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Study of the Periodic and Nonnegative Periodic Solutions of Functional Differential Equations via Fixed Points

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Abstract. In this paper, we study the existence of periodic and nonnegative periodic solutions of the nonlinear neutral differential equation

$$x'(t) = -a(t)h(x(t-\tau(t))) + c(t)x'(t-\tau(t)) + G(t, x(t), x(t-\tau(t))).$$

We invert this equation to construct a sum of a compact map and a large contraction which is suitable for applying the modification of Krasnoselskii's theorem. The Carathéodory condition is used for the function G.

Key Words and Phrases: Krasnoselskii-Burton's theorem, large contraction, neutral differential equation, integral equation, periodic solution, nonnegative solutions, Carathéodory conditions.

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1. Introduction

Theory of functional differential equations with delay has undergone a rapid development in the previous fifty years. We refer the readers to [1, 2, 3, 4, 6, 9, 10, 11, 12, 13] and references therein for a wealth of reference materials on the subject. More recently, researchers have given special attention to the study of equations in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, so-called neutral differential equations. In particular, qualitative analysis such as periodicity and positivity of solutions of neutral differential equations, has been studied extensively by many authors.

In the present paper, we study the existence of periodic and nonnegative periodic solutions of the nonlinear differential equation

$$x'(t) = -a(t)h(x(t-\tau(t))) + c(t)x'(t-\tau(t)) + G(t, x(t), x(t-\tau(t))), \qquad (1)$$

where a is positive real valued function, c is continuously differentiable, τ is twice continuously differentiable. The function $h : \mathbb{R} \to \mathbb{R}$ is continuous, $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. Our purpose here is to use a modification of Krasnoselskii's

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fixed point theorem due Burton (see [4], Theorem 3) to show the existence of periodic and nonnegative periodic solutions for the equation (1). Clearly, the present problem is totally nonlinear so that the variation of parameters cannot be applied directly. Then, we resort to the idea of adding and subtracting terms. As noted by Burton in [4], the added term destroys a contraction already present in part of the equation, but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process, we have to transform (1) into an integral equation written as a sum of two mappings: one is large contraction and the other is compact. After that, we use a variant of Krasnoselskii's fixed point theorem, to show the existence of periodic and nonnegative periodic solutions.

This paper is organized as follows. In Section 2, we present the inversion of (1), some definitions and Krasnoselskii-Burton's fixed point theorem. In Section 3, we present our main results on existence of periodic solutions of (1). Finally, we present our main results on existence of nonnegative periodic solutions of (1) in Section 4.

2. Preliminaries

For $T > 0$ define $\mathcal{P}_T = \{\phi : \phi \in C(\mathbb{R},\mathbb{R}),\ \phi(t+T) = \phi(t)\}$, where $C(\mathbb{R},\mathbb{R})$ is the space of all real valued continuous functions. Then $\mathcal{P}_T$ is a Banach space when it is endowed with the supremum norm

$$\|x\| = \max_{t\in[0,T]} |x(t)|.$$

In this paper we assume that

$$a(t-T) = a(t),\quad c(t-T) = c(t),\quad \tau(t-T) = \tau(t),\quad \tau(t) \geq \tau^* > 0, \tag{2}$$

where τ^* is a constant, a is positive and

$$1 - e^{-\int_{t-T}^{t} a(s)ds} \equiv \frac{1}{\eta} \neq 0. \tag{3}$$

It is interesting to note that equation (1) becomes of advanced type when $\tau(t) < 0$. Also, we assume that for all t, $0 \leq t \leq T$,

$$\tau'(t) \neq 1. \tag{4}$$

Since τ is periodic, condition (4) implies that $\tau'(t) < 1$. The function $G(t,x,y)$ is periodic in t of period T. That is

$$G(t-T,x,y) = G(t,x,y). \tag{5}$$

The following lemma is fundamental to our results.

Lemma 1. *Suppose (2)–(5) hold. If $x \in \mathcal{P}_T$, then x is a solution of the equation (1) if and only if*

$$x(t) = \eta\int_{t-T}^{t} \kappa(t,u)\, a(u)\,[x(u) - h(x(u))]\, du + \frac{c(t)}{1-\tau'(t)} x(t-\tau(t)) +$$

$$+\int_{t-\tau(t)}^{t} a(u)\, h(x(u))\, du - \eta \int_{t-T}^{t} \kappa(t,u)\, a(u) \int_{u-\tau(u)}^{u} a(s)\, h(x(s))\, ds du +$$
$$+\eta \int_{t-T}^{t} \kappa(t,u) \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] h(x(u-\tau(u)))\, du +$$
$$+\eta \int_{t-T}^{t} \kappa(t,u) \left[-b(u)\, x(u-\tau(u)) + G(u, x(u), x(u-\tau(u)))\right] du, \tag{6}$$

where

$$b(u) = \frac{(c'(u) + c(u)\, a(u))(1-\tau'(u)) + \tau''(u)\, c(u)}{(1-\tau'(u))^2}, \tag{7}$$

and

$$\kappa(t,u) = e^{-\int_u^t a(s) ds}. \tag{8}$$

Proof. Let $x \in \mathcal{P}_T$ be a solution of (1). Rewrite the equation (1) as

$$\begin{aligned}
x'(t) + a(t)\, x(t) &= a(t)\, x(t) - a(t)\, h(x(t)) + a(t)\, h(x(t)) - \\
&- a(t)\, h(x(t-\tau(t))) + c(t)\, x'(t-\tau(t)) + G(t, x(t), x(t-\tau(t))) = \\
&= a(t)\left[x(t) - h(x(t))\right] + \frac{d}{dt}\int_{t-\tau(t)}^{t} a(s)\, h(x(s))\, ds + \\
&+ \left[\left(1-\tau'(t)\right) a(t-\tau(t)) - a(t)\right] h(x(t-\tau(t))) + \\
&+ c(t)\, x'(t-\tau(t)) + G(t, x(t), x(t-\tau(t))).
\end{aligned}$$

