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On the Minimality of Double Exponential System in Weighted Lebesgue Space

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Abstract. This paper considers double exponential linear phase system in the weighted space $L_{p,\rho}$ with power weight $\rho(\cdot)$ on the segment $[\pi,\pi]$. Under certain conditions on the weight function $\rho(\cdot)$ and on the perturbation parameters, the minimality of this system in $L_{p,\rho}$ is proved. An explicit expression for the biorthogonal system in the case of minimality is derived and its integral representation is obtained.

Key Words and Phrases: exponential system, basicity, weighted space.

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1. Introduction

The study of many partial differential equations by the Fourier method reduces to the study of perturbed trigonometric system of sines (or cosines) of the form

$$\{\sin(nt+\alpha(t))\}_{n\in N}, \tag{1}$$

where $\alpha : [0,\pi] \to R$ is some function (N is the set of all positive integers). Similar problems were studied, for example, in the papers [9-16]. To justify the Fourier method, it is necessary to study the basis properties (completeness, minimality, basicity, etc.) of these systems in different functional spaces. Complex version of these systems is a perturbed exponential system of the form

$$\left\{e^{i(nt+\beta(t)sign\, n)}\right\}_{n\in Z}, \tag{2}$$

where $\beta : [-\pi,\pi] \to R$ is some function (Z is the set of all integers). Basis properties of the systems (1) and (2) in corresponding spaces are closely linked. In Lebesgue spaces L_p they have been well studied by various mathematicians (see, for example, [9-11,17,18,22,24-32]). The case $L_\infty = C[-\pi,\pi]$ was treated in [43]. In the context of differential equations, there has recently been a growing interest in Lebesgue spaces $L_{p(\cdot)}$ with the variable rate of summability $p(\cdot)$ and Morrey spaces $L^{p,\alpha}$. Problems of approximation in these

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spaces have also begun to be studied. Basicity problems of the systems (1), (2) in $L_{p(\cdot)}$ were studied in [34, 35], while basicity of the classical exponential linear phase system in Morrey spaces was studied in [38, 39]. Note that the study of basicity properties of the systems (1), (2) in weighted spaces $L_{p,\rho}$ is equivalent to the one of similar properties of (1), (2) with corresponding degenerate coefficients in the spaces L_p. That's why we can say that the study of the basicity of trigonometric systems in weighted Lebesgue spaces dates back to K.Babenko [23]. This research area was further developed in the works [19, 20, 21, 33, 36, 37, 40, 41]. The condition which allows to solve the problem of basicity of exponential system in the weighted space $L_{p,\rho} \equiv L_{p,\rho}(-\pi,\pi)$, $1 < p < +\infty$, can be found in [42]. We mean the Muckenhaupt condition with respect to the weight function $\rho(\cdot)$:

$$\sup_I \left(\frac{1}{|I|}\int_I \rho(t)\,dt\right)\left(\frac{1}{|I|}\int_I \rho^{-\frac{1}{p-1}}dt\right)^{p-1} < \infty, \tag{3}$$

where sup is taken over all intervals $I \subset [-\pi,\pi]$ and $|I|$ is the length of the interval I.

In [6, 20], the basicity of the system (2) in $L_{p,\rho}$, $1 < p < +\infty$, was studied in the case where $\beta(t) = \beta t$, $\beta \in R$ is some real parameter and $\rho(\cdot)$ has the following form

$$\rho(t) = \prod_{k=-r}^{r} |t - t_k|^{\alpha_k},$$

with $-\pi = t_{-r} < t_{-r+1} < ... < t_r = \pi$.

The class of weights, satisfying the condition (3), is denoted by A_p. It is easy to see that

$$\rho \in A_p \Leftrightarrow -1 < \alpha_k < p-1, \;\; k = \overline{-r,r}.$$

It is additionally required in [6] that the condition $\alpha_{-r} = \alpha_r$ holds, which means that the degeneration must be present at both ends of the segment $[-\pi,\pi]$. This effect does not take place in [20].

