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A Characterization of Some Alternating Groups by Their Orders and Character Degree Graphs

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Abstract. The aim of this study was to characterize some alternating groups by their orders and character degree graphs. To achieve this, G was used as a finite group. The character degree graph $\Gamma(G)$ of G is the graph whose vertices are the prime divisors of character degrees of G, and two vertices p and q are joined by an edge if $p \cdot q$ divides some character degree of G. A_n was used as an alternating group of degree n. Khosravi et. al (2014). have shown that A_n, with $n = 5, 6, 7$ are characterizable by the character degree graphs and their orders. The results of this study achieved the conclusion of characterizing the alternating group A_n, where $n = 8, 9, 10$, by using its character degree graph and order. In particular, the alternating groups A_9 and A_{10} are not unique determined by their character degree graphs and their orders.

Key Words and Phrases: character degree graph, alternating groups, simple groups, group order.

2010 Mathematics Subject Classifications: 20C15, 20C33

1. Introduction

In this paper, all groups investigated are finite. Let G be a finite group and $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of G. Let $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ denote the set of character degrees of G.

In [9], the concept of character degree graph was introduced. The graph that has been most widely invested is the graph $\Gamma(G)$ whose vertices are the prime divisors of character degrees of the group G and two vertices p and q are joined by an edge if pq divides the character degree of G. Let $L_n(q)$ denote the projective special linear group of degree n over a finite field of order q. Khosravi et. al. in [4] proved that the group $L_2(p^2)$, where p is a prime, is characterizable by the degree graph and order. Khosravi et. al. in [3] studied the simple groups of order less than 6000 by using the character degree graph and order. Let A_n be the alternating group of degree n. We know that A_n with $n = 5, 6, 7$ is characterized by the degree graph and its order. The only remaining alternating group A_8 whose character degree graph is not complete, has not been characterized by considering the character degree graph and its order. So we prove the following theorem.

Main Theorem 1. *Let G be a group such that $|G| = |A_8| = 2^6 \cdot 3^2 \cdot 5 \cdot 7 = 20160$ and $\Gamma(G) = \Gamma(A_8)$. Then G is isomorphic to A_8.*

In fact, not all simple groups are characterized by their character degree graphs and their orders. For instance, $U_3(3)$ whose order is 6048, is not characterizable by its degree graph and its order (see [3, Remark 1]).

Note the fact that the only pairs of simple groups of the same order are $A_8, L_3(4)$ and $PSp(2n, q)$, $P\Omega O(2n+1, q)$, where $n \geq 3$ and q is odd. These groups are determined by their smallest character degree larger than 1 [6]. Therefore there are some simple groups which are determined by their orders and their character degree graphs. We know that A_7 is characterized by its character degree graph and order. But for the alternating A_9 with $\Gamma(A_9)$ complete, what's the influence of its character degree graph and order on the structure of groups? We will try to answer this question.

Main Theorem 2. *Let G be a group such that $|G| = |A_9| = 2^6 \cdot 3^4 \cdot 5 \cdot 7 = 181440$ and $\Gamma(G) = \Gamma(A_9)$. Then G has one of the following structures:*

(1) $G = H \times A_7$, where H is a group of order 72.

(2) $G = H \times (Z_2.A_7)$, where H is a group of order 36.

(3) $G = H \times S_7$, where H is a group of order 36 and S_n is a symmetric group of degree n.

(4) $G = (Z_3 \rtimes Z_3) \times A_8$.

(5) $G = Z_3 \times SL_3(4)$.

(6) $G = A_9$

Also we give the structure of groups under the condition that $\Gamma(A_{10}) = \Gamma(G)$ and order $|A_{10}| = |G|$. Obviously, it has more complicated structures.

Main Theorem 3. *Let G be a group such that $|G| = |A_{10}| = 2^7 \cdot 3^4 \cdot 5^2 \cdot 7 = 1814400$ and $\Gamma(G) = \Gamma(A_{10})$. Then G has one of the following structures:*

(1) $G = H \times A_7$, where H is a group of order 720.

(2) $G = H \times (Z_2.A_7)$, where H is a group of order 360.

(3) $G = H \times S_7$, where H is a group of order 360.

(4) $G = H \times L_3(4)$ where $|H| = 90$ and $2 \in \mathrm{cd}(H)$.

(5) $G = H \times (Z_2.L_3(4))$ where $|H| = 45$.

(6) $G = H \times (S_3.L_3(4))$ where $|H| = 15$.

(7) $G = Z_3 \times SL_3(4)$.

(8) $G = H \times A_8$*, where* H *is a group of order 90.*

(9) $G = H \times (Z_2.A_8)$*, where* H *is a group of order 45.*

(10) $G = H \times S_8$*, where* H *is a group of order 45.*

(11) $G = H \times A_9$*, where* $|H| = 10$.

(12) $G = Z_5 \times (Z_2.A_9)$.

(13) $G = Z_5 \times S_9$.

