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Boundedness of Generalized Fractional Integral Operators From the Morrey Space $L_{1,\phi}(X;\mu)$ to the Campanato Space $\mathcal{L}_{1,\psi}(X;\mu)$ Over Non-doubling Measure Spaces

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Abstract. The present paper supplements our earlier works. Our goal in the present paper is to establish the boundedness of generalized fractional integral operators from the Morrey space $L_{1,\phi}(X;\mu)$ to the Campanato space $\mathcal{L}_{1,\psi}(X;\mu)$ over non-doubling measure spaces (X,d,μ). What is new in the present paper is that μ satisfies a minimal condition; $0=\mu(\{x\})<\mu(B(x,r))<\infty$ for all $x\in X$ and $r>0$. We first review some elementary facts on the fractional integral operators, generalized Morrey spaces, and analysis on metric measure spaces.

Key Words and Phrases: Morrey space, Riesz potential, non-doubling measure.

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1. Introduction

We consider the modified fractional integral operator $\tilde{I}_{\rho,\mu,\tau}$, which we define in (2) below, on a connected separable metric measure space (X,d,μ) to supplement our recent works [6, 28] in this paper, where ρ and τ denote its generalized order and its modification parameter, respectively. By $B(x,r)$ we denote the open ball centered at $x\in X$ of radius $r>0$. We write $d(x,y)$ for the distance between the points x and y in X. For simplicity, for any $x\in X$ and any $r>0$, we assume

$$\mu(\{x\})=0<\mu(B(x,r))<\infty. \tag{1}$$

The linear operator $\tilde{I}_{\rho,\mu,\tau}$ acts on the generalized Morrey space $L_{1,\phi}(X;\mu)$, whose norm is given by (5) below. We aim to justify the condition on the order function ρ proposed in our earlier papers [6, 28]. Define the modified fractional integral operator of generalized order ρ with modification parameter τ by:

$$\tilde{I}_{\rho,\mu,\tau}f(x)\equiv\int_{X\setminus\{x\}}K(x,y)f(y)\,d\mu(y), \tag{2}$$

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where a ball $\mathbb{B} = B(x_0, 1)$ with the basepoint $x_0 \in X$ is fixed and the integral kernel is given by:

$$K(x,y) \equiv \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} - \frac{\rho(d(x_0,y))}{\mu(B(x_0,\tau d(x_0,y)))}\chi_{X\setminus\mathbb{B}}(y) \quad (x,y \in X).$$

From the viewpoint of applications, it is natural to postulate some other weak conditions proposed by Pérez [22]:

$$\int_0^1 \frac{\rho(t)}{t}\,dt < \infty, \tag{3}$$

$$\sup_{r/2\le s\le r} \rho(s) \le C_\rho \int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t}\,dt \quad (r>0) \tag{4}$$

for some $C_\rho > 0$ and $0 < k_1 < k_2 \le 2k_1 < \infty$. We denote the set of all the functions satisfying (3) and (4) by $\mathcal{G}_0$.

Next, we define the generalized Morrey space $L_{1,\phi}(X;\mu)$ and generalized Campanato spaces $\mathcal{L}_{1,\phi}(X;\mu)$. For a function $\phi : (0,\infty) \to (0,\infty)$, let the generalized Morrey space $L_{1,\phi}(X;\mu)$ be the set of all functions $f \in L^1_{\mathrm{loc}}(X;\mu)$ such that

$$\|f\|_{L_{1,\phi}(X;\mu)} \equiv \sup_{z\in X,\, r>0} \frac{1}{\phi(r)\mu(B(z,2r))} \int_{B(z,r)} |f(x)|\,d\mu(x) < \infty, \tag{5}$$

and let the generalized Campanato space $\mathcal{L}_{1,\phi}(X;\mu)$ be the set of all functions $f \in L^1_{\mathrm{loc}}(X;\mu)$ such that

$$\|f\|_{\mathcal{L}_{1,\phi}(X;\mu)} \equiv \sup_{z\in X,\, r>0} \inf_{c_{B(z,r)}\in\mathbb{C}} \frac{1}{\phi(r)\mu(B(z,18r))} \int_{B(z,r)} |f(x) - c_{B(z,r)}|\,d\mu(x) < \infty.$$

