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Abstract Parabolic Initial Boundary Value Problems with Singular Data and with Values in Interpolation Spaces

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Abstract. We consider abstract initial boundary value problems for parabolic differential-operator equations on the rectangle $[0,T] \times [0,1]$ with singular data. We use our previous results on norm-estimates of solutions and $\mathcal{R}$-boundedness of some sets of boundary value problems for abstract elliptic equations with a parameter on $[0,1]$ in a *UMD* Banach space. Unique solvability of these problems is proved in the Sobolev spaces of vector-valued functions with values in some interpolation spaces. The corresponding estimates for the solutions are also established. We also show completeness of elementary solutions of abstract parabolic boundary value problems. Abstract results are provided by a relevant application to parabolic PDEs. In some cases, the boundary conditions may contain the intermediate points of the interval $[0,1]$ or may be integro-differential.

Key Words and Phrases: abstract parabolic equation, singular data, UMD Banach space, interpolation space, completeness, elementary solutions.

2010 Mathematics Subject Classifications: 47F05, 47G20, 35P10, 35K20, 35K35, 35K90

1. Introduction and basic notations

In our work [4], we have studied abstract parabolic initial boundary value problems in the form (5)–(7)below in the space $L_q((0,T);L_p((0,1);E))$, where E is a *UMD* Banach space. The local boundary conditions (6) are simple, but abstract operators $B(x)$ and $A(x)$ in (5) are rather general. We have proved maximal L_q-regularity property for the problem. We have been able to do that because, first, we have studied in the same paper the corresponding abstract elliptic boundary value problems depending on a parameter. We have proved $\mathcal{R}$-boundedness property for the corresponding resolvent sets, which was a core and non-trivial part of the paper. Then, in our another paper [6], we have treated a problem in the form (1)–(3)below with very general non-local boundary conditions (2). In [6], we set more restrictions on $B(x)$ and $A(x) = A + A_1(x)$ with a constant operator main term. We have not succeeded to prove there the corresponding $\mathcal{R}$-boundedness property. We only proved norm-boundedness. Therefore, we did not get the maximal L_q-regularity property, but we have investigated well-posedness of the problem in the space

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$C([0,T];L_p((0,1);E))$, where E is a *UMD* Banach space. Then, completeness of elementary solutions of the homogeneous problem corresponding to (1)–(2) has been proved. In both of the above studies, the solvability spaces and the problem data are regular.

In the present paper, using the corresponding results of S. G. Pyatkov and M. V. Uvarova [9], we continue the investigation of the same problems, but with spaces and data singular in some sense. We essentially use here our previous studies and results of [4], [5] and [6]. So, following [9], we assume that the Cauchy data and the right-hand side of the equation, as $t \to 0$, are singular in some sense: the initial data belong to some "negative" space and the right-hand side belongs to the space L_p with weight t^α, where α can be rather large. Problems for parabolic equations (including the Navier-Stokes system) with singular data were considered, e.g., by H. Amann in [1], [2].

Let us give necessary definitions and notations. We consider complex Banach spaces.

If E and F are Banach spaces, $B(E,F)$ denotes the Banach space of all bounded, linear operators from E into F with the norm equal to the operator norm; moreover, $B(E) := B(E,E)$. The spectrum of a linear operator A in E is denoted by $\sigma(A)$, its resolvent set by $\rho(A)$. The domain and range of an operator A are denoted by $D(A)$ and $R(A)$, respectively. The resolvent of an operator A is denoted by $R(\lambda, A) := (\lambda I - A)^{-1}$.

A Banach space E is said to be of **class** *HT*, if the Hilbert transform is bounded on $L_p(\mathbb{R};E)$ for some (and then all) $p > 1$. Here the Hilbert transform H of a function $f \in S(\mathbb{R};E)$, the Schwartz space of rapidly decreasing E-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f,$$

i.e., $(Hf)(t) := \frac{1}{\pi} \lim\limits_{\varepsilon \to 0} \int_{|\tau|>\varepsilon} \frac{f(t-\tau)}{\tau} d\tau$. These spaces are also often called *UMD* **Banach spaces**, where the *UMD* stands for the property of *unconditional martingale differences.*

Definition 1. *Let E be a Banach space, and A be a closed linear operator in E. The operator A is called* **sectorial** *if the following conditions are satisfied:*

(1) $\overline{D(A)} = E$, $\overline{R(A)} = E$, $(-\infty, 0) \subset \rho(A)$;

(2) $\|\lambda(\lambda I + A)^{-1}\| \le M$ *for all* $\lambda > 0$ *and some* $M < \infty$.

Definition 2. *Let E and F be Banach spaces. A family of operators $\mathcal{T} \subset B(E,F)$ is called* $\mathcal{R}$**-bounded**, *if there is a constant $C > 0$ and $p \ge 1$ such that for each natural number n, $T_j \in \mathcal{T}$, $u_j \in E$ and for all independent, symmetric, $\{-1,1\}$-valued random variables ε_j on $[0,1]$ (e.g., the Rademacher functions $\varepsilon_j(t) = \operatorname{sign}\sin(2^j \pi t)$), the inequality*

$$\Big\| \sum_{j=1}^{n} \varepsilon_j T_j u_j \Big\|_{L_p((0,1);F)} \le C \Big\| \sum_{j=1}^{n} \varepsilon_j u_j \Big\|_{L_p((0,1);E)},$$

is valid. The smallest such C is called $\mathcal{R}$**-bound** *of $\mathcal{T}$ and is denoted by $\mathcal{R}\{\mathcal{T}\}$.*

Definition 3. *A sectorial operator A is called* $\mathcal{R}$**-sectorial** *if*

$$\mathcal{R}_A(0) := \mathcal{R}\{\lambda(\lambda I + A)^{-1} \ : \ \lambda > 0\} < \infty.$$

The number

$$\phi_A^{\mathcal{R}} := \inf\{\theta \in (0, \pi) \ : \ \mathcal{R}_A(\pi - \theta) < \infty\},$$

where $\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda I + A)^{-1} \ : \ |\arg \lambda| \leq \theta\}$*, is called an* $\mathcal{R}$**-angle** *of the operator* A*.*

Definition 4. *Let E be a Banach space. The space E has a property (α) if there is a constant $C > 0$ such that, $\forall n \in \mathbb{N}$, $\forall \alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \leq 1$, $\forall u_{ij} \in E$, $1 \leq i, j \leq n$,*

$$\int_0^1 \int_0^1 \Big\| \sum_{i=1}^n \sum_{j=1}^n \varepsilon_i(t)\varepsilon_j(s)\alpha_{ij}u_{ij} \Big\| dtds \leq C \int_0^1 \int_0^1 \Big\| \sum_{i=1}^n \sum_{j=1}^n \varepsilon_i(t)\varepsilon_j(s)u_{ij} \Big\| dtds.$$

For more details about above definitions, we refer the reader, e.g., to [3] and [8].

For the closed operator A in E, the domain of definition $D(A^n)$ of the operator A^n, $n \in \mathbb{N}$, becomes a Banach space $E(A^n)$ with respect to the norm

$$\|u\|_{E(A^n)} := \Big(\sum_{k=0}^n \|A^k u\|^2 \Big)^{\frac{1}{2}}.$$

The operator A^n from $E(A^n)$ into E is bounded.

For the Banach spaces F and E, introduce the Banach space $W_p^n((0,1); F, E)$, $1 < p < \infty$, $n \in \mathbb{N}$, of vector-valued functions with the finite norm

$$\|u\|_{W_p^n((0.1); F, E)} := \Big(\int_0^1 \|u(x)\|_F^p dx + \int_0^1 \|u^{(n)}(x)\|_E^p dx \Big)^{\frac{1}{p}}.$$

We write $W_p^n((0,1); E) := W_p^n((0,1); E, E)$.