Multiply both sides of the above equation by $\exp\left(\int_0^t a(s)\, ds\right)$ and then integrate from $t-T$ to t, to obtain

$$\begin{aligned}
&\int_{t-T}^{t} \left[x(u)\, e^{\int_0^u a(s) ds}\right]' du = \\
&= \int_{t-T}^{t} a(u)\left[x(u) - h(x(u))\right] e^{\int_0^u a(s) ds} du + \\
&+ \int_{t-T}^{t} \left[\frac{d}{du}\int_{u-\tau(u)}^{u} a(s)\, h(x(s))\, ds\right] e^{\int_0^u a(s) ds} du + \\
&+ \int_{t-T}^{t} \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] h(x(u-\tau(u)))\, e^{\int_0^u a(s) ds} du + \\
&+ \int_{t-T}^{t} \left[c(u)\, x'(u-\tau(u)) + G(u, x(u), x(u-\tau(u)))\right] e^{\int_0^u a(s) ds} du.
\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}
&x(t)\, e^{\int_0^t a(s) ds} - x(t-T)\, e^{\int_0^{t-T} a(s) ds} = \\
&= \int_{t-T}^{t} a(u)\left[x(u) - h(x(u))\right] e^{\int_0^u a(s) ds} du +
\end{aligned}$$

$$+\int_{t-T}^{t}\left[\frac{d}{du}\int_{u-\tau(u)}^{u}a(s)\,h(x(s))\,ds\right]e^{\int_0^u a(s)ds}du+$$
$$+\int_{t-T}^{t}\left[\left(1-\tau'(u)\right)a(u-\tau(u))-a(u)\right]h(x(u-\tau(u)))\,e^{\int_0^u a(s)ds}du+$$
$$+\int_{t-T}^{t}\left[c(u)\,x'(u-\tau(u))+G(u,x(u),x(u-\tau(u)))\right]e^{\int_0^u a(s)ds}du.$$

By dividing both sides of the above equation by $\exp\left(\int_0^t a(s)\,ds\right)$ and taking into account the fact that $x(t)=x(t-T)$, we obtain

$$\begin{aligned}
x(t) &= \\
&=\eta\int_{t-T}^{t}a(u)\left[x(u)-h(x(u))\right]e^{-\int_u^t a(s)ds}du+\\
&+\eta\int_{t-T}^{t}\left[\frac{d}{du}\int_{u-\tau(u)}^{t}a(s)\,h(x(s))\,ds\right]e^{-\int_u^t a(s)ds}du+\\
&+\eta\int_{t-T}^{t}\left[\left(1-\tau'(u)\right)a(u-\tau(u))-a(u)\right]h(x(u-\tau(u)))\,e^{-\int_u^t a(s)ds}du+\\
&+\eta\int_{t-T}^{t}\left[c(u)x'(u-\tau(u))+G(u,x(u),x(u-\tau(u)))\right]e^{-\int_u^t a(s)ds}du,
\end{aligned}\qquad (9)$$

where η is given by (3). Rewrite

$$\begin{aligned}
&\int_{t-T}^{t}c(u)\,x'(u-\tau(u))\,e^{-\int_u^t a(s)ds}du=\\
&=\int_{t-T}^{t}\left(1-\tau'(u)\right)x'(u-\tau(u))\,\frac{c(u)}{1-\tau'(u)}e^{-\int_u^t a(s)ds}du.
\end{aligned}$$

Integration by parts in the above integral with

$$U=\frac{c(u)}{1-\tau'(u)}e^{-\int_u^t a(s)ds}\quad\text{and } dV=\left(1-\tau'(u)\right)x'(u-\tau(u)),$$

yields

$$\begin{aligned}
&\int_{t-T}^{t}c(u)\,x'(u-\tau(u))\,e^{-\int_u^t a(s)ds}du=\\
&=\frac{1}{\eta}\times\frac{x(t-\tau(t))\,c(t)}{1-\tau'(t)}-\int_{t-T}^{t}b(u)\,x(u-\tau(u))\,e^{-\int_u^t a(s)ds}du,
\end{aligned}\qquad (10)$$

where $b(u)$ is given by (7). In the same way we obtain the integral

$$\int_{t-T}^{t}\left[\frac{d}{du}\int_{u-\tau(u)}^{u}a(s)\,h(x(s))\,ds\right]e^{-\int_u^t a(s)ds}du=$$

$$
\begin{aligned}
&= \left[\int_{u-\tau(u)}^{u} a(s)\,h(x(s))\,ds e^{-\int_u^t a(s)ds}\right]_{t-T}^{t} - \\
&- \int_{t-T}^{t}\left[\int_{u-\tau(u)}^{u} a(s)\,h(x(s))\,ds\right] a(u) e^{-\int_u^t a(s)ds} du = \\
&= \left[\int_{t-\tau(t)}^{t} a(s)h(x(s))\,ds - \int_{t-T-\tau(t)}^{t-T} a(s)\,h(x(s))\,ds e^{-\int_{t-T}^{t} a(s)ds}\right] - \\
&- \int_{t-T}^{t}\left[\int_{u-\tau(u)}^{u} a(s)\,h(x(s))\,ds\right] a(u)\, e^{-\int_u^t a(s)ds} du = \\
&= \frac{1}{\eta}\int_{t-\tau(t)}^{t} a(u)\,h(x(u))\,du - \\
&- \int_{t-T}^{t}\left[\int_{u-\tau(u)}^{u} a(s)\,h(x(s))\,ds\right] a(u)\, e^{-\int_u^t a(s)ds} du. \qquad (11)
\end{aligned}
$$

Then substituting (10) and (11) into (9) we obtain (6). The converse implication is easily obtained and the proof is complete.◄

Now, we give some definitions to be used in the sequel.

