$$-\frac{\cos\alpha}{2\lambda}\left[\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\sin\lambda\mu^+(x)+\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\sin\lambda\mu^-(x)\right]. \tag{8}$$

When $q(x)\equiv 0$, the eigenvalues of the boundary value problem (1)-(4) can be found from the equation

$$F_0(\lambda)=\varphi_0(\pi,\lambda)\cos\beta+\varphi_0'(\pi,\lambda)\sin\beta=0$$

as

$$\lambda_n^0=n+\psi(n),\quad \sup_n|\psi(n)|<+\infty,\quad (n=0,\pm1,\pm2,\cdots) \tag{9}$$

(see [11]).

Before giving the below theorem, it will be useful to mention that the multiplicity of an eigenvalue of a boundary value problem is defined to be the number of linearly independent eigenfunctions corresponding to this eigenvalue.

**Theorem 1.** *The boundary value problem (1)-(4) can have only simple eigenvalues.*

*Proof.* Let $\tilde\lambda$ be an eigenvalue of the boundary value problem (1)-(4) and $\tilde\psi(x,\tilde\lambda)$ be a corresponding eigenfunction. By (2) and (5)

$$W\left\{\tilde\psi\left(0,\tilde\lambda\right),\varphi\left(0,\tilde\lambda\right)\right\}=\begin{vmatrix}\tilde\psi\left(0,\tilde\lambda\right) & \sin\alpha\\ \tilde\psi'\left(0,\tilde\lambda\right) & -\cos\alpha\end{vmatrix}=0$$

and according to Theorem II. 2. 2 (see [14]) the functions $\tilde\psi(x,\tilde\lambda)$ and $\tilde\varphi(x,\tilde\lambda)$ are linearly dependent on $[0,\pi]$. Hence it follows that $\varphi\left(x,\tilde\lambda\right)$ is an eigenfunction for the boundary value problem (1)-(4) and all the eigenfunctions of this boundary value problem which correspond to the eigenvalue $\tilde\lambda$ are pairwise linearly dependent. ◄

**Theorem 2.** *The eigenvalues of the boundary value problem (1)-(4) have the following form:*

$$\lambda_n=\lambda_n^0+\frac{d_n}{\lambda_n^0}+\frac{k_n}{n}, \tag{10}$$

*where $d_n$ is a bounded sequence and $k_n\in l_2$.*

*Proof.* First, let us note that we assume in this proof that the function $q(x)$ has a bounded derivative. It is evident that by substituting (6) into (7) we have

$$F(\lambda)=F_0(\lambda)+$$

$$+\frac{\cos\beta}{4\lambda}\left(1+\frac{1}{\alpha}\right)\int_0^\pi\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\sin\lambda\left(\mu^+(t)-\mu^+(\pi)\right)dt+$$

$$+\frac{\cos\beta}{4\lambda}\left(1+\frac{1}{\alpha}\right)\int_0^\pi\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\sin\lambda\left(\mu^-(t)-\mu^+(\pi)\right)dt+$$

$$+\frac{\cos\beta}{4\lambda}\left(1-\frac{1}{\alpha}\right)\int_0^\pi\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\sin\lambda\left(\mu^+(t)-\mu^-(\pi)\right)dt+$$

$$+\frac{\cos\beta}{4\lambda}\left(1-\frac{1}{\alpha}\right)\int_0^\pi\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\sin\lambda\left(\mu^-(t)-\mu^-(\pi)\right)dt+$$

$$+\frac{\alpha\sin\beta}{4}\left(1-\frac{1}{\alpha}\right)\int_0^\pi\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\cos\lambda\left(\mu^+(t)-\mu^-(\pi)\right)dt+$$

$$+\frac{\alpha\sin\beta}{4}\left(1-\frac{1}{\alpha}\right)\int_0^\pi\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\cos\lambda\left(\mu^-(t)-\mu^-(\pi)\right)dt+$$

$$-\frac{\alpha\sin\beta}{4}\left(1+\frac{1}{\alpha}\right)\int_0^\pi\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\cos\lambda\left(\mu^+(t)-\mu^+(\pi)\right)dt+$$

$$-\frac{\alpha\sin\beta}{4}\left(1+\frac{1}{\alpha}\right)\int_0^\pi\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\cos\lambda\left(\mu^-(t)-\mu^+(\pi)\right)dt. \qquad (11)$$