In this paper, the minimality of the exponential system

$$\left\{e^{i\left(n+\frac{\beta}{2} sign\, n\right)t}\right\}_{n\in Z},$$

is studied in the weighted space $L_{p,\rho}$, $1 < p < +\infty$, where $\beta \in C$ is a complex parameter. Unlike [6], an explicit expression for the biorthogonal system is built and its integral representation is obtained.

2. Preliminaries. Main lemma

Consider the following double exponential system:

$$\left\{e^{i[(n+\beta_1)t+\gamma]}; e^{-i[(k+\beta_2)t+\gamma_2]}\right\}_{n\in Z_+;k\in N}, \tag{4}$$

where $\beta_k = Re\beta_k + iIm\beta_k,\ \gamma_k = Re\,\gamma_k + iIm\,\gamma_k,\ k = 1,2,$ are complex parameters, $Z_+ = \{0\} \bigcup N$. We assume that the weight function $\rho(\cdot)$ has the following form

$$\rho(t) = \prod_{k=-r}^{r} |t - t_k|^{\alpha_k},$$

where $-\pi = t_{-r} < t_{-r+1} < ... < t_0 = 0 < ... < t_r = \pi, \quad \{\alpha_k\}_{k=-\overline{r,r}} \subset R$ are some numbers. We consider the weighted space $L_{p,\rho},\quad 1 < p < +\infty$, with the norm $\|\cdot\|_{p,\rho}$:

$$\|f\|_{p,\rho} = \left(\int_{-\pi}^{\pi} |f(t)|^p \rho(t)\, dt\right)^{1/p}.$$

It is easy to see that the basicity properties of the system (4) in $L_{p,\rho}$ are equivalent to those of the system

$$\left\{e^{i(n+\beta_1)t}; e^{-i(k+\beta_2)t}\right\}_{n\in Z_+;k\in N}, \tag{5}$$

in $L_{p,\rho}$. We put $g(t) = e^{\frac{i}{2}(\beta_2-\beta_1)t}$. It is evident that $\exists\delta > 0$:

$$0 < \delta \le |g(t)| \le \delta^{-1} < +\infty,\ \ \forall t \in [-\pi, \pi].$$

Multiplying the system (5) by the function $g(t)$, we immediately obtain that the basicity properties of the system (5) in $L_{p,\rho}$ are equivalent to those of the system

$$\left\{e^{i\left(n+\frac{\beta}{2} sign\, n\right)t}\right\}_{n\in Z}, \tag{6}$$

in $L_{p,\rho}$, $\beta = \beta_1 + \beta_2$. Thus, the study of basicity properties of the system (4) in $L_{p,\rho}$ is reduced to the study of corresponding properties of the system (6) in $L_{p,\rho}$.

Let $\beta \in C$ be some complex number. We will assume throughout this paper that $(1+z)^\beta$ is some fixed branch of multivalued analytic function $(1+z)^\beta$ on the complex plane with the cut along the half axis $(-\infty,-1) \subset R$ on the real axis and let

$$(1+z)^{-\beta} = \frac{1}{(1+z)^\beta}.$$

Similarly, we define a branch z^β of a multivalued function z^β on C with the cut along $(-\infty, 0) \subset R$ and $z^{-\beta} = \frac{1}{z^\beta}$.

We will essentially use the following main lemma in the proof of our main results.