(14) $G = H \times SL_3(4)$*, where* H *is a group of order 30.*

(15) $G = Z_3 \times J_2$

(16) $G = A_{10}$.

It follows from Main Theorems 2 and 3 that the groups A_9 and A_{10} are not uniquely determined by the degree graphs and their orders.

2. Notation and some preliminary results

We introduce some notation which will be used to prove the main theorem. Let S_n and A_n be the symmetric and alternating groups of degree n, respectively. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If n is an integer and r is a prime divisor of n, then we write either $n_r = r^a$ or $r^a \| n$ if $r^a \mid n$ but $r^{a+1} \nmid n$. Let G be a group and r is a prime, then denote the set of Sylow r-subgroups G_r of G by $\mathrm{Syl}_r(G)$. If H is a characteristic subgroup of G, we write $H \,\mathrm{ch}\, G$. All other notation is standard (see [1]).

Lemma 1. *Let* $A \trianglelefteq G$ *be abelian. Then* $\chi(1)$ *divides* $|G : A|$ *for all* $\chi \in \mathrm{Irr}(G)$.

Proof. See Theorem 6.5 of [2]. ◄

Lemma 2. *Let* $N \trianglelefteq G$ *and let* $\chi \in \mathrm{Irr}(G)$*. Let* θ *be an irreducible constituent of* χ_N *and suppose that* $\theta_1, \cdots, \theta_t$ *are distinct conjugates of* θ *in* G*. Then* $\chi_N = e \sum_{i=1}^{t} \theta_i$*, where* $e = [\chi_N, \theta]$ *and* $t = |G : I_G(\theta)|$*. Also* $\theta(1) \mid \chi(1)$ *and* $\frac{\chi(1)}{\theta(1)} \mid \frac{|G|}{|N|}$.

Proof. See Theorems 6.2, 6.8 and 11.29 of [2]. ◄

Lemma 3. *Let* G *be a non-soluble group. Then* G *has a normal series* $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$*, such that* K/H *is a direct product of isomorphic non-abelian simple groups and* $|G/K| \mid |\mathrm{Out}(K/H)|$.

Proof. See Lemma 1 of [13]. ◄

Lemma 4. *Let G be a finite soluble group of order $p_1^{a_1}p_2^{a_2}\cdots p_n^{a_n}$, where $p_1, p_2, \cdots, p_n$ are distinct primes. If $kp_n + 1 \nmid p_i^{a_i}$ for each $i \leq n-1$ and $k > 0$, then the Sylow p_n-subgroup is normal in G.*

Proof. See Lemma 2 of [14]. ◄

We also need the structure of non-abelian simple group whose largest prime divisor is less than 7.

Lemma 5. *If S is a finite non-abelian simple group such that $\pi(S) \subseteq \{2,3,5,7\}$, then S is isomorphic to one of the following simple groups in Table 1.*

Proof. [15]. ◄

Table 1. Finite non-abelian simple groups S with $\pi(S) \subseteq \{2,3,5,7\}$

S	Order of S	Out(S)	S	Order of S	Out(S)
A_5	$2^2\cdot3\cdot5$	2	$L_2(49)$	$2^4\cdot3\cdot5^2\cdot7^2$	2^2
$L_2(7)$	$2^3\cdot3\cdot7$	2	$U_3(5)$	$2^4\cdot3^2\cdot5^3\cdot7$	S_3
A_6	$2^3\cdot3^2\cdot5$	2^2	A_9	$2^6\cdot3^4\cdot5\cdot7$	2
$L_2(8)$	$2^3\cdot3^2\cdot7$	3	J_2	$2^7\cdot3^3\cdot5^2\cdot7$	2
A_7	$2^3\cdot3^2\cdot5\cdot7$	2	$S_6(2)$	$2^9\cdot3^4\cdot5\cdot7$	1
$U_3(3)$	$2^5\cdot3^3\cdot7$	2	A_{10}	$2^7\cdot3^4\cdot5^2\cdot7$	2
A_8	$2^6\cdot3^2\cdot5\cdot7$	2	$U_4(3)$	$2^7\cdot3^6\cdot5\cdot7$	D_8
$L_3(4)$	$2^6\cdot3^2\cdot5\cdot7$	D_{12}	$S_4(7)$	$2^8\cdot3^2\cdot5^2\cdot7^4$	2
$U_4(2)$	$2^6\cdot3^4\cdot5$	2	$O_8^+(2)$	$2^{12}\cdot3^5\cdot5^2\cdot7$	S_3

3. The proofs of Main Theorems

In this section, we present the proofs of main theorems separately.

We know from [3], that the alternating groups A_n with $n = 5, 6, 7$ are characterizable by their character degree graphs and orders. So in the following, we consider alternating groups A_n with $n = 8, 9, 10$ separately by using the character degree graphs and their orders.