Now letting

$$\lambda_n=\lambda_n^0+\epsilon_n, \qquad (12)$$

substituting (12) into (11), then taking $F_0(\lambda_n^0)=0$ into account as $n\to\infty$ and calculating all the integrals above we have

$$\epsilon_n\approx\frac{d_n}{\lambda_n^0}+\frac{k_n}{n}, \qquad (13)$$

where

$$d_n=$$

$$\frac{-\cos\beta}{4\left(\lambda_n^0\right)^2\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1+\frac{1}{\alpha}\right)\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\cos\lambda_n^0\left(\mu^-(\pi)-\mu^+(\pi)\right)+$$

$$+\frac{\cos\beta}{4\left(\lambda_n^0\right)^2\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1+\frac{1}{\alpha}\right)\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\cos\lambda_n^0\left(2a-\mu^+(\pi)\right)+$$

$$+\frac{\cos\beta}{4\left(\lambda_n^0\right)^2\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1+\frac{1}{\alpha}\right)\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\left[1-\cos\lambda_n^0\mu^+(\pi)\right]-$$

$$-\frac{\cos\beta}{4\left(\lambda_n^0\right)^2\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1-\frac{1}{\alpha}\right)\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\cos\lambda_n^0\left(\mu^+(\pi)-\mu^-(\pi)\right)+$$

$$+\frac{\cos\beta}{4\left(\lambda_n^0\right)^2\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1-\frac{1}{\alpha}\right)\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\cos\lambda_n^0\mu^-(\pi)+$$

$$+\frac{\cos\beta}{4\left(\lambda_n^0\right)^2\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1-\frac{1}{\alpha}\right)\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\left[1-\cos\lambda_n^0\left(1-\mu^+(\pi)\right)\right]-$$

$$-\frac{\alpha\sin\beta}{4\lambda_n^0\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1-\frac{1}{\alpha}\right)\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\left[\sin\lambda_n^0\left(\mu^-(\pi)-\mu^+(\pi)\right)\right]+$$

$$-\frac{\alpha\sin\beta}{4\lambda_n^0\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1-\frac{1}{\alpha}\right)\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\left[\sin\lambda_n^0\left(2a-\mu^+(\pi)\right)\right]+$$

$$+\frac{\alpha\sin\beta}{4\lambda_n^0\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1+\frac{1}{\alpha}\right)\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\left[\sin\lambda_n^0\left(\mu^+(\pi)\right)\right]-$$

$$-\frac{\alpha\sin\beta}{4\lambda_n^0\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1+\frac{1}{\alpha}\right)\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\left[\sin\lambda_n^0\left(\mu^-(\pi)-\mu^+(\pi)\right)\right]+$$

$$+\frac{\alpha\sin\beta}{4\lambda_n^0\sqrt{\rho(t)}\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\left(1+\frac{1}{\alpha}\right)\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi(t-\Delta(t),\lambda)\left[\sin\lambda_n^0\left(2a-\mu^+(\pi)\right)\right]$$