Lemma 1. *Let $Re\beta > -1$. Then the following Cauchy integral formulas hold*

$$J_m^-(z) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-i(\beta+m)\theta}\left(1+e^{i\theta}\right)^\beta}{e^{i\theta}-z} d\theta \equiv \begin{cases} 0, & |z| < 1, \\ -z^{-m-\beta-1}(1+z)^\beta, & |z| > 1, \end{cases}$$

$$J_m^+(z) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i(m+1)\theta}\left(1+e^{i\theta}\right)^\beta}{e^{i\theta}-z} d\theta \equiv \begin{cases} 0, & |z| > 1, \\ z^m (1+z)^\beta, & |z| < 1, \end{cases}$$

$$\forall m \in Z_+.$$

Proof. Consider an expression $J_m^+(z)$. Make the change of variables

$$tg\frac{\theta}{2} = t \Rightarrow e^{i\theta} = \frac{1+it}{1-it}.$$

We have

$$J_m^+(z) = \int_{-\infty}^{+\infty} \frac{(1+it)^{m+1}(1-it)^{-m-1}}{\frac{2^{-\beta}}{(1-it)^{-\beta}}\left(\frac{1+it}{1-it} - z\right)} \frac{2dt}{(1-it)(1+it)} =$$

$$= 2^{\beta+1}\int_{-\infty}^{+\infty} (1-it)^{-\beta}(1-it)^{-m-1}(1+it)^m [1+it-z+izt]^{-1} dt =$$

$$= \frac{2^{\beta+1}}{i(1+z)}\int_{-\infty}^{+\infty} (1-it)^{-\beta}(1-it)^{-m-1}\left[t - \frac{i(1-z)}{1+z}\right]^{-1}(1+it)^m dt.$$

Let $Rez = x$, $Imz = y$. We obtain

$$Im\left(i\frac{1-z}{1+z}\right) = \frac{1-x^2-y^2}{(1+x)^2+y^2} > 0, \qquad |z| < 1,$$

and it is evident that

$$Im\left(i\frac{1-z}{1+z}\right) < 0, \qquad |z| > 1.$$

Denote the integrand function by $F(w)$, $w \in C$:

$$F(w) = (1-iw)^{-\beta-m-1}(1+iw)^m\left(w - i\frac{1-z}{1+z}\right)^{-1}.$$

It is obvious that for large values of $|w|$ the following estimation holds

$$|F(w)| \leq \frac{M}{|w|^{2+Re\beta}},$$

where $M > 0$ is some constant. Applying Theorem 5.3 from monograph [1] (see p. 127), we obtain that

$$J_m^+(z) = \frac{2^{\beta+1}}{i(1+z)} 2\pi i \operatorname*{Res}_{t=i\frac{1-z}{1+z}}\left[(1-it)^{-\beta-m-1}(1+it)^m\left(t - i\frac{1-z}{1+z}\right)^{-1}\right] =$$

$$= \frac{2^{\beta+2}\pi}{1+z}\left(\frac{2}{1+z}\right)^{-\beta-m-1}\left(\frac{2z}{1+z}\right)^m = 2\pi z^m(1+z)^\beta, \quad |z| < 1,$$

since for $|z| < 1$, the only pole of the function $F(w)$ in the upper half-plane is $w = i\frac{1-z}{1+z}$. Similar reasoning yields $J_m^+(z) \equiv 0$, $|z| > 1$, since for $|z| > 1$ the function $F(w)$ has no poles in the upper half-plane.

The formula for $J_m^-(z)$ is proved in a similar way. ◄

3. Minimality in $L_{p,\rho}$

Consider the following system of functions

$$\vartheta_n^+(t) = \frac{e^{-i\frac{\beta}{2}t}}{2\pi}\left(1+e^{it}\right)^{\beta}\sum_{k=0}^{n} C_{-\beta}^{n-k} e^{-ikt}, \;\; n \in Z_+;$$
$$\vartheta_m^-(t) = -\frac{e^{-i\frac{\beta}{2}t}}{2\pi}\left(1+e^{it}\right)^{\beta}\sum_{k=1}^{m} C_{-\beta}^{m-k} e^{ikt}, \;\; m \in N;$$

where

$$C_{-\gamma}^{k} = \frac{\gamma(\gamma-1)\dots(\gamma-k+1)}{k!},$$

is a binomial coefficient. Denote

$$e_n^+(t) \equiv e^{i\left(n+\frac{\beta}{2}\right)t}, \;\; n \in Z_+; \;\; e_k^-(t) \equiv e^{-i\left(n+\frac{\beta}{2}\right)t}, \;\; k \in N.$$