Let E_0 and E_1 be two Banach spaces continuously embedded into the Banach space E : $E_0 \subset E$, $E_1 \subset E$. Two such spaces are called an **interpolation couple** $\{E_0, E_1\}$. Then, a standard **real interpolation space** $(E_0, E_1)_{\theta,p}$, $0 < \theta < 1$, $p \geq 1$, is defined (for the exact definitions and properties we refer the reader, e.g., to the book by H. Triebel [10]).

Remark 1. *All the proofs below may seem "very short". This is because we essentially use the results of our previous works [4], [5], and [6], where the most laborious calculations have been already done.*

2. Abstract parabolic initial boundary value problems and application to parabolic initial boundary value problems

We will use the corresponding results of S. G. Pyatkov and M. V. Uvarova [9], so let us state here some notation and definitions of [9]. Let X be a Banach space. Define $L_{p,t^{\delta_0}}((0,T);X)$ as the space of strongly measurable functions defined on the interval $(0,T)$ with values in X and satisfying the condition

$$\|u\|^p_{L_{p,t^{\delta_0}}((0,T);X)} = \int_0^T t^{\delta_0 p}\|u(t)\|^p_E dt < \infty.$$

If $\delta_0 = 0$, then we get just $L_p((0,T);X)$. So, in fact, $\|u\|^p_{L_{p,t^{\delta_0}}((0,T);X)} = \|t^{\delta_0}u\|^p_{L_p((0,T);X)}$.

Let now L be a linear closed operator in X with the dense domain $D(L)$. The operator L is called **positive** if $(-\infty, 0] \subset \rho(L)$ and there exists a positive constant C such that

$$\|R(\lambda, L)\| \leq C(1+|\lambda|)^{-1}, \quad \forall\lambda \in (-\infty, 0].$$

In fact, in what follows, it suffices to assume that the operator $e^{i\varphi}L$ is positive for some $\varphi \in [0, 2\pi)$.

We have already defined $E(L^n)$, $n \in \mathbb{N}$. Obviously, for a positive operator, the norm of $E(L^n)$, $n > 0$, coincides with the norm $\|L^n u\|$, i.e., $\|u\|_{E(L^n)} = \|L^n u\|$, and the space $E(L^n)$ can be also defined as the completion of X in this norm. The definition of the space $E(L^n)$ with integer $n < 0$ can be found, e.g., in [7, Section 5]. In particular, when E is also reflexive, the space $E(L^n)$, $n < 0$, coincides with the dual space of $E((L^*)^{-n})$, where L^* denotes the adjoint operator of L.

By the real interpolation method, we construct the space $B^s_q(X) = (E(L^m), E(L^n))_{\theta,q}$, where $1 \leq q \leq \infty$, $n < s < m$, n and m are integers, $0 < \theta < 1$, and $s = m(1-\theta) + n\theta$. Note that here real s can be negative! For the properties of the spaces $B^s_q(X)$, we refer the reader to [7, Section 5], [10, subsections 1.14 and 1.15.4], and [9].

Consider now, in a Banach space E, the following abstract initial boundary value problem for a parabolic differential-operator equation

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + B(x)\frac{\partial u(t,x)}{\partial x} + Au(t,x) + A_1(x)u(t,x) + \\ + \gamma u(t,x) = f(t,x), \quad (t,x) \in (0,T) \times (0,1), \tag{1}$$

$$\alpha_k \frac{\partial^{m_k} u(t,0)}{\partial x^{m_k}} + \beta_k \frac{\partial^{m_k} u(t,1)}{\partial x^{m_k}} + \sum_{s=1}^{N_k} T_{ks} u(t, x_{ks}) = 0, \quad t \in (0,T),\ k = 1,2, \tag{2}$$

$$u(0,x) = u_0(x), \quad x \in (0,1), \tag{3}$$

where $m_k \in \{0,1\}$; $\gamma > 0$; α_k, β_k are complex numbers; $x_{ks} \in [0,1]$; $B(x)$, $A_1(x)$, for $x \in [0,1]$, and A, T_{ks} are, in general, unbounded operators in E.

In the Banach space $X = L_p((0,1);E)$, $p>1$, consider an operator L defined by the equalities

$$\begin{aligned} &D(L) := W_p^2\big((0,1);E(A),E;L_k u = 0,\ k=1,2\big),\\ &Lu := u''(x) - B(x)u'(x) - Au(x) - A_1(x)u(x) - \gamma u(x), \end{aligned} \tag{4}$$

where $L_k u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{s=1}^{N_k} T_{ks} u(x_{ks}),\ k=1,2$. From the proof below it follows that the operator $e^{i\pi}L$, for some sufficiently large $\gamma>0$, is positive. Therefore, the construction of the spaces $B_q^s(X) := B_q^s(L_p)$ with respect to the above operator L in (4) is justified.

Theorem 1. *Let the following conditions be satisfied:*

(1) an operator A is closed, densely defined and invertible in a UMD Banach space E;

(2) $\mathcal{R}\{\lambda R(\lambda,A)\ :\ |\arg\lambda|\ge\beta\}<\infty$, for some $0<\beta<\frac{\pi}{2}$; †

(3) the embedding $E(A)\subset E$ is compact;

(4) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$;

(5) for any $\varepsilon>0$ and for almost every $x\in[0,1]$, there exists $C(\varepsilon)>0$ such that

$$\begin{aligned} &\|B(x)u\| \le \varepsilon\|A^{\frac12}u\| + C(\varepsilon)\|u\|, \quad u\in D(A^{\frac12}),\\ &\|A_1(x)u\| \le \varepsilon\|Au\| + C(\varepsilon)\|u\|, \quad u\in D(A); \end{aligned}$$

for $u\in D(A^{\frac12})$ the function $B(x)u$ and for $u\in D(A)$ the function $A_1(x)u$ are measurable on $[0,1]$ in E;

(6) if $m_k=0$, then $T_{ks}=0$; if $m_k=1$, then for $\varepsilon>0$ and $u\in(E(A),E)_{\frac{1}{2p},p}$, $p\in(1,\infty)$,

$$\begin{aligned} &\|T_{ks}u\|_{(E(A),E)_{\frac12+\frac{1}{2p},p}} \le \varepsilon\|u\|_{(E(A),E)_{\frac{1}{2p},p}} + C(\varepsilon)\|u\|,\\ &\|T_{ks}u\| \le \varepsilon\|u\|_{(E(A),E)_{\frac12,p}} + C(\varepsilon)\|u\|; \end{aligned}$$

(7) $f\in L_q((0,T);B_q^s(L_p))\cap L_{q,t^{\delta_0}}((0,T);B_q^{s+\delta_0}(L_p))$, for some $\delta_0>0$, $q\in(1,\infty)$, $s\in\mathbb{R}$;

(8) $u_0\in B_q^{s+1-1/q}(L_p)$.

Then, there exists sufficiently large $\gamma>0$ in (1) such that the problem (1)–(3) has a unique solution $u(t,x)$ with $u\in W_q^1((0,T);B_q^s(L_p))\cap L_q((0,T);B_q^{s+1}(L_p))$. Moreover, $Lu, u_t\in L_{q,t^{\delta_0}}((0,T);B_q^{s+\delta_0}(L_p))$, and the following estimate holds:

$$\begin{aligned} &\|u\|_{W_q^1((0,T);B_q^s(L_p))} + \|u\|_{L_q((0,T);B_q^{s+1}(L_p))} + \|t^{\delta_0}u_t\|_{L_q((0,T);B_q^{s+\delta_0}(L_p))} + \|t^{\delta_0}Lu\|_{L_q((0,T);B_q^{s+\delta_0}(L_p))}\\ &\quad\le C\big(\|f\|_{L_q((0,T);B_q^s(L_p))} + \|t^{\delta_0}f\|_{L_q((0,T);B_q^{s+\delta_0}(L_p))} + \|u_0\|_{B_q^{s+1-1/q}(L_p)}\big). \end{aligned}$$

† In fact, conditions (1) and (2) are equivalent to saying that A is invertible $\mathcal{R}$-sectorial operator in E with the $\mathcal{R}$-angle $\phi_A^{\mathcal{R}}<\beta$.