and

$$\begin{aligned}
k_n &= \\
&- \frac{\cos\beta}{4\left(\lambda_n^0\right)^2\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\int_0^\pi\frac{1}{\sqrt{\rho(t)}}\left(1+\frac{1}{\sqrt{\rho(t)}}\right)\left[q'(t)\varphi(t-\Delta(t),\lambda)+q(t)\varphi'(t-\Delta(t),\lambda)\right]\cdot\\
&\cdot \cos\lambda_n^0\left(\mu^+(t)-\mu^+(\pi)\right)dt+\\
&+ \frac{\cos\beta}{4\left(\lambda_n^0\right)^2\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\int_0^\pi\frac{1}{\sqrt{\rho(t)}}\left(1-\frac{1}{\sqrt{\rho(t)}}\right)\left[q'(t)\varphi(t-\Delta(t),\lambda)+q(t)\varphi'(t-\Delta(t),\lambda)\right]\cdot\\
&\cdot \cos\lambda_n^0\left(\mu^-(t)-\mu^+(\pi)\right)dt-\\
&- \frac{\cos\beta}{4\left(\lambda_n^0\right)^2\left[\dot{F}_0\left(\lambda_n^0\right)\right]}\int_0^\pi\frac{1}{\sqrt{\rho(t)}}\left(1+\frac{1}{\sqrt{\rho(t)}}\right)\left[q'(t)\varphi(t-\Delta(t),\lambda)+q(t)\varphi'(t-\Delta(t),\lambda)\right]\cdot
\end{aligned}$$

$$\cdot \quad \cos \lambda_n^0 \left(\mu^+(t) - \mu^+(\pi)\right) dt +$$

$$+ \quad \frac{\cos\beta}{4\left(\lambda_n^0\right)^2 \left[\dot{F}_0\left(\lambda_n^0\right)\right]} \int_0^\pi \frac{1}{\sqrt{\rho(t)}} \left(1 + \frac{1}{\sqrt{\rho(t)}}\right) \left[q'(t)\varphi(t-\Delta(t),\lambda) + q(t)\varphi'(t-\Delta(t),\lambda)\right] \cdot$$

$$\cdot \quad \cos \lambda_n^0 \left(\mu^-(t) - \mu^-(\pi)\right) dt +$$

$$+ \quad \frac{\sin\beta}{4\lambda_n^0 \left[\dot{F}_0\left(\lambda_n^0\right)\right]} \int_0^\pi \frac{1}{\sqrt{\rho(t)}} \left(1 + \frac{1}{\sqrt{\rho(t)}}\right) \left[q'(t)\varphi(t-\Delta(t),\lambda) + q(t)\varphi'(t-\Delta(t),\lambda)\right] \cdot$$

$$\cdot \quad \sin \lambda_n^0 \left(\mu^+(t) - \mu^-(\pi)\right) dt -$$

$$- \quad \frac{\sin\beta}{4\lambda_n^0 \left[\dot{F}_0\left(\lambda_n^0\right)\right]} \int_0^\pi \frac{1}{\sqrt{\rho(t)}} \left(1 - \frac{1}{\sqrt{\rho(t)}}\right) \left[q'(t)\varphi(t-\Delta(t),\lambda) + q(t)\varphi'(t-\Delta(t),\lambda)\right] \cdot$$

$$\cdot \quad \sin \lambda_n^0 \left(\mu^-(t) - \mu^-(\pi)\right) dt +$$

$$+ \quad \frac{\sin\beta}{4\lambda_n^0 \left[\dot{F}_0\left(\lambda_n^0\right)\right]} \int_0^\pi \frac{1}{\sqrt{\rho(t)}} \left(1 + \frac{1}{\sqrt{\rho(t)}}\right) \left[q'(t)\varphi(t-\Delta(t),\lambda) + q(t)\varphi'(t-\Delta(t),\lambda)\right] \cdot$$

$$\cdot \quad \sin \lambda_n^0 \left(\mu^+(t) - \mu^+(\pi)\right) dt -$$

$$- \quad \frac{\sin\beta}{4\lambda_n^0 \left[\dot{F}_0\left(\lambda_n^0\right)\right]} \int_0^\pi \frac{1}{\sqrt{\rho(t)}} \left(1 - \frac{1}{\sqrt{\rho(t)}}\right) \left[q'(t)\varphi(t-\Delta(t),\lambda) + q(t)\varphi'(t-\Delta(t),\lambda)\right] \cdot$$

$$\cdot \quad \sin \lambda_n^0 \left(\mu^-(t) - \mu^+(\pi)\right) dt.$$

It can easily be seen that $d_n$ is a bounded sequence. $k_n$ can be reduced to

$$\xi\left(\lambda\right) := \int_0^\pi R(t) e^{i\lambda t} dt,$$

where $R(t) \in L_2(0,\pi)$. It is clear from [12] (p. 66) that $\xi_n = \xi\left(\lambda_n\right) \in l_2$. By virtue of this we have $k_n \in l_2$. ◀