There are various conditions on ϕ. According to the observation by Nakai [13, p.446], it is standard to consider the following classes $\mathcal{G}$ and $\mathcal{G}_1$ as conditions on ϕ: Let $\mathcal{G}$ be the set of all functions from $(0,\infty)$ to itself with the doubling condition; that is, there exists a constant $c_\phi \ge 1$ such that

$$\frac{1}{c_\phi} \le \frac{\phi(r)}{\phi(s)} \le c_\phi \quad \text{for} \quad r,s>0 \quad \text{with} \quad \frac{1}{2} \le \frac{r}{s} \le 2. \tag{6}$$

Let $\mathcal{G}_1$ be the set of all almost decreasing functions in $\mathcal{G}$. Remark that we deal with three classes $\mathcal{G}, \mathcal{G}_0, \mathcal{G}_1$ in this paper.

Here we verify that the integral inequality (8) below is natural by establishing the boundedness of the generalized modified fractional integral operator $\tilde{I}_{\rho,\mu,\tau}$ from the generalized Morrey spaces $L_{1,\phi}(X;\mu)$ to the generalized Campanato space $\mathcal{L}_{1,\psi}(X;\mu)$ over non-doubling metric measure spaces.

Theorem 1. *Let $\varepsilon > 0$. Let $\rho \in \mathcal{G}_0$, $\phi \in \mathcal{G}_1$, and $\psi \in \mathcal{G}_1$. If, in addition, there exists a constant $C > 0$ such that*

$$\left|\frac{\rho(d(z,y))}{\mu(B(z,4d(z,y)))} - \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))}\right| \le C\left(\frac{d(x,z)}{d(y,z)}\right)^{\varepsilon}\frac{\rho(d(z,y))}{\mu(B(z,4d(z,y)))}, \tag{7}$$

for all $x, y, z \in X$ with $2d(x,z) < d(y,z)$ and there exists a constant $C' > 0$ such that

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + r^{\varepsilon}\int_r^{\infty}\frac{\rho(t)\phi(t)}{t^{1+\varepsilon}}dt \le C'\psi(r) \quad (r > 0), \tag{8}$$

then $\tilde{I}_{\rho,\mu,4}$ is bounded from $L_{1,\phi}(X;\mu)$ to $\mathcal{L}_{1,\psi}(X;\mu)$.

This result extends [16, Theorem 3.3] to a metric measure setting. Thus, we can say that the underlying metric measure space (X, d, μ) does not affect the condition on ϕ, ψ, and ρ. Remark also that the metric measure space (X, d, μ) does not satisfy the doubling condition $\mu(B(x, 2r)) \le C\mu(B(x, r))$; we overcome this disadvantage by introducing a parameter τ.

Now let us investigate the relationship between the operator $\tilde{I}_{\rho,\mu,\tau}$ and the ones appearing in our earlier papers. The authors in [6] investigated the boundedness property of the generalized Riesz potential I_ρ given by:

$$I_\rho f(x) \equiv \int_{\mathbb{R}^n}\frac{\rho(|x-y|)}{|x-y|^n}f(y)\,dy \quad (x \in X).$$

The authors in [6] obtained some necessary and sufficient conditions on the boundedness of I_ρ on generalized Morrey spaces. A natural passage to the metric measure setting is the following generalized Riesz potential $I_{\rho,\mu,\tau}$ defined by:

$$I_{\rho,\mu,\tau} f(x) \equiv \int_{X\setminus\{x\}}\frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))}f(y)\,d\mu(y) \quad (x \in X).$$

To investigate the super-critical case, we consider the modified version (2).

We organize the remaining part of the present paper as follows: Section 2 is dedicated to the proof of Theorem 1. In Section 3, we overview the recent motivation of generalization we made in the present paper. In Section 3.1, we consider why the doubling metric measure spaces are not sufficient. We explain why we need to introduce the function ϕ in Section 3.2. We survey the researches on the boundedness of I_ρ on generalized Morrey spaces in Section 3.3. Finally, in Section 3.4, we present examples of ρ in the context of the partial differential equations.

2. Proof of Theorem 1

Let $z \in X$, $r > 0$, and $f \in L_{1,\phi}(X;\mu)$ be fixed. We have to show that

$$\frac{1}{\mu(B(z,18r))}\int_{B(z,r)}|\tilde{I}_{\rho,\mu,4}f(x) - c_B|\,d\mu(x) \le C\psi(r)\|f\|_{L_{1,\phi}(X;\mu)}, \tag{9}$$

for some constant $c_B = c_{B(z,r)}$.