Definition 1. *The map $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy Carathéodory conditions with respect to $L^1[0,T]$ if the following conditions hold:*

(i) For each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t,z)$ is Lebesgue measurable.

(ii) For almost all $t \in [0,T]$, the mapping $z \mapsto f(t,z)$ is continuous on $\mathbb{R}^n$.

(iii) For each $r > 0$, there exists $\alpha_r \in L^1([0,T],\mathbb{R})$ such that for almost all $t \in [0,T]$ and for all z such that $|z| < r$, we have $|f(t,z)| \leq \alpha_r(t)$.

T. A. Burton observed that Krasnoselskii's result can be more attractive in applications with some changes and formulated Theorem 1 below (see [5] for the proof).

Definition 2. *Let $(\mathcal{M}, d)$ be a metric space and assume that $B : \mathcal{M} \to \mathcal{M}$. B is said to be a large contraction, if for φ, $\psi \in \mathcal{M}$, with $\varphi \neq \psi$, we have $d(B\varphi, B\psi) < d(\varphi,\psi)$, and if $\forall \epsilon > 0$, $\exists \delta < 1$ such that*

$$[\varphi,\psi \in \mathcal{M}, \quad d(\varphi,\psi) \geq \epsilon] \Longrightarrow d(B\varphi, B\psi) < \delta d(\varphi,\psi).$$

It is proved in [5] that a large contraction defined on a closed bounded and complete metric space has a unique fixed point.

Theorem 1 (Krasnoselskii-Burton)**.** *Let $\mathcal{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathcal{B}, \|.\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathcal{M}$ such that*

(i) A is completely continuous,

(ii) B is large contraction,

(iii) $x, y \in \mathcal{M}$ implies $Ax + By \in \mathcal{M}$.

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

3. Existence of periodic solutions

To apply Theorem 1, we need to define a Banach space $\mathcal{B}$, a closed bounded convex subset $\mathcal{M}$ of $\mathcal{B}$ and construct two mappings: one is a completely continuous and the other is large contraction. So, let $(\mathcal{B},\ \|.\|) = (\mathcal{P}_T,\ \|.\|)$ and

$$\mathcal{M} = \{\varphi \in \mathcal{P}_T,\ \|\varphi\| \leq L\}, \tag{12}$$

with $L \in (0, 1]$. For $x \in \mathcal{M}$, let the mapping H be defined by

$$H(x) = x - h(x), \tag{13}$$

and by (6), define the mapping $S : \mathcal{P}_T \to \mathcal{P}_T$ as follows

$$\begin{aligned}(S\varphi)(t) = {}& \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, H(\varphi(u))\, du + \frac{c(t)}{1-\tau'(t)} \varphi(t-\tau(t)) + \\ & + \int_{t-\tau(t)}^{t} a(u)\, h(\varphi(u))\, du - \eta \int_{t-T}^{t} \kappa(t,u)\, a(u) \int_{u-\tau(u)}^{u} a(s)\, h(\varphi(s))\, ds du + \\ & + \eta \int_{t-T}^{t} \kappa(t,u) \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u)))\, du + \\ & + \eta \int_{t-T}^{t} \kappa(t,u) \left[-b(u)\, \varphi(u-\tau(u)) + G(u, \varphi(u), \varphi(u-\tau(u)))\right] du. \end{aligned} \tag{14}$$

Then, we express the above equation as

$$(S\varphi)(t) = (A\varphi)(t) + (B\varphi)(t),$$

where $A, B : \mathcal{P}_T \to \mathcal{P}_T$ are given by

$$\begin{aligned}(A\varphi)(t) = {}& \frac{c(t)}{1-\tau'(t)} \varphi(t-\tau(t)) + \int_{t-\tau(t)}^{t} a(u)\, h(\varphi(u))\, du - \\ & - \eta \int_{t-T}^{t} \kappa(t,u)\, a(u) \int_{u-\tau(u)}^{u} a(s)\, h(\varphi(s))\, ds du + \\ & + \eta \int_{t-T}^{t} \kappa(t,u) \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u)))\, du + \\ & + \eta \int_{t-T}^{t} \kappa(t,u) \left[-b(u)\, \varphi(u-\tau(u)) + G(u, \varphi(u), \varphi(u-\tau(u)))\right] du, \end{aligned} \tag{15}$$

and

$$(B\varphi)(t) = \eta \int_{t-T}^{t} \kappa(t,u)\, a(u)\, H(\varphi(u))\, du. \tag{16}$$

We will assume that the following conditions hold.

(H1) $a \in L^1[0, T]$ is bounded.