Proof. Using (4), rewrite the problem (1)-(3) in the form

$$u'(t) = Lu(t) + f(t),$$
$$u(0) = u_0,$$

to which we want to apply [9, Theorem 4] (see Theorem 7 in the Appendix). To this end, it suffices to verify that, for some $\alpha > 0$,

$$\|R(\lambda, L)\|_{B(X)} \leq C(1 + |\lambda|)^{-1}, \quad |\arg \lambda| \leq \frac{\pi}{2} + \alpha.$$

In turn, the latter inequality (with $\alpha = \frac{\pi}{2} - \beta$), which is the most difficult part in the proof, follows from our earlier work [5, Theorem 6] (with $\varphi = \pi - \beta$; see Theorem 9 in the Appendix) for sufficiently large $\gamma > 0$. ◄

Remark 2. *Note that there is no the term $\gamma u(x)$ in the equation in (30) and the result of Theorem 9 is obtained in the sector $|\arg \lambda| \leq \varphi$ for sufficiently large $|\lambda|$. Taking $\gamma > 0$ sufficiently large in our operator L in (4), we get the corresponding result of Theorem 9 in the whole sector.*

For the resolvent of the operator L in (4), we have proved only a norm-estimate ([5, Theorem 6]; see Theorem 9 in the Appendix) and not an $\mathcal{R}$-boundedness condition. That is why, for problem (1)–(3), we were able to use [9, Theorem 4] (see Theorem 7 in the Appendix), not [9, Theorem 5] (see Theorem 8 in the Appendix), where the $\mathcal{R}$-boundedness condition for L is claimed. On the other hand, in [4] we have considered a problem with rather simple boundary conditions, but with wider classes of the operators in the equation, for which we have succeeded to prove the $\mathcal{R}$-boundedness condition. Then, the next theorem is proved in a similar way as the previous one. The only thing is that we use [9, Theorem 5] (see Theorem 8 in the Appendix), and now, condition (1) of Theorem 8 follows from [4, Theorem 2.3].

So let's consider in a Banach space E the following abstract initial boundary value problem for a parabolic differential-operator equation

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + B(x)\frac{\partial u(t,x)}{\partial x} + A(x)u(t,x) + \\ + \gamma u(t,x) = f(t,x), \quad (t,x) \in (0,T) \times (0,1), \tag{5}$$

$$\frac{\partial^{m_1} u(t,0)}{\partial x^{m_1}} = 0, \ \frac{\partial^{m_2} u(t,1)}{\partial x^{m_2}} = 0, \quad t \in (0,T), \tag{6}$$

$$u(0,x) = u_0(x), \quad x \in (0,1), \tag{7}$$

where $m_k \in \{0,1\}$; $\gamma > 0$; $B(x)$, $A(x)$, for $x \in [0,1]$, are, in general, unbounded operators in E.

In the Banach space $X = L_p((0,1);E)$, $p > 1$, consider an operator $\tilde{L}$ defined by the equalities

$$\begin{aligned} D(\tilde{L}) &:= W_p^2\big((0,1); E(A), E; \tilde{L}_k u = 0, \ k = 1,2\big), \\ \tilde{L}u &:= u''(x) - B(x)u'(x) - A(x)u(x) - \gamma u(x), \end{aligned} \tag{8}$$

where $\tilde{L}_1 u := u^{(m_1)}(0), \tilde{L}_2 u := u^{(m_2)}(1)$. One constructs the spaces $B_q^s(X) := B_q^s(L_p)$ with respect to the above operator $\tilde{L}$ in (8).

Theorem 2. *Let the following conditions be satisfied:*

(1) for any $x \in [0,1]$, the operator $A(x)$ is closed, densely defined and invertible in a UMD Banach space E with property (α) and there exist $C(x) > 0$, $0 < \beta(x) < \pi$, such that

$$\|\lambda R(\lambda, A(x))\| \leq C(x), \quad |\arg \lambda| \geq \beta(x);$$

(2) the domains $D(A(x)) := D(A)$ and $D(A(x)^{1/2}) := D(A^{1/2})$ are independent of $x \in [0,1]$;

(3) for any $x \in [0,1]$, the operator $B(x) \in B(E(A^{1/2}), E)$; and if $x = 0$ or $x = 1$, then, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\|B(x)u\| \leq \varepsilon \|A^{\frac{1}{2}} u\| + C(\varepsilon)\|u\|, \quad u \in D(A^{\frac{1}{2}});$$

(4) the maps $x \to A(x)$ and $x \to B(x)$ belong to $C([0,1]; B(E(A), E))$ and $C([0,1]; B(E(A^{1/2}), E))$, respectively;

(5) $\forall x \in [0,1]$, $\forall \lambda \in \mathbb{C}$, with $Re(\lambda) \geq 0$, $\forall \sigma \in \mathbb{R}$, the operator $\lambda + \sigma^2 + i\sigma B(x) + A(x)$ is a bijection between $E(A)$ and E, and $(\lambda + \sigma^2 + i\sigma B(x) + A(x))^{-1} \in B(E)$.

(6) $\forall x \in [0,1]$, the families of operators $\{(\lambda + \sigma^2)(\lambda + \sigma^2 + i\sigma B(x) + A(x))^{-1} : Re(\lambda) \geq 0, \sigma \in \mathbb{R}\}$ and $\{A(x)(\lambda + \sigma^2 + i\sigma B(x) + A(x))^{-1} : Re(\lambda) \geq 0, \sigma \in \mathbb{R}\}$ in $B(E)$ are $\mathcal{R}$-bounded;

(7) $f \in L_q((0,T); B_q^s(L_p)) \cap L_{q,t^{-s}}((0,T); L_p((0,1); E))$, for some $q \in (1,\infty)$, $s \leq 0$, $s \neq 1/q - 1$;

(8) $u_0 \in B_q^{s+1-1/q}(L_p)$.

Then, there exists sufficiently large $\gamma > 0$ in (5) such that the problem (5)–(7) has a unique solution $u(t,x)$ with $u \in W_q^1((0,T); B_q^s(L_p)) \cap L_q((0,T); B_q^{s+1}(L_p))$. Moreover, $Lu, u_t \in L_{q,t^{-s}}((0,T); L_p((0,1); E))$, and the following estimate holds:

$$\|u\|_{W_q^1((0,T);B_q^s(L_p))} + \|u\|_{L_q((0,T);B_q^{s+1}(L_p))} +$$
$$+ \|t^{-s} u_t\|_{L_q((0,T);L_p((0,1);E))} + \|t^{-s} \tilde{L} u\|_{L_q((0,T);L_p((0,1);E))} \leq$$
$$\leq C\big(\|f\|_{L_q((0,T);B_q^s(L_p))} + \|t^{-s} f\|_{L_q((0,T);L_p((0,1);E))} + \|u_0\|_{B_q^{s+1-1/q}(L_p)}\big).$$

Note that the boundary conditions (6) are much more simple than the boundary conditions (2), but there is an advantage about the operators in the equation (5). The main operator $A(x)$ in (5) depends on x. The operator $B(x)$ in (5) satisfies the corresponding inequality (see condition (3) of the last theorem) only at the endpoints of the interval $[0,1]$

in contrast to condition (5) of Theorem 1, where the inequality is claimed for almost every $x \in [0,1]$. This means, in fact, that the operator $B(x)$ is compact from $E(A^{\frac{1}{2}})$ into E for almost every $x \in [0,1]$. For the problem (5)–(7) it is claimed that $B(x)$ is only bounded from $E(A^{\frac{1}{2}})$ into E, for $x \in (0,1)$, and is compact only for $x = 0,1$!