To this end, we let $f_1 \equiv \chi_{B(z,4r)} f$ and $f_2 \equiv f - f_1$. Define

$$c_{B,1} \equiv -\int_{B(z,4r)} \frac{\rho(d(x_0,y))}{\mu(B(x_0,4d(x_0,y)))} \chi_{X\setminus\mathbb{B}}(y) f(y)\, d\mu(y),$$

$$c_{B,2} \equiv \tilde{I}_{\rho,\mu,4} f_2(z) \quad \text{and} \quad c_B \equiv c_{B,1} + c_{B,2}.$$

We claim that c_B does the job by proving that

$$\frac{1}{\mu(B(z,18r))} \int_{B(z,r)} |\tilde{I}_{\rho,\mu,4} f_1(x) - c_{B,1}|\, d\mu(x) \le C\psi(r) \|f\|_{L_{1,\phi}(X;\mu)}, \tag{10}$$

and that

$$\frac{1}{\mu(B(z,18r))} \int_{B(z,r)} |\tilde{I}_{\rho,\mu,4} f_2(x) - c_{B,2}|\, d\mu(x) \le C\psi(r) \|f\|_{L_{1,\phi}(X;\mu)}. \tag{11}$$

First we deal with f_1. We have

$$\begin{aligned}
&\frac{1}{\mu(B(z,18r))} \int_{B(z,r)} |\tilde{I}_{\rho,\mu,4} f_1(x) - c_{B,1}|\, d\mu(x) \le \\
&\le \frac{1}{\mu(B(z,18r))} \int_{B(z,r)} \left(\int_{B(z,4r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)|\, d\mu(y) \right) d\mu(x) \le \\
&\le \frac{1}{\mu(B(z,18r))} \int_{B(z,r)} \left(\int_{B(x,5r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)|\, d\mu(y) \right) d\mu(x).
\end{aligned}$$

Let us concentrate on the inner integral. We calculate that

$$\begin{aligned}
&\int_{B(x,5r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)|\, d\mu(y) \le \\
&\le \sum_{k=1}^{\infty} \int_{B(x,2^{4-k}r)\setminus B(x,2^{3-k}r)} \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} |f(y)|\, d\mu(y) \le \\
&\le \sum_{k=1}^{\infty} \left(\sup_{t\in[2^{3-k}r,2^{4-k}r]} \rho(t) \right) \frac{1}{\mu(B(x,2^{5-k}r))} \int_{B(x,2^{4-k}r)} |f(y)|\, d\mu(y).
\end{aligned} \tag{12}$$

Meanwhile

$$\begin{aligned}
&\int_{B(z,r)} \left(\frac{1}{\mu(B(x,2^{5-k}r))} \int_{B(x,2^{4-k}r)} |f(y)|\, d\mu(y) \right) d\mu(x) = \\
&= \int_{B(z,r)} \left(\frac{1}{\mu(B(x,2^{5-k}r))} \int_X \chi_{B(x,2^{4-k}r)}(y) |f(y)|\, d\mu(y) \right) d\mu(x) = \\
&= \int_X \left(\int_{B(z,r)} \frac{1}{\mu(B(x,2^{5-k}r))} \chi_{B(x,2^{4-k}r)}(y) |f(y)|\, d\mu(x) \right) d\mu(y).
\end{aligned}$$

Note that if $d(x,y) < 2^{4-k}r$, then $B(x,2^{5-k}r) \supset B(y,2^{4-k}r)$. Thus

$$\begin{aligned}
&\int_{B(z,r)} \left(\frac{1}{\mu(B(x,2^{5-k}r))} \int_{B(x,2^{4-k}r)} |f(y)|\, d\mu(y) \right) d\mu(x) \le \\
&\le \int_X \left(\frac{1}{\mu(B(y,2^{4-k}r))} \int_{B(z,r)} \chi_{B(x,2^{4-k}r)}(y)|f(y)|\, d\mu(x) \right) d\mu(y) = \\
&= \int_X \left(\frac{1}{\mu(B(y,2^{4-k}r))} \int_{B(z,r)} \chi_{B(y,2^{4-k}r)}(x)|f(y)|\, d\mu(x) \right) d\mu(y) = \\
&= \int_X \frac{\mu(B(z,r) \cap B(y,2^{4-k}r))}{\mu(B(y,2^{4-k}r))} |f(y)|\, d\mu(y).
\end{aligned}$$