Assume that $Re\beta > -1$. The expansion of the function $(1+z)^{-\beta} J_m^+(z)$(which is analytic on $|z| < 1$) in powers of z is

$$(1+z)^{-\beta} J_m^+(z) = \sum_{n=0}^{\infty} a_{n;m}^+ z^n,$$

where

$$a_{n;m}^+ = \int_{-\pi}^{\pi} e^{i\left(m+\frac{\beta}{2}\right)t} \vartheta_n^+(t)\, dt.$$

On the other hand, it follows from Lemma 1 that

$$(1+z)^{-\beta} J_m^+(z) \equiv z^m, \quad |z| < 1.$$

Comparing the corresponding coefficients, we arrive at the following equalities:

$$\int_{-\pi}^{\pi} e_m^+(t)\, \vartheta_n^+(t)\, dt = \delta_{nm}, \quad \forall n, m \in Z_+.$$

Expanding the function $(1+z)^{-\beta} J_m^+(z)$ in powers of z^{-1} at infinity, we obtain

$$(1+z)^{-\beta} J_m^+(z) = \sum_{n=1}^{\infty} b_{n;m}^+ z^{-n}, \quad |z| > 1,$$

where

$$b_{n;m}^+ = \int_{-\pi}^{\pi} e^{i\left(m+\frac{\beta}{2}\right)t} \vartheta_n^-(t)\, dt, \quad m \in Z_+, \quad n \in N.$$

It is easy to see that

$$\lim_{|z|\to\infty} (1+z)^{-\beta} J_m^+(z) = 0.$$

On the other hand, again, due to Lemma 1, we have

$$(1+z)^{-\beta} J_m^+(z) \equiv 0, \quad |z| > 1.$$

These two expansions imply

$$\int_{-\pi}^{\pi} e^{i\left(m+\frac{\beta}{2}\right)t}\vartheta_n^-(t)\,dt=0,\quad \forall m\in Z_+,\quad \forall n\in N.$$

The relations

$$\begin{array}{l}\int_{-\pi}^{\pi} e_m^-(t)\,\vartheta_n^+(t)\,dt=0,\quad m\in N,\quad n\in Z_+;\\ \int_{-\pi}^{\pi} e_m^-(t)\,\vartheta_n^-(t)\,dt=\delta_{nm},\qquad \forall n,m\in N,\end{array}$$

can be proved similarly.

As a result, we get the validity of the following statement.

Proposition 1. *Let $Re\beta>-1$. Then for all admissible values of indices n and m the relations*

$$\int_{-\pi}^{\pi} e_n^{\pm}(t)\,\vartheta_m^{\pm}(t)\,dt=\delta_{nm},\qquad \int_{-\pi}^{\pi} e_n^{\pm}(t)\,\vartheta_m^{\mp}(t)\,dt=0,$$

hold.

Consider the following proposition.

Proposition 2. *Let $1<p<+\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then $(L_{p,\rho})^*=L_{q,\rho}$ and every functional $\vartheta^*\in(L_{p,\rho})^*$ is represented, in terms of uniquely determined for it function $\vartheta\in L_{q,\rho}$, by the expression*

$$\vartheta^*(f)=\int_{-\pi}^{\pi} f\bar{\vartheta}\rho dt,\quad \forall f\in L_{p,\rho}.$$

Now define the following system of functions

$$h_n^{\pm}(t)=\rho^{-1}(t)\,\overline{\vartheta_n^{\pm}(t)}.$$

It is easy to see that the system $\{h_n^{\pm}\}$ belongs to the space $L_{q,\rho}$ when

$$\alpha_k<\frac{1}{q-1},\quad k=\overline{-r+1,r-1};\ Re\beta-\frac{\alpha_{\pm r}}{p}>-\frac{1}{q}.$$

This follows directly from the representation of $\{\vartheta_n^{\pm}\}$ and from the relation

$$\int_{-\pi}^{\pi}\left|h_n^{\pm}\right|^q\rho dt=\int_{-\pi}^{\pi}\rho^{1-q}\left|\vartheta_n^{\pm}\right|^q dt.$$

Taking into account that $\frac{1}{q-1}=\frac{p}{q}$, we obtain the following theorem from Propositions 1 and 2.