Observe also, that under the conditions of the last theorem, we get for the operator $\tilde{L}$ the $\mathcal{R}$-boundedness condition (1) in Theorem 8 for $|\arg\lambda| \le \frac{\pi}{2}$. But, using, e.g., [8, Lemma 2.21], we get that there exists $\alpha \in (0, \frac{\pi}{2})$ such that the corresponding $\mathcal{R}$-boundedness condition is also fulfilled for $|\arg\lambda| \le \frac{\pi}{2} + \alpha$.

Show now some application of Theorem 1. Let $\Omega := (0,1) \times G$, where $G \subset \mathbb{R}^r$, $r \ge 2$ be a bounded open domain with an $(r-1)$-dimensional boundary ∂G which locally admits rectification, and let us consider in the domain $(0,T) \times \Omega$ a very nonclassical parabolic initial boundary value problem (with integro-differential terms in the equation and unbounded operators and the values of the unknown function in intermediate points in boundary conditions)

$$\begin{aligned} D_t u(t,x,y) - D_x^2 u(t,x,y) + b(x,y) D_x u(t,x,y) + \int_G c(x,y,z) D_x u(t,x,z) dz - \\ - \sum_{s,j=1}^{r} a_{sj}(y) D_s D_j u(t,x,y) + \sum_{j=1}^{r} b_j(x,y) D_j u(t,x,y) + b_0(x,y) u(t,x,y) + \\ + \sum_{\ell=0}^{1} \sum_{j=1}^{r} \int_G c_{\ell j}(x,y,z) D_{z_j}^{\ell} u(t,x,z) dz + \gamma u(t,x,y) = \\ = f(t,x,y), \quad (t,x,y) \in (0,T) \times (0,1) \times G, \end{aligned} \tag{9}$$

$$L_k u := \alpha_k D_x^{m_k} u(t,0,y) + \beta_k D_x^{m_k} u(t,1,y) + \sum_{s=1}^{N_k} T_{ks} u(t, x_{ks}, \cdot) = 0, \quad (t,y) \in (0,T) \times G, \quad k = 1,2, \tag{10}$$

$$L_0 u := \sum_{j=1}^{r} c_j(y') D_j u(t,x,y') + c_0(y') u(t,x,y') = 0, \quad (t,x,y') \in (0,T) \times (0,1) \times \partial G, \tag{11}$$

$$u(0,x,y) = u_0(x,y), \quad (x,y) \in (0,1) \times G, \tag{12}$$

where $D_t := \frac{\partial}{\partial t}$, $D_x := \frac{\partial}{\partial x}$, $D_{z_j} := \frac{\partial}{\partial z_j}$, $D_j := -i\frac{\partial}{\partial y_j}$, $m_k \in \{0,1\}$, $\gamma > 0$, α_k, β_k are complex numbers, $y := (y_1, \dots, y_r)$, $x_{ks} \in [0,1]$, T_{ks} are, in general, unbounded operators in $L_h(G)$, $1 < h < \infty$. Let m be the order of the differential boundary operator L_0 in (11), i.e., $m = 0$ if all $c_j(y') \equiv 0$, $j = 1, \dots, r$ (and then $c_0(y') \ne 0$, $\forall y' \in \partial G$), and $m = 1$ if at least one of $c_j(y')$, $j = 1, \dots, r$, is not identically zero.

In this section we will also consider the Besov spaces

$$B^s_{h,p}(G) := (W_h^{s_0}(G), W_h^{s_1}(G))_{\theta,p},$$

where $0 \le s_0, s_1$ are integers, $W_h^n(G)$ stands for the Sobolev space, $0 < \theta < 1$, $1 < h < \infty$, $1 < p < \infty$ and $s = (1-\theta)s_0 + \theta s_1$.

Theorem 3. *Let the following conditions be satisfied:*

(1) (smoothness conditions) $|a_{sj}(y) - a_{sj}(z)| \le C|y-z|^\delta$ *for some* $C > 0$ *and* $\delta \in (0,1)$, $\forall y, z \in \overline{G}$; $b, b_j, b_0 \in L_\infty(\overline{\Omega})$; $c, c_{\ell j} \in L_\infty(\overline{\Omega \times G})$; $c_j, c_0 \in C^{2-m}(\partial G)$; $\partial G \in C^2$;

(2) (ellipticity condition for the below operator A) for $y \in \overline{G}$, $\sigma \in \mathbb{R}^r$, $|\arg\lambda| \ge \beta$, *for some* $0 < \beta < \frac{\pi}{2}$, $|\sigma| + |\lambda| \neq 0$, *we have*

$$\lambda + \sum_{s,j=1}^{r} a_{sj}(y)\sigma_s\sigma_j \neq 0;$$

(3) (Lopatinskii-Shapiro condition for the below operator A) y' *is any point on* ∂G, *the vector* σ' *is a tangent and* σ *is a normal vector to* ∂G *at the point* $y' \in \partial G$. *Consider the following ordinary differential problem, for* $|\arg\lambda| \ge \beta$ *with* β *from condition (2):*

$$\Big[\lambda + \sum_{s,j=1}^{r} a_{sj}(y')\Big(\sigma'_s - i\sigma_s\frac{d}{dt}\Big)\Big(\sigma'_j - i\sigma_j\frac{d}{dt}\Big)\Big]u(t) = 0, \quad t > 0, \tag{13}$$

$$\sum_{j=1}^{r} c_j(y')\Big(\sigma'_j - i\sigma_j\frac{d}{dt}\Big)u(t)\Big|_{t=0} = \omega, \quad for\ m = 1, \tag{14}$$

$$u(0) = \omega, \quad for\ m = 0; \tag{15}$$

it is required that for $m = 1$ *the problem (13), (14) (for* $m = 0$ *the problem (13), (15)) have one and only one solution, with all its derivatives tending to zero as* $t \to \infty$ *for any number* $\omega \in \mathbb{C}$; ‡

(4) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$;

(5) if $m_k = 0$, *then* $T_{ks} = 0$; *if* $m_k = 1$, *then, for* $\varepsilon > 0$ *and* $u \in B^{2-\frac{1}{p}}_{h,p}(G; L_0u = 0, m < 2 - \frac{1}{p} - \frac{1}{h})$,

$$\|T_{ks}u\|_{B^{1-\frac{1}{p}}_{h,p}(G)} \le \varepsilon\|u\|_{B^{2-\frac{1}{p}}_{h,p}(G)} + C(\varepsilon)\|u\|_{L_h(G)},$$
$$\|T_{ks}u\|_{L_h(G)} \le \varepsilon\|u\|_{B^{1}_{h,p}(G)} + C(\varepsilon)\|u\|_{L_h(G)},$$

where $p \neq \frac{h}{h-1}$ *and* $p, h \in (1,\infty)$, *or* $p = \frac{h}{h-1}$ *and* $m = 0$.§

‡Recall that, in the case $m = 0$, the boundary condition (11) is transformed into the Dirichlet boundary condition $u(t,x,y') = 0$, $(t,x,y') \in (0,T) \times (0,1) \times \partial G$.

§In the case where $p = \frac{h}{h-1} = 2$ and $m = 1$, $B^{\frac{3}{2}}_{2,2}(G; L_0u \in \widetilde{B}^{\frac{1}{2}}_{2,2}(G))$ (see [10, Theorem 4.3.3]) should be written instead of $B^{\frac{3}{2}}_{2,2}(G; L_0u = 0, m < 1)$. Here is the definition: $\widetilde{B}^s_{q,p}(G) := \{u \mid u \in B^s_{q,p}(\mathbb{R}^r),\ \mathrm{supp}(u) \subset \overline{G}\}$.

(6)

$$f \in L_q((0,T); B^s_q(L_p)) \cap L_{q,t^{\delta_0}}((0,T); B^{s+\delta_0}_q(L_p)),$$

for some $\delta_0 > 0$, $q \in (1,\infty)$, $s \in \mathbb{R}$; ¶

(7) $u_0 \in B^{s+1-1/q}_q(L_p)$.