3.1. The proof of Main Theorem 1

Proof. It is easy to get from [1, p. 22] that

$$\mathrm{cd}(A_8) = \{1, 7, 14, 20, 21, 28, 35, 45, 56, 64, 70\}$$

and

$$|G| = |A_8| = 2^6\cdot3^2\cdot5\cdot7 = 20160.$$

It follows that $\Gamma(G)$ is the graph with vertex set $\{2,3,5,7\}$ and there is an edge between the vertices 5 and 7. So there is a character $\chi \in \mathrm{Irr}(G)$ with $5\cdot7 \mid \chi(1)$.

It is easy to get $O_5(G) = 1$ and $O_7(G) = 1$. In fact, if $O_7(G) \neq 1$, then since $|G_7| = 7$, $O_7(G)$ is a normal Sylow 7-subgroup of G of order 7 and so $O_7(G)$ is abelian. Then by

Lemma 1, for all $\chi \in \mathrm{Irr}(G)$, $\chi(1) \mid |G : O_7(G)|$, a contradiction. Similarly we can prove that $O_5(G) = 1$.

We first assume that G is soluble and M is a minimal normal subgroup of G. Then M is an elementary abelian p-group. where $p = 2$ or $p = 3$. Note that $|G|_p = p$ for $p = 5, 7$ and in $\Gamma(G)$, there is a character $\chi \in \mathrm{Irr}(G)$ such that $5 \cdot 7$ divides $\chi(1)$. Therefore we consider the following two cases.

(1) Let M be a 3-group.

Since there is an edge between 3 and 7, then $|M| = 3$. Let H/M be a Hall subgroup of G/M of order $2^6 \cdot 5 \cdot 7$. Then $|G/M : H/M| = 3$. It follows that $(G/M)/(L/M) \hookrightarrow S_3$, where S_3 is the symmetric group of degree 3 and $L/M = \mathrm{Core}_{G/M}(H/M) := \bigcap_{gM \in G/M}(H/M)^{gM}$, the core of H/M in G/M. So $|L/M| = |H_G/M| = 2^6 \cdot 5 \cdot 7$, or $|L/M| = 2^5 \cdot 5 \cdot 7$. In what follows, two cases are considered.

(1.1) $|L/M| = 2^5 \cdot 5 \cdot 7$.
Then $L = H_G = H$ and $|G : L| = 3$. Let $\theta \in \mathrm{Irr}(L)$ with $e = [\chi_L, \theta] \neq 0$. Then $5 \cdot 7 = et\theta(1)$, where $t = [G : I_G(\theta)]$. Since e and t are divisors of $|G : L| = 3$. Therefore $e = t = 1$ and so $\chi_L = \theta$. Let $\eta \in \mathrm{Irr}(M)$ be such that $e' = [\theta_M, \eta] \neq 0$. Therefore $\theta(1) = e't'$, where e' and t' are divisors of $L : M$, and $t' = [L : I_L(\eta)]$. Since there are three linear characters, then $t' \leq 3$ and so $t' = 1$. It follows that $e' = 5 \cdot 7$. Therefore $(5 \cdot 7)^2 \leq |L : M| = 2^5 \cdot 5 \cdot 7$, a contradiction.

(1.2) $|L/M| = 2^6 \cdot 5 \cdot 7$.
Similarly as in the Case 1.1 above, we can have that $(e'_2, t'_2) = (2 \cdot 7, 1)$, $(e'_3, t'_3) = (2 \cdot 5, 1)$, $(e'_4, t'_4) = (6, 1)$ or $(2, 3)$, $(e'_5, t'_5) = (15, 1)$ or $(5, 3)$ and $(e'_7, t'_7) = (21, 1)$ or $(7, 3)$. In these cases, $\sum e'_i t'_i$ is equal to at most 2123 which is less than $2^6 \cdot 5 \cdot 7$. It means that there is a character χ such that $\chi(1) = pqr$ is the product of three different primes p, q, r of $|L|$. Since $\pi(L) = \{2, 3, 5, 7\}$, then the possible triples (p, q, r) are $(2, 3, 7)$ or $(2, 5, 7)$, because there is no edge between the vertices 3 and 5.

If $(p, q, r) = (2, 5, 7)$, then similar to the Case 1.1, we have that $e' = 2 \cdot 5 \cdot 7$ and $t' = 1$ and so $(2 \cdot 5 \cdot 7)^2 \leq 2^6 \cdot 5 \cdot 7 = |L/M|$, a contradiction.

If $(p, q, r) = (2, 3, 7)$, then $e' = 42$, $t' = 1$ or $e' = 14$, $t' = 3$. If the former, then $42^2 + 35^2 \leq |L/M| = 2^6 \cdot 5 \cdot 7$, a contradiction. If the latter, then $35^2 + 3 \cdot 14^2 + 10^2 + 7^2 \cdot 3 + 14^2 + 2^2 \cdot 3 = 2268 \leq |L/M| = 2^6 \cdot 5 \cdot 7 = 2240$, a contradiction.

(2) Let M be a 2-group.