Theorem 1. *Assume that the following inequalities hold:*

$$\begin{array}{l}Re\beta>-1;\quad -1<\alpha_k<\frac{p}{q},\quad k=\overline{-r+1,r-1};\\ -1<\alpha_{\pm r}<\frac{p}{q}+pRe\beta.\end{array}$$

Then the exponential system $\left\{e^{i\left(n+\frac{\beta}{2}sign\,n\right)t}\right\}_{n\in Z}$ is minimal in $L_{p,\rho}$, $1<p<+\infty$.

The lemma below plays a very important role in the study of orthogonal series.

Lemma 2. *The system* $\left\{\vartheta_B^{\pm}\right\}$ *has the following integral representation:*

$$\vartheta_{n-1}^{+}(t)=\frac{e^{-i\left(n+\frac{\beta}{2}\right)t}}{2\pi}\left(1+e^{it}\right)^{\beta}\left[\left(1+e^{it}\right)^{-\beta}-\frac{\sin\pi\beta}{\pi}e^{i(t+\pi)n}\int_0^1\frac{x^{n+\beta-1}}{1+xe^{it}}dx\right],$$

$$\vartheta_{n}^{-}(t)=\frac{e^{i\left(n-\frac{\beta}{2}\right)t}}{2\pi}\left(1+e^{it}\right)^{\beta}\left[\left(1+e^{-it}\right)^{-\beta}-\frac{\sin\pi\beta}{\pi}e^{i(\pi-t)n}\int_0^1\frac{x^{n+\beta-1}(1-x)^{-\beta}}{1+xe^{it}}dx\right],\ \ n\in N,$$

for $Re\beta\in(-1,1)$.

Proof. We will prove this lemma with regard to ϑ_n^- since for ϑ_n^+ it can be proved in exactly the same way. Thus, let

$$\vartheta_n^-(t)=-\frac{e^{-i\frac{\beta}{2}t}}{2\pi}\left(1+e^{it}\right)^{\beta}\sum_{k=1}^{n}C_{-\beta}^{n-k}e^{itk}.$$

Make the following transformation

$$\sum_{k=1}^{n}C_{-\beta}^{n-k}e^{ikt}=e^{int}\sum_{k=0}^{n-1}C_{-\beta}^{k}e^{-ikt}=e^{int}\left[\left(1+e^{-it}\right)^{-\beta}-\sum_{k=n}^{\infty}C_{-\beta}^{k}e^{-ikt}\right]=$$

$$=e^{int}\left[\left(1+e^{-it}\right)^{-\beta}-e^{-int}\sum_{k=0}^{\infty}C_{-\beta}^{k+n}e^{-ikt}\right]=$$

$$=e^{int}\left[\left(1+e^{-it}\right)^{-\beta}-e^{-int}\frac{(-1)^n(\beta)_n}{n!}F\left(1;n+\beta;n+1;e^{i(\pi-t)}\right)\right],$$

where

$$(\beta)_n=\beta(\beta+1)\ldots\ldots(\beta+n-1)=\frac{\Gamma(\beta+n)}{\Gamma(\beta)},$$

$\Gamma(\cdot)$ is Euler's gamma function and $F(a;b;c;z)$ is hypergeometric function. Using integral representation for hypergeometric functions (see [3], p. 72), we find

$$\sum_{k=1}^{n}C_{-\beta}^{n-k}e^{ikt}=e^{int}\left[\left(1+e^{-it}\right)^{-\beta}-e^{i(\pi-t)n}\frac{\sin\pi\beta}{\pi}\int_0^1\frac{x^{n+\beta-1}(1-x)^{-\beta}}{1+xe^{-it}}dx\right].$$

Substituting this representation into the expression for ϑ_n^-, we arrive at the required result. ◄

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References

[1] A.G. Sveshnikov, A.N. Tikhonov, *Theory of functions of a complex variable*, Moscow, Nauka, 1967.