Then, there exists sufficiently large $\gamma > 0$ *in (9) such that the problem (9)–(12) has a unique solution* $u(t,x,y)$ *with* $u \in W^1_q((0,T); B^s_q(L_p)) \cap L_q((0,T); B^{s+1}_q(L_p))$. *Moreover,* $Lu, u_t \in L_{q,t^{\delta_0}}((0,T); B^{s+\delta_0}_q(L_p))$, *and the following estimate holds:*

$$\begin{aligned}&\|u\|_{W^1_q((0,T);B^s_q(L_p))}+\\ &+\|u\|_{L_q((0,T);B^{s+1}_q(L_p))} + \|t^{\delta_0}u_t\|_{L_q((0,T);B^{s+\delta_0}_q(L_p))} + \|t^{\delta_0}Lu\|_{L_q((0,T);B^{s+\delta_0}_q(L_p))} \leq\\ &\leq C\big(\|f\|_{L_q((0,T);B^s_q(L_p))} + \|t^{\delta_0}f\|_{L_q((0,T);B^{s+\delta_0}_q(L_p))} + \|u_0\|_{B^{s+1-1/q}_q(L_p)}\big).\end{aligned} \tag{16}$$

Proof. Denote $E := L_h(G)$ and consider an operator A defined by the equalities

$$D(A) := W^2_h(G; L_0u = 0), \quad Au := -\sum_{s,j=1}^{r} a_{sj}(y)D_sD_ju(y) + \gamma_0 u(y),$$

where, by [3, Theorem 8.2], there exists $\gamma_0 > 0$ such that the operator A is an $\mathcal{R}$-sectorial operator in E with the $\mathcal{R}$-angle $\phi^{\mathcal{R}}_A < \pi$. For $x \in [0,1]$, also consider operators

$$\begin{aligned}&B(x)u := b(x,y)u(y) + \int_G c(x,y,z)u(z)dz,\\ &A_1(x)u := \sum_{j=1}^{r} b_j(x,y)D_ju(y) + b_0(x,y)u(y) + \sum_{\ell=0}^{1}\sum_{j=1}^{r}\int_G c_{\ell j}(x,y,z)D^{\ell}_{z_j}u(z)dz - \gamma_0 u(y).\end{aligned}$$

Then, the problem (9)–(12) can be rewritten in the form

$$\begin{aligned}&\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + B(x)\frac{\partial u(t,x)}{\partial x} + Au(t,x) + A_1(x)u(t,x)+\\ &\qquad + \gamma u(t,x) = f(t,x), \quad (t,x) \in (0,T)\times(0,1),\\ &\alpha_k\frac{\partial^{m_k}u(t,0)}{\partial x^{m_k}} + \beta_k\frac{\partial^{m_k}u(t,1)}{\partial x^{m_k}} + \sum_{s=1}^{N_k} T_{ks}u(t,x_{ks}) = 0, \quad t \in (0,T),\ k = 1,2,\\ &u(0,x) = u_0(x), \quad x \in (0,1),\end{aligned} \tag{17}$$

where $u(t,x) := u(t,x,\cdot)$, $f(t,x) := f(t,x,\cdot)$, and $u_0(x) = u_0(x,\cdot)$ are functions with values in the *UMD* Banach space $E := L_h(G)$, i.e., in the form of the problem (1)–(3). We want now to apply Theorem 1 to problem (17). Conditions (7)-(8) of Theorem 1 follow

¶Note that here $B^s_q(L_p)$ is constructed for the operator L in (4) corresponding to the below problem (17) in the space $X = L_p((0,1); L_h(G))$.

from conditions (6)-(7). Conditions (1)-(6) of Theorem 1 follow from conditions (1)-(5). This is the most difficult part of the proof and it was done in the proofs of Theorems 7 and 8 in [5].◄

Remark 3. *Note that we have not been able to explicitly describe the spaces $B^s_q(L_p)$ in conditions (6) and (7) of Theorem 3, since their construction is rather complicated for our general problems. For more special cases, the spaces can be explicitly described (see, e.g., [9, Section 4]).*

Examples of T_{ks} (at least for $\partial G \in C^\infty$) satisfying condition (5) of Theorem 3 (for the proof see [5, p.52]):

1) $(T_{ks})(y) := \gamma_{ks} u(y)$, where $\gamma_{ks} \in \mathbb{C}$;

2) $(T_{ks})(y) := \int_G \sum_{|\ell|\le 1} T_{ks\ell}(x,y) \frac{\partial^{|\ell|} u(x)}{\partial x_1^{\ell_1} \cdots \partial x_r^{\ell_r}} dx$, where $T_{ks\ell} \in L_{t'}(G\times G)$, $\frac{1}{t'} + \frac{1}{t} = 1$, $t = \min\{h, h'\}$, $\frac{1}{h'} + \frac{1}{h} = 1$, and $T_{ks\ell}(x,y)$ are continuously differentiable with respect to y_j, $j = 1,\dots,r$ and $\frac{\partial}{\partial y_j} T_{ks\ell} \in L_{t'}(G\times G)$. So, we consider, in particular, integro-differential boundary conditions.

Let us now consider some possible application problem of our second abstract Theorem 2. As in the previous application, $\Omega = (0,1)\times G$, $G \subset \mathbb{R}^r$, $r \ge 2$. Consider, in $(0,T)\times\Omega$, the following parabolic initial boundary value problem

$$\begin{aligned} &D_t u(t,x,y) - D_x^2 u(t,x,y) + B(x,y,D_y) D_x u(t,x,y) + A(x,y,D_y) u(t,x,y) + \\ &\qquad + \gamma u(t,x,y) = f(t,x,y), \quad (t,x,y) \in (0,T)\times(0,1)\times G, \\ &D_x^{(m_1)} u(t,0,y) = 0, \ D_x^{(m_2)} u(t,1,y) = 0, \quad (t,y)\times G, \\ &B_j(y', D_y) u(t,x,y') = 0, \quad j = 1,...,m, \ (t,x,y') \in (0,T)\times(0,1)\times\partial G, \\ &u(0,x,y) = u_0(x,y), \quad (x,y) \in (0,1)\times G, \end{aligned} \tag{18}$$

where

$$A(x,y,D_y) = \sum_{|\alpha|\le 2m} a_\alpha(x,y) D_y^\alpha, \ B(x,y,D_y) = \sum_{|\alpha|\le m} b_\alpha(x,y) D_y^\alpha,$$

$$B_j(y', D_y) = \sum_{|\alpha|\le m_j} b_{j,\alpha}(y') D_y^\alpha,$$

$D_t := \frac{\partial}{\partial t}$, $D_x := \frac{\partial}{\partial x}$, $D_y := (D_1,\dots,D_r)$, $D_y^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_r^{\alpha_r}$, $D_j := \frac{\partial}{\partial y_j}$, $\alpha := (\alpha_1,\alpha_2,...,\alpha_r)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_r$, $y := (y_1,\dots,y_r)$, $0 \le m_j \le 2m-1$; $b_\alpha(x,y) \equiv 0$ if $|\alpha| = m$, for $x = 0,1$, $y \in \overline{G}$.

We do not state here all restrictions on the data of the problem as in the previous application. The reader can find them in [4, Section 4], where all necessary calculations for using Theorem 2 can be found, too. Our purpose here is just to illustrate a problem to which Theorem 2 can be applied. We would like to emphasize that due to the less restrictive condition (3) on the operator $B(x)$ in Theorem 2, in contrast to condition (5) in Theorem 1, we can consider here the operator A of order $2m$ and the operator B of order m, at least for $x \in (0,1)$. This is not a case in application of Theorem 1, where the operator B should be of order strictly less than m. Unfortunately, a remark similar to Remark 3 is also applicable here.