Notice that $y \in B(z,9r)$ in order that $B(z,r) \cap B(y,2^{4-k}r) \neq \emptyset$ for some $k = 1,2,\ldots$. Therefore,

$$\int_{B(z,r)} \left(\frac{1}{\mu(B(x,2^{5-k}r))} \int_{B(x,2^{4-k}r)} |f(y)|\, d\mu(y) \right) d\mu(x) \le \int_{B(z,9r)} |f(y)|\, d\mu(y). \tag{13}$$

Thus, it follows from (12), (13), and (4), that

$$\begin{aligned}
&\frac{1}{\mu(B(z,18r))} \int_{B(z,r)} |\tilde{I}_{\rho,\mu,4} f_1(x) - c_{B,1}|\, d\mu(x) \le \\
&\le \sum_{k=1}^{\infty} \left(\sup_{t \in [2^{3-k}r, 2^{4-k}r]} \rho(t) \right) \frac{1}{\mu(B(z,18r))} \int_{B(z,9r)} |f(y)|\, d\mu(y) \le \\
&\le C \sum_{k=1}^{\infty} \int_{k_1 2^{4-k}r}^{k_2 2^{4-k}r} \frac{\rho(s)}{s}\, ds \frac{1}{\mu(B(z,18r))} \int_{B(z,9r)} |f(y)|\, d\mu(y) \le \\
&\le C\phi(r) \int_0^{8k_2 r} \frac{\rho(s)}{s}\, ds \cdot \|f\|_{L_{1,\phi}(X;\mu)} \le \\
&\le C\psi(r) \|f\|_{L_{1,\phi}(X;\mu)}.
\end{aligned}$$

In summary, we obtain (10).

We deal with f_2. We shall consider

$$\begin{aligned}
&\tilde{I}_{\rho,\mu,4} f_2(x) - c_{B,2} = \\
&= \int_{X \setminus B(z,4r)} \left(\frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} - \frac{\rho(d(z,y))}{\mu(B(z,4d(z,y)))} \right) f(y)\, d\mu(y) \quad (x \in B(z,r)).
\end{aligned}$$

By the triangle inequality and (7), we have

$$\begin{aligned}
&|\tilde{I}_{\rho,\mu,4} f_2(x) - c_{B,2}| \le \qquad (14) \\
&\le \int_{X \setminus B(z,4r)} \left| \frac{\rho(d(x,y))}{\mu(B(x,4d(x,y)))} - \frac{\rho(d(z,y))}{\mu(B(z,4d(z,y)))} \right| |f(y)|\, d\mu(y) \le
\end{aligned}$$

$$\le C \int_{X\setminus B(z,4r)} \left(\frac{d(x,z)}{d(y,z)}\right)^{\varepsilon} \frac{\rho(d(z,y))}{\mu(B(z,4d(z,y)))}|f(y)|\,d\mu(y).$$

By the dyadic decomposition and (4), we obtain

$$\begin{aligned}
&\int_{X\setminus B(z,4r)} \left(\frac{d(x,z)}{d(y,z)}\right)^{\varepsilon} \frac{\rho(d(z,y))}{\mu(B(z,4d(z,y)))}|f(y)|\,d\mu(y) = \\
&= \sum_{k=1}^{\infty} \int_{B(z,2^{k+2}r)\setminus B(z,2^{k+1}r)} \left(\frac{d(x,z)}{d(y,z)}\right)^{\varepsilon} \frac{\rho(d(z,y))}{\mu(B(z,4d(z,y)))}|f(y)|\,d\mu(y) \le \\
&\le \sum_{k=1}^{\infty} \frac{1}{2^{k\varepsilon}} \frac{1}{\mu(B(z,2^{k+3}r))} \sup_{t\in[2^{k+1}r,2^{k+2}r]} \rho(t) \int_{B(z,2^{k+2}r)} |f(y)|\,d\mu(y) \le \\
&\le C\sum_{k=1}^{\infty} \int_{2^{k+2}k_1 r}^{2^{k+3}k_2 r} \frac{\phi(2^{k+2}r)}{2^{k\varepsilon}}\rho(s)\frac{ds}{s} \frac{1}{\phi(2^{k+2}r)\mu(B(z,2^{k+3}r))} \int_{B(z,2^{k+2}r)} |f(y)|\,d\mu(y) \le \\
&\le C\|f\|_{L_{1,\phi}(X;\mu)} \sum_{k=1}^{\infty} \int_{2^{k+2}k_1 r}^{2^{k+3}k_2 r} \frac{\phi(2^{k+2}r)}{2^{k\varepsilon}}\rho(s)\frac{ds}{s}.
\end{aligned}$$