Similar to the Case 1.1, $e = t = 1$ and $\chi_L = \theta$. Let $\eta \in \mathrm{Irr}(M)$ be such that $e' = [\theta_M, \eta] \neq 0$. Therefore $\theta(1) = e't'$, where e' and t' are divisors of $|G : L| = 2^{6-k}$, and $t' = [L : I_L(\eta)]$. As $\theta(1)^2 \leq |L| = 2^k \cdot 3 \cdot 5 \cdot 7$ and there is an edge between 2 and 3, $k = 4, 5$.

If $|L| = 2^4 \cdot 3^3 \cdot 5 \cdot 7$, let $\eta \in \mathrm{Irr}(M)$ be such that $e' = [\theta_M, \eta] \neq 0$. Then $\theta(1) = 5 \cdot 7 = e't'$, where $t' = [L : I_L(\eta)]$. Also M has 16 linear characters and so $t' \leq 16$.

Therefore $e' = 35$ and $t' = 1$. It follows that $35^2 \leq |L : M| = 3^2 \cdot 5 \cdot 7$. Similarly as $|L| = 2^5 \cdot 5 \cdot 7$, we can rule out.

Therefore G is insoluble and so by Lemma 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.

We will prove that $5, 7 \in \pi(K/H)$. Assume the contrary. Then by [5, Lemma 6(d)] and [8, Lemma 2.13], $|\mathrm{Out}(K/H)|$ is divisible by neither 5 nor 7. If the primes 5 and 7 belong to $\pi(H)$, then by Burnside's theorem K/H is soluble since $\pi(K/H) = \{2, 3\}$ and $|G_p| = p$ where $p = 5, 7$, a contradiction. If 5 divides the order $|H|$ but $7 \nmid |H|$, then $7 \in \pi(K/H)$(otherwise K/H is soluble by Burnside's theorem) and G_5 is characteristic in H. We also get a contradiction by Lemma 1. Similarly, $7 \nmid |H|$.

Therefore by Lemma 5 and considering group order of A_8, K/H is isomorphic to one of the simple groups: A_7, A_8 or $L_3(4)$.

If $K/H \cong A_7$, then $A_7 \leq G/H \leq \mathrm{Aut}(A_7)$. If $G/H \cong A_7$, then there is an edge between the vertices 2 and 3 in $\Gamma(G)$, a contradiction since $\mathrm{cd}(A_7) = \{1, 6, 10, 14, 15, 21, 35\}$. Similarly, we can rule out when $G/H \cong S_7$.

If $K/H \cong L_3(4)$, then $L_3(4) \leq G/H \leq \mathrm{Aut}(L_3(4))$. If $G/H \cong L_3(4)$, then $H = 1$ and so $G \cong L_3(4)$. But $\Gamma(L_3(4))$ has no edge between the vertices 2 and 7, a contradiction since $\mathrm{cd}(L_3(4)) = \{1, 20, 35, 45, 63, 64\}$ by [1, p. 24]. For the remaining cases, order consideration rules out.

If $K/H \cong A_8$, then $A_8 \leq G/H \leq S_8$. If $G/H \cong A_8$, then $H = 1$ and so $G \cong A_8$. If $G/H \cong S_8$, then order consideration rules out.

This completes the proof. ◄

Corollary 1. *Let G be a finite group with* $\mathrm{cd}(G) = \mathrm{cd}(A_8)$ *and* $|G| = |A_8|$. *Then G is isomorphic to A_8.*

Proof. Since G_7 is a Sylow 7-subgroup of G with order 7, then $O_7(G) = 1$. In fact, if $O_7(G) \neq 1$, then there is a character χ such that $\chi(1) = 70$. So $\chi(1) \mid |G : O_7(G)|$ by Lemma 1. Similarly, $O_5(G) = 1$.

Assume that G is soluble and let M be a normal minimal subgroup of G. Then M is an elementary abelian p-group. From the above arguments, we have $p = 2, 3$. If $p = 2$, then $|M| \geq 2$ and since M is abelian, there is no character χ such that $\chi(1) = 64 \mid |G : M|$, a contradiction. If $p = 3$, then similarly, there is no character χ such that $\chi(1) = 9 \mid |G : M|$, a contradiction.

Therefore G is insoluble and so by Lemma 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By [5, Lemma 6(d)] and [8, Lemma 2.13], $|\mathrm{Out}(K/H)|$ is divisible by neither 5 nor 7. Also $5, 7 \nmid |H|$ since $O_5(G) = 1 = O_7(G)$. Hence K/H is isomorphic to A_7, A_8 or $L_3(4)$. If $K/H \cong A_7$, then $\Gamma(G)$ is complete, a contradiction since $\Gamma(A_7)$ is complete. If $K/H \cong L_3(4)$, then $G \cong L_3(4)$, a contradiction since the vertices 2 and 7 are joined by an edge. If $K/H \cong A_8$, then $G \cong A_8$, this is the desired result.