3. Completeness of elementary solutions of abstract parabolic boundary value problems and application to parabolic initial boundary value problems

Consider the problem (1)–(3) with the homogeneous equation (1), i.e.,

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + B(x)\frac{\partial u(t,x)}{\partial x} + Au(t,x) + A_1(x)u(t,x) + \\ + \gamma u(t,x) = 0, \quad (t,x) \in (0,T) \times (0,1), \tag{19}$$

$$\alpha_k \frac{\partial^{m_k} u(t,0)}{\partial x^{m_k}} + \beta_k \frac{\partial^{m_k} u(t,1)}{\partial x^{m_k}} + \sum_{s=1}^{N_k} T_{ks} u(t, x_{ks}) = 0, \quad t \in (0,T),\ k = 1,2, \tag{20}$$

$$u(0,x) = u_0(x), \quad x \in (0,1), \tag{21}$$

where $m_k \in \{0,1\}$; $\gamma > 0$; α_k, β_k are complex numbers; $x_{ks} \in [0,1]$; $B(x)$, $A_1(x)$, for $x \in [0,1]$, and A, T_{ks} are, in general, unbounded operators in E.

Combining Theorem 1 and [6, Theorem 3.1] (or [6, Theorem 3.2]), we can get the following theorems about an approximation of the unique solution of the problem (19)–(21) by linear combinations of elementary solutions of (19)–(20). Recall that, e.g., by [11, Lemma 2.1/1], a function of the form

$$u_{ik_i}(t,x) := e^{\lambda_i t}\left(\frac{t^{k_i}}{k_i!}u_{i0}(x) + \frac{t^{k_i-1}}{(k_i-1)!}u_{i1}(x) + \cdots + u_{ik_i}(x)\right), \tag{22}$$

becomes the **elementary solution** of (19)–(20) if and only if $u_{i0}(x), u_{i1}(x), \ldots, u_{ik_i}(x)$ is a chain of root functions of the spectral problem

$$L(\lambda)u := \lambda u(x) - u''(x) + B(x)u'(x) + Au(x) + A_1(x)u(x) - \gamma u(x) = 0, \tag{23}$$

$$L_k u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{s=1}^{N_k} T_{ks} u(x_{ks}) = 0,\ k = 1,2, \tag{24}$$

corresponding to the eigenvalue $\lambda = \lambda_i$. The maximal possible k_i for fixed i is denoted by $i_{\max}$. In such a way, a system of functions $\{u_{ik_i}(0,x)\}$, $i = 1,2,...,$ $k_i = 0,1,...,i_{\max}$, will give us a system of all root functions corresponding to all eigenvalues of the problem (23)–(24). For the exact definitions of the eigenvalues and the root functions of the problem (23)–(24) see [6, Section 3].

First, consider the Hilbert spaces setting, i.e., we will use Theorems 1 and [6, Theorem 3.1]. So, we will denote by H the space E and by $H(A)$ the space $E(A)$ in order to distinguish between Hilbert and Banach spaces. We also refer the reader for the definitions of the singular numbers s_j and approximation numbers $\tilde{s}_j$ to [6, Section 3].

Note that the spaces $B^s_q(L_2)$ and $B^s_q(L_p)$ in the below Theorems 4 and 5, respectively, are constructed for the operator L given in (4), while in Theorem 6 the spaces $B^s_q(L_2)$ are constructed for the operator L in (4) corresponding to the problem (17) in the space $X = L_2((0,1); L_h(G))$.

Theorem 4. *Let the following conditions be fulfilled:*

(1) α_k, β_k *are complex numbers;* $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$*;* $x_{ks} \in [0,1]$*;*

(2) the embedding $H(A) \subset H$ *is compact and for some* $t > 0$*, for the embedding operator* J*, we have that the singular numbers* $s_j(J; H(A), H) \leq Cj^{-t}$*,* $j = 1, 2, \ldots$*;*

(3) the operator A *is closed, densely defined in a Hilbert space* H *and for some* φ*, such that* $\max\left\{\frac{2\pi}{2+t}, \frac{\pi}{2}\right\} < \varphi < \pi$*,*

$$\|R(\lambda, A)\| \leq C(1 + |\lambda|)^{-1}, \quad |\arg\lambda| \geq \pi - \varphi;$$

(4) for any $\varepsilon > 0$ *and for almost every* $x \in [0,1]$*, there exists* $C(\varepsilon) > 0$ *such that*

$$\|B(x)u\| \leq \varepsilon\|A^{\frac{1}{2}}u\| + C(\varepsilon)\|u\|, \quad u \in D(A^{\frac{1}{2}}),$$
$$\|A_1(x)u\| \leq \varepsilon\|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A);$$

for $u \in D(A^{\frac{1}{2}})$ *the function* $B(x)u$ *and for* $u \in D(A)$ *the function* $A_1(x)u$ *are measurable on* $[0,1]$ *in* H*;*

(5) if $m_k = 0$*, then* $T_{ks} = 0$*; if* $m_k = 1$*, then for* $\varepsilon > 0$ *and* $u \in (H(A), H)_{\frac{1}{4},2}$*,*

$$\|T_{ks}u\|_{(H(A),H)_{\frac{3}{4},2}} \leq \varepsilon\|u\|_{(H(A),H)_{\frac{1}{4},2}} + C(\varepsilon)\|u\|,$$
$$\|T_{ks}u\| \leq \varepsilon\|u\|_{(H(A),H)_{\frac{1}{2},2}} + C(\varepsilon)\|u\|;$$

(6) $u_0 \in W^2_2\big((0,1); H(A), H; L_k u = 0,\ k = 1,2\big)$*, where* $L_k u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{s=1}^{N_k} T_{ks}u(x_{ks}),\ k = 1,2.$

Then, there exists sufficiently large $\gamma > 0$ *in (19) such that the problem (19)–(21) has a unique solution* $u(t,x)$ *with* $u(t,x) \in W^1_q((0,T); B^s_q(L_2)) \cap L_q((0,T); B^{s+1}_q(L_2))$*, for any* $1 < q < \infty$*,* $s < 1/q$*, and there exist numbers* $C^n_{ik_i}$ *such that for the solution* $u(t,x)$ *we have,* $\forall\delta_0 > 0$

$$\lim_{n\to\infty}\Bigg(\Big\|u(t,x) - \sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i}u_{ik_i}(t,x)\Big\|_{W^1_q((0,T);B^s_q(L_2))} + \\ + \Big\|u(t,x) - \sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i}u_{ik_i}(t,x)\Big\|_{L_q((0,T);B^{s+1}_q(L_2))} + \\ + \Big\|t^{\delta_0}\frac{\partial}{\partial t}\Big(u(t,x) - \sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i}u_{ik_i}(t,x)\Big)\Big\|_{L_q((0,T);B^{s+\delta_0}_q(L_2))}\Bigg) = 0, \quad (25)$$

where $u_{ik_i}(t,x)$ *are elementary solutions (22) of (19)–(20).*

Proof. By [6, Theorem 3.1], where we take $A_1(x)-\gamma I$ instead of $A_1(x)$, a system of root functions of the problem (23)–(24) is complete in $W_2^2\big((0,1);\, H(A), H; L_k u = 0,\ k=1,2\big)$. Hence, there exist numbers $C^n_{ik_i}$ such that

$$\lim_{n\to\infty}\Big\|u_0(x)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i} u_{ik_i}(0,x)\Big\|_{W_2^2((0,1);H(A),H)}=0, \tag{26}$$