Note that (8), together with the doubling property of ϕ, yields

$$\sum_{k=1}^{\infty} \int_{2^{k+2}k_1 r}^{2^{k+3}k_2 r} \frac{\phi(2^{k+2}r)}{2^{k\varepsilon}}\rho(s)\frac{ds}{s} \le Cr^{\varepsilon} \sum_{k=1}^{\infty} \int_{2^{k+2}k_1 r}^{2^{k+3}k_2 r} \frac{\phi(s)}{s^{\varepsilon}}\rho(s)\frac{ds}{s} \le C\psi(r).$$

Thus

$$\int_{X\setminus B(z,4r)} \left(\frac{d(x,z)}{d(y,z)}\right)^{\varepsilon} \frac{\rho(d(z,y))}{\mu(B(z,4d(z,y)))}|f(y)|\,d\mu(y) \le C\psi(r)\|f\|_{L_{1,\phi}(X;\mu)}. \tag{15}$$

From (14) and (15), it follows that

$$|\tilde{I}_{\rho,\mu,4} f_2(x) - c_{B,2}| \le C\psi(r)\|f\|_{L_{1,\phi}(X;\mu)} \quad (x \in B(z,r)). \tag{16}$$

If we integrate (16) over $B(z,r)$, then we obtain (11). Combining (10) and (11), we obtain (9).

3. Historical remarks

3.1. Function spaces with non-doubling measures

Recall again that a Radon measure μ is said to be doubling, if there exists a constant $C > 0$ such that

$$\mu(B(x,2r)) \le C\mu(B(x,r)), \tag{17}$$

for all $x \in \operatorname{supp}(\mu)(= X)$ and $r > 0$; otherwise μ is said to be non-doubling.

A recent trend on analysis is to work on an infinite dimensional normed space. Unfortunately, the doubling condition is too strong a postulate as the folllowing proposition shows.

Proposition 1. *Let X be an infinite dimensional normed space. Then there is no non-zero doubling Radon measure μ on X.*

Proof. Assume that such μ exists. Since X is infinite dimensional, we can choose $\{x_j\}_{j=1}^{\infty}$ in a unit ball B and $\delta \in (0,1)$ such that $\|x_j - x_k\| > 2\delta$ as long as $j > k$. Consider a disjoint collection of balls $\{B(x_j,\delta)\}_{j=1}^{\infty}$. A geometric observation shows

$$B \subset \frac{2}{\delta} B(x_j,\delta).$$

Thus

$$C\mu(B) \geq \mu(2B) \geq \sum_{j=1}^{\infty} \mu\left(B(x_j,\delta)\right) \geq C \sum_{j=1}^{\infty} \mu\left(\frac{2}{\delta}B(x_j,\delta)\right) \geq C \sum_{j=1}^{\infty} \mu(B),$$

implying that $\mu(B) = 0$. As a result, μ does not charge

$$X = \bigcup_{j=1}^{\infty} j \cdot B.$$

This is a contradiction.

In connection with the $5r$-covering lemma, the doubling condition had been a key condition in harmonic analysis. However, Nazarov, Treil and Volberg showed that the doubling condition (17) is not necessary by using the modified maximal operator [19, 20]; see also [7, 9, 10, 24, 25, 26, 27, 30] as well as a textbook [32].

3.2. The generalized Morrey space

Next, let us recall the definition of generalized Morrey spaces in $\mathbb{R}^n$. We denote by $B(z,r)$ the ball $\{x \in \mathbb{R}^n : |x - z| < r\}$ with center z and radius $r > 0$, and by $|B(z,r)|$ its Lebesgue measure, i.e. $|B(z,r)| = \omega_n r^n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. For $1 \leq p < \infty$ and a doubling function $\phi : (0,\infty) \to (0,\infty)$, let the Morrey space $L_{p,\phi}(\mathbb{R}^n)$ be the family of all $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ such that $\|f\|_{L_{p,\phi}} < \infty$, where

$$\|f\|_{L_{p,\phi}} \equiv \sup_{z \in \mathbb{R}^n,\, r>0} \frac{1}{\phi(r)} \left(\frac{1}{|B(z,r)|} \int_{B(z,r)} |f(x)|^p \, dx \right)^{1/p}. \tag{18}$$