## 3. Asymptotics of the Eigenfunctions

This section aims to show the boundedness of the eigenfunctions and then get the asymptotic formulas for them. Let $\Delta_0 = \max_{[0,\pi]} \Delta(t)$ and $q_\pi = \int_0^\pi |q(t)|\, dt$, extend the function $\phi(x)$ continuously to the interval $[-\Delta_0, 0]$ and let $\phi_0 = \max_{[-\Delta_0,0]} |\phi(t)|$.

**Lemma 2.** *As $\lambda > 4q_\pi$, the solution $\varphi(x,\lambda)$ satisfies the following inequality:*

$$|\varphi(x,\lambda)| \le max\left\{\frac{2}{q_\pi}\sqrt{25\pi^2 \sin^2\alpha + \cos^2\alpha}, |\sin\alpha|\, \phi_0\right\}, \quad \left(-\Delta_0 \le x \le \pi\right). \tag{14}$$

*Proof.* Let $B_\lambda = \max_{[0,\pi]} |\varphi(x,\lambda)|$. From (6) it is clear that

$$|\varphi(x,\lambda)| \leq$$

$$\leq \frac{1}{2}\left|1+\frac{1}{\sqrt{\rho(x)}}\right| \left|\left(\cos\lambda\mu^+(x)+\cos\lambda\mu^-(x)\right)\sin\alpha + \left(\sin\lambda\mu^+(x)+\sin\lambda\mu^-(x)\right)\frac{\cos\alpha}{\lambda}\right| +$$
$$+ \frac{1}{4|\lambda|}\left|1+\frac{1}{\sqrt{\rho(x)}}\right| \int_0^x \left|1+\frac{1}{\sqrt{\rho(t)}}\right| |q(t)|\,|\varphi(t-\Delta(t),\lambda)| \cdot$$
$$\cdot \; |\sin\lambda\left(\mu^+(t)-\mu^+(x)\right)+\sin\lambda\left(\mu^-(t)-\mu^+(x)\right)+$$
$$+ \sin\lambda\left(\mu^+(t)-\mu^-(x)\right)+\sin\lambda\left(\mu^-(t)-\mu^-(x)\right)|dt.$$

Hence we have

$$B_\lambda \leq 2\left|\sin\alpha + \frac{\cos\alpha}{\lambda}\right| + \frac{4}{|\lambda|}B_\lambda q_\pi.$$

From the last inequality, we derive easily that

$$B_\lambda \leq 2\sqrt{\sin^2\alpha + \frac{\cos^2\alpha}{\lambda^2}} + \frac{4}{\lambda}B_\lambda q_\pi.$$

Taking (5) into account and following the same procedure we similarly arrive at

$$B_\lambda \leq 2\sqrt{\sin^2\alpha + \frac{\cos^2\alpha}{\lambda^2}} + \frac{4}{\lambda}q_\pi\,|\sin\alpha|\,\phi_0.$$

However, in either case, if $\lambda > 4q_\pi$,

$$B_\lambda \leq max\left\{\frac{2}{q_\pi}\sqrt{25\pi^2\sin^2\alpha+\cos^2\alpha}, |\sin\alpha|\,\phi_0\right\}.$$

This yields us to (14). ◄

In what follows we shall assume that $\lambda$ is sufficiently large and we consider the four possible cases:

$$1.\sin\alpha \neq 0, \sin\beta \neq 0;$$

$$2.\sin\alpha \neq 0, \sin\beta = 0;$$

$$3.\sin\alpha = 0, \sin\beta \neq 0;$$

$$4.\sin\alpha = 0, \sin\beta = 0.$$

The following lemma can be proven easily with the help of inequality (14) and condition (4).