where $u_{ik_i}(t,x)$ are elementary solutions (22) of (19)–(20). On the other hand, $W_2^2\big((0,1);\, H(A), H;\ L_k u = 0,\ k=1,2\big) \subset B_q^{s+1-1/q}(L_2)$, for any $1<q<\infty$, $s<1/q$ (see [9, Assertion 1]). Then, from Theorem 1 (recall that in the framework of Hilbert spaces, $\mathcal{R}$-boundedness in condition (1) of Theorem 1 is just a norm-boundedness in condition (3)), we get that the problem (19)–(21) has a unique solution $u(t,x)$ such that $u(t,x)\in W_q^1((0,T);B_q^s(L_2))\cap L_q((0,T);B_q^{s+1}(L_2))$ and the corresponding estimate in Theorem 1, with $f=0$, is fulfilled. Then, $u(t,x)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i}u_{ik_i}(t,x)$ is a unique solution of the problem (19)–(21), but with the initial function

$$u_0(x)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i} u_{ik_i}(0,x),$$

and, from the corresponding estimate in Theorem 1, one gets

$$\begin{aligned}
&\Big\|u(t,x)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i} u_{ik_i}(t,x)\Big\|_{W_q^1((0,T);B_q^s(L_2))}+\\
&\quad+\Big\|u(t,x)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i} u_{ik_i}(t,x)\Big\|_{L_q((0,T);B_q^{s+1}(L_2))}+\\
&\quad+\Big\|t^{\delta_0}\frac{\partial}{\partial t}\Big(u(t,x)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i} u_{ik_i}(t,x)\Big)\Big\|_{L_q((0,T);B_q^{s+\delta_0}(L_2))}\le\\
&\quad\le C\Big\|u_0(x)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i} u_{ik_i}(0,x)\Big\|_{B_q^{s+1-1/q}(L_2)}\le\\
&\quad\le C\Big\|u_0(x)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}} C^n_{ik_i} u_{ik_i}(0,x)\Big\|_{W_2^2((0,1);H(A),H)}.
\end{aligned}\tag{27}$$

The use of (26) completes the proof.◄

Consider now problem (19)-(21) in a separable, reflexive *UMD* Banach space E.

Theorem 5. *Let the spectrum of the operator L in (4) be non empty; for some $s>0$, the approximation numbers*

$$\tilde{s}_j(J;W_p^2((0,1);E(A),E),L_p((0,1);E))\le Cj^{-s},$$

and let all the conditions of [5, Theorem 6] (see Theorem 9 in the Appendix) be satisfied. Moreover, let the condition (2) of [5, Theorem 6] is satisfied for some $\frac{2-s}{2}\pi < \varphi < \pi$ *if* $0 < s \leq 1$ *and for some* $\frac{\pi}{2} < \varphi < \pi$ *if* $s > 1$*; finally,* $u_0 \in W_p^2\big((0,1); E(A), E; L_k u = 0,\ k = 1,2\big)$*, where* $L_k u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{s=1}^{N_k} T_{ks} u(x_{ks}),\ k = 1,2.$

Then, there exists sufficiently large $\gamma > 0$ *in (19) such that the problem (19)–(21) has a unique solution* $u(t,x)$ *with* $u(t,x) \in W_q^1((0,T); B_q^s(L_p)) \cap L_q((0,T); B_q^{s+1}(L_p))$*, for any* $1 < q < \infty$*,* $s < 1/q$*, and there exist numbers* $C_{ik_i}^n$ *such that for the solution* $u(t,x)$ *we have,* $\forall \delta_0 > 0$

$$\lim_{n\to\infty} \Bigg(\Big\| u(t,x) - \sum_{i=1}^{n} \sum_{k_i=0}^{i_{\max}} C_{ik_i}^n u_{ik_i}(t,x) \Big\|_{W_q^1((0,T);B_q^s(L_p))} + + \Big\| u(t,x) - \sum_{i=1}^{n} \sum_{k_i=0}^{i_{\max}} C_{ik_i}^n u_{ik_i}(t,x) \Big\|_{L_q((0,T);B_q^{s+1}(L_p))} + + \Big\| t^{\delta_0} \frac{\partial}{\partial t} \Big(u(t,x) - \sum_{i=1}^{n} \sum_{k_i=0}^{i_{\max}} C_{ik_i}^n u_{ik_i}(t,x) \Big) \Big\|_{L_q((0,T);B_q^{s+\delta_0}(L_p))} \Bigg) = 0,$$

where $u_{ik_i}(t,x)$ *are elementary solutions (22) of (19)–(20).*

Proof. The proof is the same as that of Theorem 4. We only use [6, Theorem 3.2] instead of [6, Theorem 3.1].◄

Show an application of Theorem 4. In fact, all necessary data are given in the application part of Section 2. We just consider the homogeneous equation (instead of the nonhomogeneous equation (9))

$$D_t u(t,x,y) - D_x^2 u(t,x,y) + b(x,y) D_x u(t,x,y) + \int_G c(x,y,z) D_x u(t,x,z) dz - - \sum_{s,j=1}^{r} a_{sj}(y) D_s D_j u(t,x,y) + \sum_{j=1}^{r} b_j(x,y) D_j u(t,x,y) + b_0(x,y) u(t,x,y) + + \sum_{\ell=0}^{1} \sum_{j=1}^{r} \int_G c_{\ell j}(x,y,z) D_{z_j}^{\ell} u(t,x,z) dz + \gamma u(t,x,y) = = 0, \quad (t,x,y) \in (0,T) \times (0,1) \times G, \tag{28}$$

with boundary conditions (10)–(11) and initial condition (12).

Theorem 6. *Assume that conditions (1)-(5) (with* $p = h = 2$*) of Theorem 3 are fulfilled (conditions (2)-(3) with some* $0 < \beta < \pi - \frac{2\pi r}{2r+2}$*), and condition (6) of Theorem 4 is also fulfilled, where* $H(A) = W_2^2(G; L_0 u = 0)$*,* $H = L_2(G)$*, and* L_0, L_k *are defined in (10)–(11).*

Then, there exists sufficiently large $\gamma > 0$ *in (28) such that the problem (28), (10)–(12) has a unique solution* $u(t,x,y)$ *with* $u(t,x,y) \in W_q^1((0,T); B_q^s(L_2)) \cap L_q((0,T);\ B_q^{s+1}(L_2))$*,* *for any* $1 < q < \infty$, $s < 1/q$, *and there exist numbers* $C^n_{ik_i}$ *such that for the solution* $u(t,x,y)$ *we have,* $\forall \delta_0 > 0$

$$\lim_{n\to\infty}\Big(\Big\|u(t,x,y)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}}C^n_{ik_i}u_{ik_i}(t,x,y)\Big\|_{W^1_q((0,T);B^s_q(L_2))}+$$
$$+\Big\|u(t,x,y)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}}C^n_{ik_i}u_{ik_i}(t,x,y)\Big\|_{L_q((0,T);B^{s+1}_q(L_2))}+$$
$$+\Big\|t^{\delta_0}\frac{\partial}{\partial t}\Big(u(t,x,y)-\sum_{i=1}^{n}\sum_{k_i=0}^{i_{\max}}C^n_{ik_i}u_{ik_i}(t,x,y)\Big)\Big\|_{L_q((0,T);B^{s+\delta_0}_q(L_2))}\Big)=0,$$

where $u_{ik_i}(t,x,y) := e^{\lambda_i t}\left(\frac{t^{k_i}}{k_i!}u_{i0}(x,y)+\frac{t^{k_i-1}}{(k_i-1)!}u_{i1}(x,y)+\cdots+u_{ik_i}(x,y)\right)$ *are elementary solutions of the system (28), (10)–(11), i.e.,* $u_{i0}(x,y), u_{i1}(x,y), \ldots, u_{ik_i}(x,y)$ *is a chain of root functions of the spectral problem corresponding to (28), (10)–(11).*