This completes the proof. ◄

3.2. The proof of Main Theorem 2

Proof. We can get from [1, p. 37], that $\mathrm{cd}(A_9) = \{1, 8, 21, 28, 35, 42, 48, 56, 84, 105, 120, 162, 168, 189, 216\}$. Therefore $\Gamma(G)$ is a complete graph with vertex set $\{2, 3, 5, 7\}$. Then there is a character $\chi \in \mathrm{Irr}(G)$ such that $5 \cdot 7 \mid \chi(1)$. If $O_5(G) \neq 1$, then since $|G_5| = 5$, $O_5(G)$ is a normal Sylow 5-subgroup of G and so for all $\chi \in \mathrm{Irr}(G)$, $\chi(1) \mid |G : O_5(G)|$, contradicting Lemma 1. Therefore $O_5(G) = 1$. Similarly, $O_7(G) = 1$.

Suppose that G is soluble and let M be a minimal normal subgroup of G. Then M is an elementary abelian normal p-group. Since $O_5(G) = O_7(G) = 1$, then either $p = 2$ or $p = 3$. Since $\Gamma(G)$ is complete, then there is a character β such that $6 \mid \beta(1)$ and so, $|M| \mid 2^5$ or $|M| \mid 3^3$. We consider two cases.

Case 1. Let M be a 3-group.

Then $|M| = 3^a$ with $1 \leq a \leq 3$ and so $|G/M| = 2^6 \cdot 3^{4-a} \cdot 5 \cdot 7$. Let H/M be a Hall subgroup of order $2^6{\cdot}5{\cdot}7$. Then $|G/M : H/M| = 3^{4-a}$ and so $G/H_G \rightharpoonup S_{3^{4-a}}$. Let $L = H_G$. Then the order of L/M is equal to $|L/M| = |H_G/M| = 2^6{\cdot}5{\cdot}7$ or $|L/M| = 2^5{\cdot}5{\cdot}7$. Similar to the Case 1 of Theorem 1, we only consider the following cases: $(p, q, r) = (2, 3, 5)$ and $(p, q, r) = (3, 5, 7)$ with $|L/M| = |H_G/M| = 2^6 \cdot 5 \cdot 7$.

(1.1) $(p, q, r) = (2, 3, 5)$ and $|L/M| = |H_G/M| = 2^6 \cdot 5 \cdot 7$.

In this case, there is a character such that $\chi(1) = 2 \cdot 3 \cdot 5$. Let $\theta \in \mathrm{Irr}(L)$ with $e = [\chi_L, \theta] \neq 0$. Then $2 \cdot 3 \cdot 5 = et\theta(1)$, where $t = [G : I_G(\theta)]$. Since e and t are divisors of $|G : L| = 3^{4-a}$ where $a \in \{1, 2, 3\}$. Therefore $(e, t) = (1, 1)$, $(3, 10)$ or $(1, 30)$.

Let $(e, t) = (1, 1)$. Then $\chi_l = \theta$. Let $\eta \in \mathrm{Irr}(M)$ be such that $e' = [\theta_M, \eta] \neq 0$. Therefore $\theta(1) = e't'$, where e' and t' are divisors of $|L : M|$, and $t' = [L : I_L(\eta)]$. Since there are 3^a linear characters, then $t' \leq 3^a$ and so $t' = 1$, $e' = 2 \cdot 3 \cdot 5$ or $t' = 3$, $t' = 10$. Similar to the Case 1.1 of Theorem 1, $(e'_2, t'_2) = (2 \cdot 7, 1)$, $(e'_3, t'_3) = (2 \cdot 5, 1)$, $(e'_4, t'_4) = (6, 1)$ or $(2, 3)$, $(e'_5, t'_5) = (15, 1)$ or $(5, 3)$ and $(e'_7, t'_7) = (21, 1)$ or $(7, 3)$. Therefore $e'^2 \cdot t' + \sum_{i=2}^{7} e_i'^2 \cdot t'_i \leq 2240$, a contradiction. If the latter, we also can rule out similarly.

Also we can rule out "$(e, t) = (3, 10)$ or $(1, 30)$".

(1.2) $(p, q, r) = (3, 5, 7)$ and $|L/M| = |H_G/M| = 2^6 \cdot 5 \cdot 7$.

Similar to the Case 1.1 of Theorem 2, we have either $t' = 1$ and $e = 105$ or $t' = 3$ and $e' = 35$. If the former, then $(105)^2 \leq |L/M| = 2^6 \cdot 5 \cdot 7$, a contradiction. If the latter, then $3 \cdot (35)^2 \leq 2^6 \cdot 5 \cdot 7$, also a contradiction.

Case 2. Let M be a 2-group.

Then since there is an edge between 2 and 3 in $\Gamma(G)$, $|M| = 2^a$ with $1 \leq a \leq 5$. Hence we have $|G/M| = 2^{6-a} \cdot 3^4 \cdot 5 \cdot 7$. Let H/M be a Hall subgroup of order $3^4 \cdot 5 \cdot 7$ of G/M.