If $\phi \equiv 1$, then $L_{p,\phi}(\mathbb{R}^n) \sim L^{\infty}(\mathbb{R}^n)$ with norm coincidence by virtue of the Lebesgue differentiation theorem. When $\phi(r) \equiv r^{-\lambda/p}$ $(r > 0)$, $L_{p,\phi}(\mathbb{R}^n)$ coincides with $L^{p,\lambda}(\mathbb{R}^n)$ defined by Adams [1].

3.3. The fractional integral operator I_α on generalized Morrey spaces

If $\rho(r) \equiv r^\alpha, r > 0$ for $0 < \alpha < n$, then $I_\rho f$ coincides with the classical Riesz potential of order α. Let us recall the Riesz potential I_α of order $\alpha \in (0, n)$ for a locally Lebesgue integrable function f on $\mathbb{R}^n$. We define $I_\alpha f$ by:

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \quad (x \in \mathbb{R}^n).$$

The operator I_α is also called the fractional integral operator.

Adams [1, Theorem 3.1] showed that there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{q,\lambda}} \le C\|f\|_{L^{p,\lambda}},$$

provided that the parameters p, q, λ satisfy

$$1 < p < q < \infty,\ 0 < \lambda \le n,\ -\frac{\lambda}{p} + \frac{\alpha}{n} = -\frac{\lambda}{q}.$$

If $\lambda = n$, then this is the Hardy-Littlewood-Sobolev theorem asserting that the fractional integral operator I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$; see also [23, 3, 12, 21, 11, 17, 4, 8, 31, 5] for a series of the study of the behavior of I_α in generalized Morrey spaces.

3.4. The condition on ρ

Nakai introduced the generalized Riesz potential $I_\rho f$ in [14]. Nakai investigated the boundedness of $I_\rho f$ of functions in $L_{p,\phi}(\mathbb{R}^n)$ in [8] assuming that ρ is a doubling function. For the boundedness of $I_\rho f$, we also refer the reader to [15, 5, 28].

We remark that Pérez introduced the class $\mathcal{G}_0$ in [22]. As the following example shows, $\mathcal{G}_0$ is a condition more suitable than the doubling condition.

Example 1. *In view of* [2], *we see that* $(1-\Delta)^{-\alpha/2}$ *falls under the scope of our main results, when* ρ *is given by*

$$\rho(r) := \frac{r^{\frac{n+\alpha}{2}}}{2^{\frac{n+\alpha-2}{2}}\pi^{n/2}\Gamma(\alpha/2)} K_{\frac{n-\alpha}{2}}(r), \quad (r > 0)$$

and $K_{\frac{n-\alpha}{2}}(r)$ *is the modified Bessel function of the third kind. As* $r \downarrow 0$, *when* $0 < \alpha < n$,

$$\rho(r) \sim \frac{1}{2^\alpha \pi^{\frac{n}{2}}}\Gamma\left(\frac{\alpha}{2}\right)^{-1}\Gamma\left(\frac{n-\alpha}{2}\right) r^\alpha,$$

when $\alpha = n$

$$\rho(r) \sim \frac{r^n}{2^{n-1}\pi^{\frac{n}{2}}}\Gamma\left(\frac{\alpha}{2}\right)^{-1}\log\frac{1}{r},$$

and when $\alpha > n$,

$$\rho(r) \sim \frac{r^n}{2^n \pi^{\frac{n}{2}}}\Gamma\left(\frac{\alpha}{2}\right)^{-1}\Gamma\left(\frac{\alpha-n}{2}\right).$$

See [2, (4.2)]. *Furthermore, as* $r \to \infty$,

$$\rho(r) \sim \frac{r^{\frac{\alpha+n-1}{2}} e^{-r}}{2^{\frac{n+\alpha-1}{2}} \pi^{\frac{n-1}{2}} \Gamma\left(\frac{\alpha}{2}\right)}.$$

See [2, (4.3)]. *The above estimates mean that we have* (4) *with* $k_1 = 1/4$ *and* $k_2 = 1/2$. *Note that* $\rho \in \mathcal{G}$ *implies* (4). *See also* [29, Remark 2.2] *and* [18, Lemma 2.5].

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