(H2) h is locally Lipschitz continuous. Then for x, $y \in \mathcal{M}$ there exist a constant $E > 0$ such that

$$|h(x) - h(y)| \leq E \|x - y\|.$$

(H3) G satisfies Carathéodory conditions with respect to $L^1[0,T]$.

(H4) There exists periodic functions g_1, g_2, $g_3 \in L^1[0,T]$, with period T, such that

$$|G(t,x,y)| \leq g_1(t)|x| + g_2(t)|y| + g_3(t).$$

Now, we need the following assumptions

$$\beta_1\beta_2(EL + |h(0)|) \leq \frac{\gamma_1}{2}L, \tag{17}$$

where $\beta_1 = \max_{t\in[0,T]}|\tau(t)|$ and $\beta_2 = \max_{t\in[0,T]}\{a(t)\}$,

$$|b(t)| \leq \gamma_2 a(t), \tag{18}$$

$$\left(\left(1 - \tau'(t)\right)a(t - \tau(t)) + a(t)\right)(EL + |h(0)|) \leq \gamma_3 La(t), \tag{19}$$

$$g_1(t)L + g_2(t)L + g_3(t) \leq \gamma_4 La(t), \tag{20}$$

$$\gamma_5 = \max_{t\in[0,T]}\left|\frac{c(t)}{1 - \tau'(t)}\right|, \tag{21}$$

$$J[\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5] \leq 1, \tag{22}$$

where γ_i, $1 \leq i \leq 5$ and J are positive constants with $J \geq 3$.

Lemma 2. *For A defined in (15), suppose that (2)–(5), (17)–(22) and (H1)–(H4) hold. Then $A : \mathcal{M} \to \mathcal{M}$.*

Proof. Let A be defined by (15). First, by (2) and (5), a change of variables in (15) shows that $(A\varphi)(t + T) = (A\varphi)(t)$. That is, if $\varphi \in \mathcal{P}_T$, then $A\varphi$ is periodic with period T. By (H2) we obtain

$$|h(x)| \leq E|x| + |h(0)|.$$

Now let $\varphi \in \mathcal{M}$. By (17)–(22) and (H1)–(H4) we have

$$\begin{aligned}
&|(A\varphi)(t)| \leq \\
&\leq \left|\frac{c(t)}{1 - \tau'(t)}\right||\varphi(t - \tau(t))| + \int_{t-\tau(t)}^{t} a(u)|h(\varphi(u))|du + \\
&+ \eta\int_{t-T}^{t}\kappa(t,u)a(u)\int_{u-\tau(u)}^{u} a(s)|h(\varphi(s))|dsdu + \\
&+ \eta\int_{t-T}^{t}\kappa(t,u)\left[\left(1 - \tau'(u)\right)a(u - \tau(u)) + a(u)\right]|h(\varphi(u - \tau(u)))|du +
\end{aligned}$$

$$+\eta\int_{t-T}^{t}\kappa\left(t,u\right)\left(\left|b\left(u\right)\right|\left|\varphi\left(u-\tau\left(u\right)\right)\right|+\left|G\left(u,\varphi\left(u\right),\varphi\left(u-\tau\left(u\right)\right)\right)\right|\right)du\leq$$
$$\leq\gamma_5L+\beta_1\beta_2\left(EL+\left|h\left(0\right)\right|\right)+$$
$$+\eta\int_{t-T}^{t}\kappa\left(t,u\right)a\left(u\right)\beta_1\beta_2\left(EL+\left|h\left(0\right)\right|\right)du+$$
$$+\eta\int_{t-T}^{t}\kappa\left(t,u\right)\left[\left(1-\tau'\left(u\right)\right)a\left(u-\tau\left(u\right)\right)+a\left(u\right)\right]\left(EL+\left|h\left(0\right)\right|\right)du+$$
$$+\eta\int_{t-T}^{t}\kappa\left(t,u\right)a\left(u\right)\gamma_2Ldu+$$
$$+\eta\int_{t-T}^{t}\kappa\left(t,u\right)\left[g_1\left(t\right)\left|\varphi\left(t\right)\right|+g_2\left(t\right)\left|\varphi\left(t-\tau\left(t\right)\right)\right|+g_3\left(t\right)\right]du\leq$$
$$\leq\gamma_1L+\gamma_2L+\gamma_3L+\gamma_4L+\gamma_5L\leq\frac{L}{J}\leq L.$$

That is $A\varphi\in\mathcal{M}$.◀

Lemma 3. *For $A:\mathcal{M}\to\mathcal{M}$ defined in (15), suppose that (2)–(5), (17)–(22) and (H1)–(H4) hold. Then A is completely continuous.*

Proof. We show that A is continuous in the supremum norm. Let $\varphi_n\in\mathcal{M}$, where n is a positive integer such that $\varphi_n\to\varphi$ as $n\to\infty$. Then