If $3 \leq a \leq 5$, then $\frac{G}{H_G} \hookrightarrow S_{2^{6-a}}$. Let $L = H_G$. Then $|L/M| = 3^4 \cdot 5 \cdot 7$ or $|L/M| = 3^3 \cdot 5 \cdot 7$. By Lemma 4, G_7M/M is normal in L/M. Since G_7 is normal in G_7M, then $G_7M = G_7 \times M$. It follows that G_7 is normal in L. As $L \operatorname{ch} G$, G_7 is normal in G, a contradiction.

If $1 \leq a \leq 2$, then by [11], $\frac{G}{C_G(M)} \lessapprox GL(a,3)$, where $GL(n,q)$ is the general linear group of degree n over finite field of order q. Therefore $G = CG(M)$, $|C_G(M)| = 2^5 \cdot 3^4 \cdot 5 \cdot 7$, $C_G(M)| = 2^2 \cdot 3^3 \cdot 5 \cdot 7$. Let $C = C_G(M)$.

(2.1) Let $|G| = |C|$. In this case, there is a 2-group which is normal in G. So we rule out this case as the minimality of M.

(2.2) Let $|C| = 2^5 \cdot 3^4 \cdot 5 \cdot 7$. Then we can rule out it as the Case 1.1 of Theorem 1.

(2.3) Let $|C| = 2^2 \cdot 3^3 \cdot 5 \cdot 7$. Then G_7 is normal in C by Lemma 4. Since $C \operatorname{ch} G$, then G_7 is normal in G, a contradiction.

Therefore G is insoluble and so by Lemma 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.

Similarly as in the proof of Theorem 1, the primes 5 and 7 divide $|K/H|$. Therefore by Lemma 5 and considering group orders, K/H is isomorphic to one of the simple groups: A_7, A_8, $L_3(4)$ or A_9.

If $K/H \cong A_7$, then $A_7 \leq G/H \leq \mathrm{Aut}(A_7)$. We know that $\mathrm{cd}(A_7) = \{1, 6, 10, 14, 15, 21, 35\}$ and that $\Gamma(G)$ and $\Gamma(A_7)$ are complete.

(a) $G/H \cong A_7$. Then $|H| = 72$ and so the two possibilities for G are $G = H \times A_7$ and $G = M \times (Z_2.A_7)$, where M is a group of order 36.

In order to prove our claims, note that the order of $\mathrm{Aut}(H)$ is smaller than the order of A_7. Now let $C = C_G(H)$. Then $N_G(H)/C_G(H) = G/C$ is isomorphic to a subgroup of the automorphism group of H. It means that $|G/C| < |A_7|$ and $C \nleq H$. Thus $CH > H$(if $CH = H$, then G/C is embedded in a subgroup of $\mathrm{Aut}(H)$, but the order of G/C is larger than that of $\mathrm{Aut}(H)$, a contradiction.) and $G = CH$. Let $D = C \cap H$. Then C centralizes D and since $D \leq C$, H centralizes D. Therefore $D \leq Z(G)$. Since $G/H = CH/H \cong C/D$ is isomorphic to A_7 and $D \leq Z(C)$. Since the Schur multiplier of A_7 has order 2, then $C' \cap D$ has order 1 or 2. If $C' \cap D = 1$, then $C' = A_7$ and $C = D \times C'$. In this case, A_7 is a direct factor of G. If $|C' \cap D| = 2$, then $C' = Z_2.A_{124}$ and C' is a direct factor of G.

(b) $G/H \cong S_7$. Then $|H| = 36$ and so $G = H \times S_7$.

If $K/H \cong L_3(4)$, then $L_3(4) \leq G/H \leq \mathrm{Aut}(L_3(4))$. If $G/H \cong L_3(4)$, then $|H| = 9$. But since $\mathrm{cd}(L_3(4)) = \{1, 20, 35, 45, 63, 64\}$ by [1, p. 24], $\Gamma(L_3(4))$ has no edge between the vertices 2 and 7, a contradiction. If $G/H \cong SL_3(4)$, then $|H| = 3$ and by [10], $\mathrm{cd}(SL_3(4)) = \{1, 15, 20, 21, 35, 45, 63, 64, 84, 105\}$ and so $\Gamma(SL_3(4))$ is complete. It follows that there is only one possible group for this case. If $G/H \cong Z_3.L_3(4)$, then $|H| = 3$ and so $G = Z_3 \times (Z_3.L_3(4))$. Also in this case, $\Gamma(Z_3.L_3(4))$ has no edge between the vertices 2 and 7, a contradiction.

If $K/H \cong A_8$, then $A_8 \leq G/H \leq S_8$. We know that $\mathrm{cd}(A_8) = \{1, 7, 14, 20, 21, 28, 35, 45, 56, 64, 70\}$ and so there is no edge between the vertices 2 and 3.