[2] F.D. Gakhov, *Boundary value problems*, Moscow, Nauka, 1977.

[3] H. Bateman, A. Erdelyi, *Higher transcendental functions*, Moscow, **2**, 1977.

[4] B.T. Bilalov, S.G. Veliev, *Bases of the eigenfunctions of two discontinuous differential operators*, Differ. Uravn., **42(10)**, 2006, 1428-1430.

[5] E.I. Moiseev, *On the basis property of sine and cosine systems in a weighted space*, Differ. Uravn., **34(1)**, 1998, 40-44.

[6] S.S. Pukhov, A.M. Sedletskii, *Bases of exponentials, sines, and cosines in weighted spaces on a finite interval*, Dokl. Akad. Nauk, **425(4)**, 2009, 452-455.

[7] I.I. Danilyuk, *Nonregular boundary value problems in the plane*, Moscow, Nauka, 1975.

[8] J. Garnett, *Bounded analytic functions*, Moscow, Mir, 1984.

[9] A.V. Bitsadze, *On a system of functions*, Uspekhi Mat. Nauk, **5(4:38)**, 1950, 154-155.

[10] S.M. Ponomarev, *An eigenvalue problem*, Dokl. Akad. Nauk SSSR, **249(5)**, 1979, 1068-1070.

[11] S.M. Ponomarev, *On the theory of boundary value problems for equations of mixed type in three-dimensional domains*, Dokl. Akad. Nauk SSSR, **246(6)**, 1979, 1303-1306.

[12] E.I. Moiseev, *Application of the method of the separation of variables for solving equations of mixed type*, Diff. Uravn., **26(7)**, 1990, 1160-1172.

[13] E.I. Moiseev, *On the solution of a nonlocal boundary value problem by the spectral method*, Differ. Uravn., **35(8)**, 1999, 1094-1100.

[14] V.A. Il'in, E.I. Moiseev, *On systems consisting of subsets of root functions of two different boundary value problems*, Trudy Mat. Inst. Steklov., **201(15)**, 1992, 219-230.

[15] L.V. Kritskov, *Necessary Condition for the Uniform Minimality of Kostyuchenko Type Systems*, Azerb. J. of Math., **5(1)**, 2015, 97-103.

[16] E.I. Moiseev, N.O. Taranov, *Solution of a Gellerstedt problem for the Lavrentev-Bitsadze equation*, Differ. Uravn., **45(4)**, 2009, 543-548.

[17] E.I. Moiseev, *The basis property for systems of sines and cosines*, Dokl. Akad. Nauk SSSR, **275(4)**, 1984, 794-798.

[18] E.I. Moiseev, *On the basis property of a system of sines*, Diff. Uravn.,**23(1)**, 1987, 177-179.

[19] E.I. Moiseev, *On the differential properties of expansions in a system of sines and cosines*, Diff. Uravn., **32(1)**, 1996, 117-126.

[20] E.I. Moiseev, *On the basis property of sine and cosine systems in a weighted space*, Diff. Uravn., **34(1)**, 1998, 40-44.

[21] E.I. Moiseev, *The basis property of a system of eigenfunctions of a differential operator in a weighted space*, Diff. Uravn., **35(2)**, 1999, 200-205.

[22] A.M. Sedletskii *Biorthogonal expansions of functions in exponential series on intervals of the real axis*, Uspekhi Mat. Nauk, **37(5:227)**, 1982, 51-95.