$$\left|\left(A\varphi_n\right)\left(t\right)-\left(A\varphi\right)\left(t\right)\right|\leq$$
$$\leq\left|\frac{c\left(t\right)}{1-\tau'\left(t\right)}\right|\left|\varphi_n\left(t-\tau\left(t\right)\right)-\varphi\left(t-\tau\left(t\right)\right)\right|+$$
$$+\int_{t-\tau(t)}^{t}a\left(u\right)\left|h\left(\varphi_n\left(u\right)\right)-h\left(\varphi\left(u\right)\right)\right|du+$$
$$+\eta\int_{t-T}^{t}\kappa\left(t,u\right)a\left(u\right)\int_{u-\tau(u)}^{u}a\left(s\right)\left|h\left(\varphi_n\left(s\right)\right)-h\left(\varphi\left(s\right)\right)\right|dsdu+$$
$$+\eta\int_{t-T}^{t}\kappa\left(t,u\right)\left[\left(1-\tau'\left(u\right)\right)a\left(u-\tau\left(u\right)\right)-a\left(u\right)\right]\times$$
$$\times\left|h\left(\varphi_n\left(u-\tau\left(u\right)\right)\right)-h\left(\varphi\left(u-\tau\left(u\right)\right)\right)\right|du+$$
$$+\eta\int_{t-T}^{t}\kappa\left(t,u\right)\left|b\left(u\right)\right|\left|\varphi_n\left(u-\tau\left(u\right)\right)-\varphi\left(u-\tau\left(u\right)\right)\right|du+$$
$$+\eta\int_{t-T}^{t}\kappa\left(t,u\right)\left|G\left(u,\varphi_n(u),\varphi_n\left(u-\tau\left(u\right)\right)\right)-G\left(u,\varphi\left(u\right),\varphi\left(u-\tau\left(u\right)\right)\right)\right|du.$$

By the Dominated Convergence Theorem, $\lim_{n\to\infty}\left|\left(A\varphi_n\right)\left(t\right)-\left(A\varphi\right)\left(t\right)\right|=0$. Then A is continuous.

We next show that A is completely continuous. Let $\varphi\in\mathcal{M}$. Then, by Lemma 2, we see that

$$\left\|A\varphi\right\|\leq L,$$

and so the family of functions $A\varphi$ is uniformly bounded. Again, let $\varphi \in \mathcal{M}$. Without loss of generality, we can pick $\omega < t$ such that $t - \omega < T$. Then

$$
\begin{aligned}
&\left|(A\varphi)(t) - (A\varphi)(\omega)\right| \leq \\
&\leq \left| \frac{c(t)}{1-\tau'(t)} \varphi(t-\tau(t)) - \frac{c(\omega)}{1-\tau'(\omega)} \varphi(\omega - \tau(\omega)) \right| + \\
&+ \left| \int_{t-\tau(t)}^{t} a(u) h(\varphi(u)) du - \int_{\omega-\tau(\omega)}^{\omega} a(u) h(\varphi(u)) du \right| + \\
&+ \eta \left| \int_{t-T}^{t} \kappa(t,u) a(u) \int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds du - \right. \\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u) a(u) \int_{u-\tau(u)}^{u} a(s) h(\varphi(s)) ds du \right| + \\
&+ \eta \left| \int_{t-T}^{t} \kappa(t,u) \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u))) du - \right. \\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u) \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u))) du \right| + \\
&+ \eta \left| \int_{t-T}^{t} \kappa(t,u) b(u) \varphi(u-\tau(u)) du - \int_{\omega-T}^{\omega} \kappa(\omega,u) b(u) \varphi(u-\tau(u)) du \right| + \\
&+ \eta \left| \int_{t-T}^{t} \kappa(t,u) G(u,\varphi(u),\varphi(u-\tau(u))) du - \right. \\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u) G(u,\varphi(u),\varphi(u-\tau(u))) du \right|.
\end{aligned}
$$

Since (H1)–(H4) and (17)–(22) hold, we can rewrite

$$
\begin{aligned}
&\eta \left| \int_{t-T}^{t} \kappa(t,u) \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u))) du - \right. \\
&\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u) \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] h(\varphi(u-\tau(u))) du \right| \leq \\
&\leq \eta \int_{\omega}^{t} \kappa(t,u) \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] \left|h(\varphi(u-\tau(u)))\right| du + \\
&+ \eta \int_{\omega-T}^{\omega} \left|\kappa(t,u) - \kappa(\omega,u)\right| \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] \times \\
&\times \left|h(\varphi(u-\tau(u)))\right| du + \\
&+ \eta \int_{\omega-T}^{t-T} \kappa(\omega,u) \left[\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right] \left|h(\varphi(u-\tau(u)))\right| du \leq \\
&\leq 2\eta\beta_3 \int_{\omega}^{t} \gamma_3 L a(u) du + \eta \int_{\omega-T}^{\omega} \left|\kappa(t,u) - \kappa(\omega,u)\right| \gamma_3 L a(u) du \leq
\end{aligned}
$$

$$\leq 2\eta\beta_3\gamma_3 L \int_{\omega}^{t} a(u)\,du + \eta\gamma_3 L \int_{0}^{T} \left|\kappa(t,u) - \kappa(\omega,u)\right| a(u)\,du,$$

where $\beta_3 = \max_{u\in[t-T,t]}\{\kappa(t,u)\}$, and

$$\eta\left|\int_{t-T}^{t} \kappa(t,u)\, b(u)\,\varphi(u-\tau(u))\,du - \right.$$
$$\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u)\, b(u)\,\varphi(u-\tau(u))\,du\right| +$$
$$+\,\eta\left|\int_{t-T}^{t} \kappa(t,u)\, G(u,\varphi(u),\varphi(u-\tau(u)))\,du - \right.$$
$$\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u)\, G(u,\varphi(u),\varphi(u-\tau(u)))\,du\right| \leq$$
$$\leq 2\eta\beta_3 \int_{\omega}^{t} \left[a(u)\,\gamma_3 L + g_{\sqrt{2}L}(u)\right] du +$$
$$+\,\eta \int_{0}^{T} \left|\kappa(t,u) - \kappa(\omega,u)\right| \left[a(u)\,\gamma_3 L + g_{\sqrt{2}L}(u)\right] du,$$