**Lemma 3.** *In cases 1 and 2 we have*

$$\varphi(x,\lambda) = O(1), \tag{15}$$

*and in cases 3 and 4 we have*

$$\varphi(x,\lambda) = O\left(\frac{1}{\lambda}\right), \tag{16}$$

*on the interval* $[-\Delta_0, \pi]$.

**Lemma 4.** *In cases 1 and 2 we have*

$$\varphi'_\lambda(x,\lambda) = O\left(1\right), \tag{17}$$

*on the interval* $[-\Delta_0, \pi]$.

*Proof.* By differentiating (6) with respect to $\lambda$, with the help of (14) we have,

$$\begin{aligned}
\varphi'_\lambda(x,\lambda) = & \\
= & \frac{1}{4\lambda}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi'_\lambda(t,\lambda)\sin\lambda\left(\mu^+(t)-\mu^+(x)\right)dt + \\
+ & \frac{1}{4\lambda}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi'_\lambda(t,\lambda)\sin\lambda\left(\mu^-(t)-\mu^+(x)\right)dt + \\
+ & \frac{1}{4\lambda}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi'_\lambda(t,\lambda)\sin\lambda\left(\mu^+(t)-\mu^+(x)\right)dt + \\
+ & \frac{1}{4\lambda}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi'_\lambda(t,\lambda)\sin\lambda\left(\mu^-(t)-\mu^+(x)\right)dt + \\
& +K\left(x,\lambda\right),
\end{aligned} \tag{18}$$

where $|K\left(x,\lambda\right)| \le K_0$. Let $C_\lambda = \max\limits_{[-\Delta_0,\pi]} |\varphi'_\lambda(x,\lambda)|$. The existence of $C_\lambda$ follows from the continuity of the derivative on $[-\Delta_0, \pi]$.

From (18) we have

$$C_\lambda \le \frac{4}{\lambda} q_\pi C_\lambda + K_0.$$

Now let $\lambda > 4q_\pi$. Then $C_\lambda \le 5K_0$, and the validity of the asymptotic formula (16) follows. ◀

**Lemma 5.** *In cases 3 and 4, we have*

$$\varphi'_\lambda(x,\lambda) = O\left(\frac{1}{\lambda}\right), \tag{19}$$

*on* $[-\Delta_0, \pi]$.

*Proof.* Differentiating (6) with respect to $\lambda$ and by using (15) we obtain

$$\begin{aligned}
\varphi'_{\lambda}(x,\lambda) &= \frac{1}{4\lambda}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi'_{\lambda}(t,\lambda)\sin\lambda\left(\mu^+(t)-\mu^+(x)\right)dt+\\
&+\frac{1}{4\lambda}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi'_{\lambda}(t,\lambda)\sin\lambda\left(\mu^-(t)-\mu^+(x)\right)dt+\\
&+\frac{1}{4\lambda}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1+\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi'_{\lambda}(t,\lambda)\sin\lambda\left(\mu^+(t)-\mu^+(x)\right)dt+\\
&+\frac{1}{4\lambda}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)\int_0^x\left(1-\frac{1}{\sqrt{\rho(t)}}\right)q(t)\varphi'_{\lambda}(t,\lambda)\sin\lambda\left(\mu^-(t)-\mu^+(x)\right)dt+\\
&+\frac{1}{\lambda}K^*(x,\lambda),
\end{aligned}$$

where $|K^*(x,\lambda)|\le K_0^*$. Repeating the arguments used in the proof of Lemma 4 we arrive at (19).

**Theorem 3.** *The eigenfunctions of the boundary value problem (1)-(4) have the following form*

$$\varphi(x,\lambda_n)=\varphi_0(x,\lambda_n)+O\left(\frac{1}{n}\right). \tag{20}$$

*Proof.* Substituting formula (12) into (6) we can easily obtain the asymptotic formulas for the eigenfunctions $\varphi(x,\lambda_n)$ as in (20). ◄

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