[23] K.I. Babenko, *On conjugate functions*, Doklady Akad.Nauk SSSR, **62**, 1948, 157-160.

[24] B.T. Bilalov, *Basicity of some systems of exponents, cosines and sines*, Diff. Uravn., **26**, 1990, 10-16.

[25] B.T. Bilalov, *Uniform convergence of series in a system of sines*, Diff. Uravn., **24(1)**, 1988, 175-177.

[26] B.T. Bilalov, *The basis property of some systems of functions*, Diff. Uravn., **25(1)**, 1989, 163-164.

[27] B.T. Bilalov, *Completeness and minimality of some trigonometric systems*, Diff. Uravn., **28(1)**, 1992, 170-173.

[28] B.T. Bilalov, *On basicity of systems of exponents, cosines and sines in* L_p, Dokl. RAN, **365(1)**, 1999, 7-8.

[29] B.T. Bilalov, *On basicity of some systems of exponents, cosines and sines in* L_p, Dokl. RAN, **379(2)**, 2001, 7-9.

[30] B.T. Bilalov, *Bases of exponents, cosines and sines which are eigenfunctions of differential operators*, Diff. Uravn., **39(5)**, 2003, 1-5.

[31] B.T. Bilalov, *Bases of a system of exponentials in* L_p, Dokl. Akad. Nauk, **392(5)**, 2003, 583-585.

[32] B.T. Bilalov, *Basis properties of some system of exponents, cosines and sines*, Sibirsk. Mat. Zh., **45**, 2004, 264-273.

[33] B.T. Bilalov, S.G. Veliev, *Bases of the eigenfunctions of two discontinuous differential operators,* Diff. Uravn., **42(10)**, 2006, 1428-1430.

[34] B.T. Bilalov, Z.G. Guseinov, *A criterion for the basis property of a perturbed system of exponentials in Lebesgue spaces with a variable summability exponent*, Dokl. Akad. Nauk, **436(5)**, 2011, 586-589.

[35] B.T. Bilalov, Z.G. Guseinov, *Basicity of a system of exponents with a piece-wise linear phase in variable spaces*, Mediterr. J. Math., **9(3)**, 2012, 487-498.

[36] B.T. Bilalov, F.A. Guliyeva, On the frame properties of degenerate system of sines, Journal of Function Spaces and Applications, **2012**, Article ID 184186, 2012, 12 pages.

[37] B.T. Bilalov, Z.V. Mamedova, On the frame properties of some degenerate trigonometric systems, Dokl. Nats. Akad. Nauk Azerb., **68(5)**, 2012, 14-18. (in Russian)

[38] B.T. Bilalov, A.A. Guliyeva, *On basicity of exponential systems in Morrey-type spaces*, International Journal of Mathematics, **25(6)**, 2014, 10 pages.

[39] B.T. Bilalov, T.B. Gasymov, A.A. Guliyeva, *On solvability of Riemann boundary value problem in Morrey-Hardy classes*, Turk. J. Math., **40**, 2016, 1085-1101.

[40] B.T. Bilalov, S.R. Sadigova, *Frame properties of a part of exponential system with degenerate coefficients in Hardy classes*, Georgian Mathematical Journal, accepted to print, DOI: 10.1515/gmj-2016-0051.

[41] S.R. Sadigova, Z.A. Kasumov, *On atomic decomposition for Hardy classes with respect to degenerate exponential systems*, Proc. of the Inst. of Math. and Mech., NAS of Azerbaijan, **40(1)**, 2014, 55-67.

[42] R.A. Hunt, W.S. Young, *A weighted norm inequality for Fourier series*, Bull. Amer. Math.Soc., **80**, 1974, 274-277.

[43] A.Sh. Shukurov, *Impossibility of Power Series Expansion for Continuous Functions*, Azerb. J. of Math., **6(1)**, 2016, 122-125.

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