(a) If $G/H \cong A_8$, then $|H| = 9$. Since the possible groups of order 9 are Z_9 and $Z_3 \rtimes Z_3$, the semidirect product of Z_3 by Z_3. Hence there are groups satisfying that $\Gamma(G)$ is complete.

(b) If $G/H \cong S_8$, then order consideration rules out.

If $K/H \cong A_9$, then $G \cong A_9$ or $G \cong S_9$. If the latter, order consideration rules out. So $G \cong A_9$.

This completes the proof. ◄

3.3. The proof of Main Theorem 3

Proof. From [1, p. 48], $\mathrm{cd}(A_{10}) = \{1, 9, 35, 36, 42, 75, 84, 90, 126, 160, 210, 224, 225, 252, 288, 300, 315, 350, 384, 450, 525, 567\}$ and so $\Gamma(G)$ is complete. Since there is an edge between the vertices 5 and 7, then there is a character $\chi \in \mathrm{Irr}(G)$ such that $5 \cdot 7 \mid \chi(1)$. Similarly as in the proof of Main Theorem 2, we have $O_7(G) = 1$.

Assume that G is soluble and let M be a minimal normal subgroup of G. Then M is a normal abelian elementary p-group. Since $O_7(G) = 1$, then $p = 2$, 3 or 5. Since $\Gamma(G)$ is complete, then there is a character β such that $2 \cdot 3 \cdot 5 \mid \beta(1)$ and so, $|M| \mid 2^6$, $|M| \mid 3^3$ or $|M| = 5$. We consider three cases.

Case 1. Let M be a 5-group.

Similarly as in Case 1.1 of Theorem 1, we rule out.

Case 2. Let M be a 3-group.

Similarly as in the proof of Case 1 of Theorem 2, we can rule out.

Case 3. Let M be a 2-group.

Similarly as in the proof of Case 2 of Theorem 2, we can rule out.

Therefore G is insoluble and so by Lemma 3, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.

Similarly as in the proof of Theorem 1, the primes 5 and 7 divide $|K/H|$. Therefore by Lemma 5 and considering group orders, K/H is isomorphic to one of the simple groups: A_7, A_8, $L_3(4)$, A_9, J_2 or A_{10}.

If $K/H \cong A_7$, then $A_7 \leq G/H \leq \mathrm{Aut}(A_7)$. We know that $\mathrm{cd}(A_7) = \{1, 6, 10, 14, 15, 21, 35\}$ and that $\Gamma(G)$ and $\Gamma(A_7)$ are complete.

(a) $G/H \cong A_7$. Then $|H| = 720$ and so the two possibilities for G are $G = H \times A_7$ and $G = M \times (Z_2.A_7)$, where M is a group of order 360. In this case, we can prove it similarly as in the case "$G/H \cong A_7$" of Theorem 2.

(b) $G/H \cong S_7$. Then $|H| = 360$ and so $G = H \times S_7$.

If $K/H \cong L_3(4)$, then $L_3(4) \leq G/H \leq \mathrm{Aut}(L_3(4))$. We know that $\mathrm{Mult}(L_3(4)) = 4 \times 4 \times 3$ and $\mathrm{Out}(L_3(4))$ has the structure $2.S_3$ by [1, p. 23].

(a) Let $G/H \cong L_3(4)$. But since $\mathrm{cd}(L_3(4)) = \{1, 20, 35, 45, 63, 64\}$ by [1, p. 24], $\Gamma(L_3(4))$ has no edge between the vertices 2 and 7. We know that $\Gamma(G)$ is complete and so, we need to consider that there is an edge between 2 and 7 in $\Gamma(G)$.

(a1) Let $G/H \cong L_3(4)$. Then $G = H \times L_3(4)$ with $|H| = 90$ and $2 \in \mathrm{cd}(H)$.

(a2) Let $G/H \cong Z_2.L_3(4)$. Then $|H| = 45$ and so $G = H \times (Z_2.L_3(4))$.

(a3) Let $G/H \cong Z_4.L_3(4)$. Then order consideration rules out. Similarly we rule out $G/H \cong Z_2.S_3.L_3(4)$ and $G/H \cong Z_3.L_3(4)$.

(a4) Let $G/H \cong S_3.L_3(4)$. Then $G = H \times (S_3.L_3(4))$.

In order to prove our claims, note that the order of $\mathrm{Aut}(H)$ is smaller than the order of $L_3(4)$. Now let $C = C_G(H)$. Then $N_G(H)/C_G(H) = G/C$ is isomorphic to a subgroup of the automorphism group of H. It means that $|G/C| < |L_3(4)|$ and $C \nleq H$. Thus $CH > H$ and $G = CH$ since H is a maximal soluble subgroup of G. Let $D = C \cap H$. Then C centralizes D and since $D \leq C$, H centralizes D. Therefore $D \leq Z(G)$. Since $G/H = CH/H \cong C/D$ is isomorphic to $L_3(4)$ and $D \leq Z(C)$. Since the Schur multiplier of $L_3(4)$ has order 48, then $C' \cap D$ has order n with $n \mid 48$ and, also $n \mid 90$. We consider the following cases by using the order of H.