and

$$\eta\left|\int_{t-T}^{t} \kappa(t,u)\, a(u) \int_{u-\tau(u)}^{u} a(s)\, h(\varphi(s))\, ds du - \right.$$
$$\left. - \int_{\omega-T}^{\omega} \kappa(\omega,u)\, a(u) \int_{u-\tau(u)}^{u} a(s) h(\varphi(s))\, ds du\right| \leq$$
$$\leq 2\eta\beta_3 \int_{\omega}^{t} a(u)\,\frac{\gamma_1}{2} L du + \eta \int_{\omega-T}^{\omega} \left|\kappa(t,u) - \kappa(\omega,u)\right| a(u)\,\frac{\gamma_1}{2} L du \leq$$
$$\leq \eta\beta_3\gamma_1 L \int_{\omega}^{t} a(u)\,du + \eta\frac{\gamma_1}{2} L \int_{0}^{T} \left|\kappa(t,u) - \kappa(\omega,u)\right| a(u)\,du,$$

and

$$\left|\int_{t-\tau(t)}^{t} a(s)\, h(\varphi(s))\, ds - \int_{\omega-\tau(\omega)}^{\tau} a(s)\, h(\varphi(s))\, ds\right| =$$
$$= \left|\int_{\omega}^{t} a(s)\, h(\varphi(s))\, ds - \int_{\omega-\tau(\omega)}^{t-\tau(t)} a(s)\, h(\varphi(s))\, ds\right| \leq$$
$$\leq (EL + h(0)) \left(\int_{\omega}^{t} a(s)\,ds + \int_{\omega-\tau(\omega)}^{t-\tau(t)} a(s)\,ds\right),$$

which implies

$$\left|(A\varphi)(t) - (A\varphi)(\omega)\right| \leq$$

$$\leq \left| \frac{c(t)}{1-\tau'(t)} \varphi(t-\tau(t)) - \frac{c(\omega)}{1-\tau'(\omega)} \varphi(\omega-\tau(\omega)) \right| +$$
$$+ 2\eta\beta_3\gamma_3 L \int_\omega^t a(u)\,du + \eta\gamma_3 L \int_0^T |\kappa(t,u)-\kappa(\omega,u)|\,a(u)\,du +$$
$$+ 2\eta\beta_3 \int_\omega^t \left[a(u)\gamma_3 L + g_{\sqrt{2}L}(u)\right] du +$$
$$+ \eta \int_0^T |\kappa(t,u)-\kappa(\omega,u)| \left[a(u)\gamma_3 L + g_{\sqrt{2}L}(u)\right] du +$$
$$+ \eta\beta_3\gamma_1 L \int_\omega^t a(u)\,du + \eta \frac{\gamma_1}{2} L \int_0^T |\kappa(t,u)-\kappa(\omega,u)|\,a(u)\,du +$$
$$+ (EL + h(0)) \left(\int_\omega^t a(s)\,ds + \int_{\omega-\tau(\omega)}^{t-\tau(t)} a(s)\,ds \right).$$

Then, by the Dominated Convergence Theorem, $|(A\varphi)(t) - (A\varphi)(\omega)| \to 0$ as $t - \omega \to 0$ independently of $\varphi \in \mathcal{M}$. Thus, $(A\varphi)$ is equicontinuous. Hence, by Ascoli-Arzela's theorem, A is completely continuous.◀

Now, we state an important result of [1, Theorem 3.4] and for convenience we present below its proof. We deduce by this Theorem that the following are sufficient conditions for the mapping H given by (13) to be a large contraction on the set $\mathcal{M}$.

(H5) $h : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$.

(H6) The function h is strictly increasing on $[-L, L]$.

(H7) $\sup_{t\in(-L,L)} h'(t) \leq 1$.

Theorem 2. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying (H5)–(H7). Then the mapping H in (13) is a large contraction on the set $\mathcal{M}$.*

Proof. Let $\varphi, \psi \in \mathcal{M}$ with $\varphi \neq \psi$. Then $\varphi(t) \neq \psi(t)$ for some $t \in \mathbb{R}$. Let us denote the set of all such t by $D(\varphi, \psi)$, i.e.,

$$D(\varphi,\psi) = \{t \in \mathbb{R} : \varphi(t) \neq \psi(t)\}.$$

For all $t \in D(\varphi, \psi)$, we have

$$\begin{aligned} &|(H\varphi)(t) - (H\psi)(t)| \leq \\ &\leq |\varphi(t) - \psi(t) - h(\varphi(t)) + h(\psi(t))| \leq \\ &\leq |\varphi(t) - \psi(t)| \left| 1 - \frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} \right|. \end{aligned} \tag{23}$$

Since h is a strictly increasing function, we have

$$\frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} > 0 \text{ for all } t \in D(\varphi,\psi). \tag{24}$$

For each fixed $t \in D(\varphi, \psi)$ define the interval $I_t \subset [-L, L]$ by

$$I_t = \begin{cases} (\varphi(t), \psi(t)) & \text{if } \varphi(t) < \psi(t), \\ (\psi(t), \varphi(t)) & \text{if } \psi(t) < \varphi(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\varphi, \psi)$ there exists a real number $c_t \in I_t$ such that

$$\frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} = h'(c_t).$$

By (H6) and (H7) we have

$$0 \leq \inf_{u \in (-L,L)} h'(u) \leq \inf_{u \in I_t} h'(u) \leq h'(c_t) \leq \sup_{u \in I_t} h'(u) \leq \sup_{u \in (-L,L)} h'(u) \leq 1. \tag{25}$$

Hence, by (23)–(25) we obtain

$$|(H\varphi)(t) - (H\psi)(t)| \leq |\varphi(t) - \psi(t)| \left| 1 - \inf_{u \in (-L,L)} h'(u) \right|, \tag{26}$$

for all $t \in D(\varphi, \psi)$. This implies a large contraction in the supremum norm. To see this, choose a fixed $\epsilon \in (0, 1)$ and assume that φ and ψ are two functions in $\mathcal{M}$ satisfying

$$\epsilon \leq \sup_{t \in (-L,L)} |\varphi(t) - \psi(t)| = \|\varphi - \psi\|.$$

If $|\varphi(t) - \psi(t)| \leq \frac{\epsilon}{2}$ for some $t \in D(\varphi, \psi)$, then we get by (25) and (26) that

$$|(H\varphi)(t) - (H\psi)(t)| \leq \frac{1}{2} |\varphi(t) - \psi(t)| \leq \frac{1}{2} \|\varphi - \psi\|. \tag{27}$$

Since h is continuous and strictly increasing, the function $h\left(u + \frac{\epsilon}{2}\right) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\epsilon}{2} \leq |\varphi(t) - \psi(t)|$ for some $t \in D(\varphi, \psi)$, then by (H6) and (H7) we conclude that

$$1 \geq \frac{h(\varphi(t)) - h(\psi(t))}{\varphi(t) - \psi(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2L} \min \left\{ h\left(u + \frac{\epsilon}{2}\right) - h(u) : u \in [-L, L] \right\} > 0.$$

Hence, (23) implies

$$|(H\varphi)(t) - (H\psi)(t)| \leq (1 - \lambda) \|\varphi - \psi\|. \tag{28}$$

Consequently, combining (27) and (28) we obtain

$$|(H\varphi)(t) - (H\psi)(t)| \leq \delta \|\varphi - \psi\|, \tag{29}$$

where

$$\delta = \max\left\{ \frac{1}{2}, 1 - \lambda \right\}.$$

The proof is complete.◄

The next result shows the relationship between the mappings H and B in the sense of large contractions. Assume that

$$\max\{|H(-L)|,|H(L)|\} \leq \frac{2L}{J}. \tag{30}$$

Lemma 4. *Let B be defined by (16), and suppose (2)–(5) and (H5)–(H7) hold. Then $B:\mathcal{M}\to\mathcal{M}$ is a large contraction.*

Proof. Obviously, B is continuous and it is easy to show that $(B\varphi)(t+T)=(B\varphi)(t)$. Let $\varphi\in\mathcal{M}$. Then

$$\begin{aligned}|(B\varphi)(t)| &\leq \int_{t-T}^{t}\kappa(t,u)\,a(u)\max\{|H(-L)|,|H(L)|\}\,du \leq \\ &\leq \frac{2L}{J} < L,\end{aligned}$$

which implies $B:\mathcal{M}\to\mathcal{M}$.

By Theorem 2, H is a large contraction on $\mathcal{M}$. Then for any $\varphi,\psi\in\mathcal{M}$ with $\varphi\neq\psi$ and for any $\epsilon>0$, from the proof of that Theorem we can find a $\delta<1$ such that

$$\begin{aligned}|(B\varphi)(t)-(B\psi)(t)| &= \left|\eta\int_{t-T}^{t}\kappa(t,u)\,a(u)\left[H(\varphi(u))-H(\psi(u))\right]du\right| \leq \\ &\leq \|\varphi-\psi\|\,\eta\int_{t-T}^{t}\kappa(t,u)\,a(u)\,du \leq \\ &\leq \delta\,\|\varphi-\psi\|.\end{aligned}$$

The proof is complete.◄

Theorem 3. *Suppose the hypotheses of Lemmas 2, 3 and 4 hold. Let $\mathcal{M}$ be defined by (12). Then the equation (1) has a T-periodic solution in $\mathcal{M}$.*

Proof. By Lemmas 2, 3, A is continuous and $A(\mathcal{M})$ is contained in a compact set. Also, from Lemma 4, the mapping B is a large contraction. Next, we show that if $\varphi,\psi\in\mathcal{M}$, we have $\|A\psi+B\varphi\|\leq L$. Let $\varphi,\psi\in\mathcal{M}$ with $\|\varphi\|,\|\psi\|\leq L$. Then

$$\begin{aligned}\|A\psi+B\varphi\| &\leq [\gamma_1+\gamma_2+\gamma_3+\gamma_4]\,L+\frac{2}{J}L \leq \\ &\leq \frac{L}{J}+\frac{2L}{J}=L.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus, there exists a fixed point $z\in\mathcal{M}$ such that $z=Az+Bz$. By Lemma 1, this fixed point is a solution of (1). Hence, (1) has a T-periodic solution.◄