If $C' \cap D = 1$, then $C' = L_3(4)$ and $C = D \times C'$. In this case, $L_3(4)$ is a direct factor of G.

If $|C' \cap D| = 2$, then $C' = Z_2.L_3(4)$ and C' is a direct factor of G.

If $|C' \cap D| = 3$, then $C' = Z_3.L_3(4)$ and C' is a direct factor of G.

If $|C' \cap D| = 6$, then $C' = Z_6.L_3(4)$ or $S_3.L_3(4)$ since there are only two types of groups of order 6: cyclic group, Z_6 and symmetric group, S_3. In this case, also C' is a direct factor of G.

(b) If $G/H \cong SL_3(4)$, then $|H| = 3$. By [10], $\mathrm{cd}(SL_3(4)) = \{1, 15, 20, 21, 35, 45, 63, 64, 84, 105\}$ and so $\Gamma(SL_3(4))$ is complete. Therefore $G = Z_3 \times SL_3(4)$.

If $K/H \cong A_8$, then $A_8 \leq G/H \leq S_8$. We know that $\mathrm{cd}(A_8) = \{1, 7, 14, 20, 21, 28, 35, 45, 56, 64, 70\}$ and so there is no edge between the vertices 2 and 3.

(a) If $G/H \cong A_8$, then $|H| = 90$. Since $\Gamma(G)$ is complete, then the possibilities for G are $G = H \times A_8$, where $3 \in \mathrm{cd}(H)$, and $G = M \times (Z_2.A_8)$, where M is a group of order 45. In this case, we can prove it similarly as in the case $G/H \cong A_7$ of Theorem 2.

(b) If $G/H \cong S_8$, then $|H| = 45$. Since $\mathrm{cd}(S_8) = \{1, 7, 14, 20, 21, 28, 35, 42, 56, 64, 70, 90\}$ [7], then $\Gamma(S_8)$ is complete and so $G = H \times S_8$, where H is a group of order 45.

If $K/H \cong A_9$, then $G/H \cong A_9$ or $G \cong S_9$.

(a) If $G/H \cong A_9$, then $|H| = 10$ and so, we have either $G = H \times A_9$ or $G = M \times (Z_2.A_9)$, where M is a group of order 5. In this case, we can prove it similarly as in the case $G/H \cong A_7$ of Theorem 2.

(b) If $G/H \cong S_9$, then $|H| = 5$ and so, $G = H \times S_9$, where H is a group of order 5.

If $K/H \cong J_2$, then $G/H \cong J_2$ or $G/H \cong \mathrm{Aut}(J_2)$. If $G/H \cong \mathrm{Aut}(J_2)$, order consideration rules out. If $G/H \cong J_2$, then $|H| = 3$ and since $\mathrm{cd}(J_2) = \{1, 14, 21, 36, 63, 70, 90, 126, 160, 175, 189, 224, 225, 288, 300, 336\}$, the graph $\Gamma(J_2)$ is complete. It follows that $G = Z_3 \times J_2$.

If $K/H \cong A_{10}$, then $G/H \cong A_{10}$ or $G/H \cong S_{10}$. If the former, $H = 1$ and so $G = A_{10}$. If the latter, order consideration rules out.

This completes the proof. ◀

4. Non character-degree-graph characterizable alternating groups

We start this section with a result of D. L. White [12, Theorem 3.1] which is concerning simple alternating groups with the same character degree graph. More precisely, he proved the following:

Lemma 6. *[12, Theorem 3.1] Let $G = A_n$, the alternating group of degree n, where $n \geq 5$. If n is not 5, 6 or 8, then $\Gamma(G)$ is complete.*

In fact, $|A_{p+r}| = |A_p| \times (p+1) \cdots \times (p+r)$ with $p \geq 11$ is a prime and $\pi((p+r)!) \subsetneqq \pi(p!)$. We let $n = (p+1) \cdots \times (p+r)$. Then $\Gamma(A_{p+r})$ and $\Gamma(A_p)$ have the same degree graph. So the influence of degree graph and order of A_{p+r} is largely dependent on the structure of groups of order n and the number r.

We know that A_5, A_7 are characterizable by degree graphs and orders. Then we put forward the following conjecture.

Conjecture 1. *Are all alternating groups A_p with $p \geq 5$ a prime, characterizable by character degree graphs and their orders?*

From Main Theorems 2 and 3, A_9 and A_{10} are not uniquely determined by the character degree graphs and their orders. If $p \geq 11$, then there are alternating group which have the same character-degree graph, and so there are at least two groups with these property, then we have the following conjecture.

Conjecture 2. *Let $p \geq 11$ be a prime and $\pi((p+r)!) \subsetneqq \pi(p!)$. Then for all alternating groups A_{p+r}, there are at least 2 groups with the same character degree graphs and orders.*

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