4. Existence of nonnegative periodic solutions

Motivated by the works [7, 8, 9], we obtain in this section the existence of a nonnegative periodic solution of (1). To apply Theorem 1, we need to define a closed, convex and bounded subset $\mathbb{M}$ of $\mathcal{P}_T$. So, let

$$\mathbb{M} = \{\phi \in \mathcal{P}_T : 0 \le \phi \le K\}, \tag{31}$$

where K is a positive constant. To simplify notation, we let

$$F(t, x(t)) = \int_{t-\tau(t)}^{t} a(s)\, h(x(s))\, ds, \tag{32}$$

and we assume for all $t \in [0, T]$, $x \in \mathbb{M}$, that there exist constants h_1, h_2,c_1, c_2, a_1, a_2, such that

$$h_1, h_2 \ge 0, \quad -h_1 x \le F(t, x) \le h_2 x, \tag{33}$$

$$c_1, c_2 \ge 0, \quad -c_1 \le \frac{c(t)}{1-\tau'(t)} \le c_2, \tag{34}$$

$$c_2 + h_2 < 1, \tag{35}$$

$$0 < a_1 \le a(t) \le a_2, \tag{36}$$

$$(h_1 + c_1)\, a_2 + (c_2 + h_2)\, a_1 \le a_1, \tag{37}$$

$$\begin{aligned}(h_1 + c_1) K a_2 \le -a(t) F(t, x) + \left[\left(1-\tau'(t)\right) a(t-\tau(t)) - a(t)\right] h(y) - \\ - b(t) y + G(t, x, y),\end{aligned} \tag{38}$$

$$\begin{aligned}&- a(t) F(t, x) + \left[\left(1-\tau'(t)\right) a(t-\tau(t)) - a(t)\right] h(y) + a(t) H(x) - b(t) y + \\ &+ G(t, x, y) \ \le \ (1 - h_2 - c_2) K a_1.\end{aligned} \tag{39}$$

Lemma 5. *Let A, B be given by (15), (16), respectively, and assume (2)–(5) and (33)–(39) hold. Then A, $B : \mathbb{M} \to \mathbb{M}$.*

Proof. Let A be defined by (15). Then, for any $\varphi \in \mathbb{M}$, we have

$$\begin{aligned}(A\varphi)(t) \le& \frac{c(t)}{1-\tau'(t)} \varphi(t-\tau(t)) + F(t, \varphi(t)) - \\ &- \eta \int_{t-T}^{t} \kappa(t, u)\, a(u)\, F(u, \varphi(u))\, du + \\ &+ \eta \int_{t-T}^{t} \kappa(t, u) \left(\left(1-\tau'(u)\right) a(u-\tau(u)) - a(u)\right) h(\varphi(u-\tau(u)))\, du + \\ &+ \eta \int_{t-T}^{t} \kappa(t, u) \left[-b(u)\, \varphi(u-\tau(u)) + G(u, \varphi(u), \varphi(u-\tau(u)))\right] du \le\end{aligned}$$

$$\le \eta \int_{t-T}^{t} \kappa(t,u)(1-h_2-c_2)Ka_1 du + h_2K + c_2K \le$$
$$\le \eta \int_{t-T}^{t} \kappa(t,u)a(u)(1-h_2-c_2)Kdu + h_2K + c_2K = K.$$

On the other hand

$$\begin{aligned}(A\varphi)(t) &\ge \eta \int_{t-T}^{t} \kappa(t,u)a_2(Kh_1+Kc_1)du - h_1K - h_1K \ge \\ &\ge \eta \int_{t-T}^{t} \kappa(t,u)a(u)(Kh_1+Kc_1)du - h_1K - h_1K = 0.\end{aligned}$$

That is, $A\varphi \in \mathbb{M}$.

Now, let B be defined by (16). Then, for any $\varphi \in \mathbb{M}$, we have

$$0 \le (B\varphi)(t) \le \eta \int_{t-T}^{t} \kappa(t,u)a(u)(1-h_2-c_2)Kdu \le K.$$

That is, $B\varphi \in \mathbb{M}$.◄

Theorem 4. *Suppose the hypotheses of Lemmas 3, 4 and 5 hold. Then the equation (1) has a nonnegative T-periodic solution x in the subset $\mathbb{M}$.*

Proof. By Lemma 3, A is completely continuous. Also, by Lemma 4, the mapping B is a large contraction. By Lemma 5, $A, B : \mathbb{M} \to \mathbb{M}$. Next, we show that if $\varphi, \psi \in \mathbb{M}$, then we have $0 \le A\psi + B\varphi \le K$. Let $\varphi, \psi \in \mathbb{M}$ with $0 \le \varphi, \psi \le K$. By (33)–(39)

$$\begin{aligned}&(A\psi)(t) + (B\varphi)(t) = \\ &= \eta \int_{t-T}^{t} \kappa(t,u)a(u)H(\varphi(u))du + \frac{c(t)}{1-\tau'(t)}\psi(t-\tau(t)) + \\ &+ F(t,\psi(t)) - \eta \int_{t-T}^{t} \kappa(t,u)a(u)F(u,\psi(u))du + \\ &+ \eta \int_{t-T}^{t} \kappa(t,u)\left[\left(1-\tau'(u)\right)a(u-\tau(u)) - a(u)\right]h(\varphi(u-\tau(u)))du + \\ &+ \eta \int_{t-T}^{t} \kappa(t,u)\left[-b(u)\psi(u-\tau(u)) + G(u,\psi(u),\psi(u-\tau(u)))\right]du \le \\ &\le \eta \int_{t-T}^{t} \kappa(t,u)a(u)(1-h_2-c_2)Kdu + h_2K + c_2K = K.\end{aligned}$$

On the other hand

$$(A\psi)(t) + (B\varphi)(t) \ge 0.$$

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus, there exists a fixed point $z \in \mathbb{M}$ such that $z = Az + Bz$. By Lemma 1, this fixed point is a solution of (1) and the proof is complete.◄

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