Proof. We use Theorem 4. As in the proof of [6, Theorem 3.3], we note that all operators and all necessary explanations are the same as in the proof of Theorem 3. As in the proof of [6, Theorem 3.3], the only thing we have to do is to check condition (2) and the corresponding restriction on φ in condition (3) of Theorem 4. In our case, they are the same as in [6, Theorem 3.1], (which have been checked in the proof of [6, Theorem 3.3]), since $\max\left\{\frac{2\pi}{2+t}, \frac{\pi}{2}\right\} = \frac{2\pi}{2+t}$ for $t = \frac{2}{r}$ and $r \geq 2$.◄

Remark 4. *A remark similar to Remark 3 is also applicable here.*

Remark 5. *This section is written only for the problem (1)–(3) and not for the problem (5)–(7). We have essentially used here our previous result [6, Theorem 3.1]. In fact, a result similar to [6, Theorem 3.1] can be also obtained for the homogeneous spectral problem corresponding to the problem (5)–(6) due to [4, Theorem 2.3]. Recall that in [4, Theorem 2.3] we have obtained the* $\mathcal{R}$*-boundedness property which implies, in particular, norm-boundedness and this is enough for proving the corresponding theorem like [6, Theorem 3.1]. So, one can rewrite this section for the problem (5)–(7) with the corresponding changes, but it would be too much in the framework of one paper.*

4. Appendix

Consider the Cauchy problem

$$\begin{aligned} &u'(t) = Lu(t) + f(t), \quad t \in (0,T),\\ &u(0) = u_0, \end{aligned} \tag{29}$$

in a Banach space X. For the definition of the spaces $B^s_q(X)$ see the beginning of Section 2.

Theorem 7. *([9, Theorem 4]) Let the following conditions be satisfied*

(1) L is a linear closed operator in X and, for some $\alpha \in (0, \frac{\pi}{2})$,

$$\|R(\lambda, L)\|_{B(X)} \leq C(1 + |\lambda|)^{-1}, \quad |\arg \lambda| \leq \frac{\pi}{2} + \alpha;$$

(2) $f \in L_q((0,T); B_q^s(X)) \cap L_{q,t^{\delta_0}}((0,T); B_q^{s+\delta_0}(X))$, for some $\delta_0 > 0$, $q \in (1, \infty)$, $s \in \mathbb{R}$;

(3) $u_0 \in B_q^{s+1-1/q}(X)$.

Then, the problem (29) has a unique solution such that

$$u \in W_q^1((0,T); B_q^s(X)) \cap L_q((0,T); B_q^{s+1}(X)).$$

Moreover, $Lu, u_t \in L_{q,t^{\delta_0}}((0,T); B_q^{s+\delta_0}(X))$, and the following estimate holds:

$$\|u\|_{W_q^1((0,T);B_q^s(X))} + \|u\|_{L_q((0,T);B_q^{s+1}(X))} + \|t^{\delta_0} u_t\|_{L_q((0,T);B_q^{s+\delta_0}(X))} + \|t^{\delta_0} Lu\|_{L_q((0,T);B_q^{s+\delta_0}(X))} \leq$$
$$\leq C\big(\|f\|_{L_q((0,T);B_q^s(X))} + \|t^{\delta_0} f\|_{L_q((0,T);B_q^{s+\delta_0}(X))} + \|u_0\|_{B_q^{s+1-1/q}(X)}\big).$$

Theorem 8. *([9, Theorem 5]) Let the following conditions be satisfied*

(1) L is a linear closed operator in a UMD Banach space X and, for some $\alpha \in (0, \frac{\pi}{2})$,

$$\mathcal{R}\{\lambda R(\lambda, L) : \ |\arg \lambda| \leq \frac{\pi}{2} + \alpha\} < \infty;$$

(2) $f \in L_q((0,T); B_q^s(X)) \cap L_{q,t^{-s}}((0,T); X)$, for some $q \in (1, \infty), s \leq 0, s \neq 1/q - 1$;

(3) $u_0 \in B_q^{s+1-1/q}(X)$.

Then, the problem (29) has a unique solution such that

$$u \in W_q^1((0,T); B_q^s(X)) \cap L_q((0,T); B_q^{s+1}(X)).$$

Moreover, $Lu, u_t \in L_{q,t^{-s}}((0,T); X)$, and the following estimate holds:

$$\|u\|_{W_q^1((0,T);B_q^s(X))} + \|u\|_{L_q((0,T);B_q^{s+1}(X))} + \|t^{-s} u_t\|_{L_q((0,T);X)} + \|t^{-s} Lu\|_{L_q((0,T);X)} \leq$$
$$\leq C\big(\|f\|_{L_q((0,T);B_q^s(X))} + \|t^{-s} f\|_{L_q((0,T);X)} + \|u_0\|_{B_q^{s+1-1/q}(X)}\big).$$

Consider, in a Banach space E, an abstract elliptic boundary value problem with a parameter

$$\begin{aligned} L(\lambda)u &:= \lambda u(x) - u''(x) + B(x)u'(x) + Au(x) + A_1(x)u(x) = f(x), \\ L_k u &:= \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{s=1}^{N_k} T_{ks} u(x_{ks}) = f_k, \ k = 1, 2, \end{aligned} \tag{30}$$

where λ is a complex parameter, $m_k \in \{0, 1\}$; α_k, β_k are complex numbers; $x_{ks} \in [0, 1]$; $B(x)$, $A_1(x)$, for $x \in [0, 1]$, and A, T_{ks} are, in general, unbounded operators in E.

Theorem 9. *([5, Theorem 6]) Let the following conditions be satisfied:*

(1) A *is a closed, densely defined operator in a UMD Banach space* E;

(2) $\mathcal{R}\{\lambda R(\lambda, A) \;:\; |\arg\lambda| \geq \pi - \varphi\} < \infty$ *for some* $0 \leq \varphi < \pi$;[‖]

(3) the embedding $E(A) \subset E$ *is compact*;

(4) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$;

(5) for any $\varepsilon > 0$ *and for almost every* $x \in [0,1]$*, there exists* $C(\varepsilon) > 0$ *such that*

$$\|B(x)u\| \leq \varepsilon\|A^{\frac{1}{2}}u\| + C(\varepsilon)\|u\|, \quad u \in D(A^{\frac{1}{2}}),$$
$$\|A_1(x)u\| \leq \varepsilon\|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A);$$

for $u \in D(A^{\frac{1}{2}})$ *the function* $B(x)u$ *and for* $u \in D(A)$ *the function* $A_1(x)u$ *are measurable on* $[0,1]$ *in* E;

(6) if $m_k = 0$*, then* $T_{ks} = 0$*; if* $m_k = 1$*, then for* $\varepsilon > 0$ *and* $u \in (E(A), E)_{\frac{1}{2p},p}$*, where* $p \in (1, \infty)$*,*

$$\|T_{ks}u\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} \leq \varepsilon\|u\|_{(E(A),E)_{\frac{1}{2p},p}} + C(\varepsilon)\|u\|,$$
$$\|T_{ks}u\| \leq \varepsilon\|u\|_{(E(A),E)_{\frac{1}{2},p}} + C(\varepsilon)\|u\|.$$

Then, the operator $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := \Big(L(\lambda)u, L_1u, L_2u\Big)$*, for* $|\arg\lambda| \leq \varphi$ *and sufficiently large* $|\lambda|$*, is an isomorphism from* $W_p^2((0,1); E(A), E)$ *onto* $L_p((0,1); E) \dotplus (E(A), E)_{\theta_1,p} \dotplus (E(A), E)_{\theta_2,p}$*, where* $\theta_k = \frac{m_k}{2} + \frac{1}{2p}$*, and with these* λ*, the following estimate holds for the solution of (30):*

$$|\lambda|\|u\|_{L_p((0,1);E)} + \|u''\|_{L_p((0,1);E)} + \|Au\|_{L_p((0,1);E)}$$
$$\leq C\Big[\|f\|_{L_p((0,1);E)} + \sum_{k=1}^{2}\Big(\|f_k\|_{(E(A),E)_{\theta_k,p}} + |\lambda|^{1-\theta_k}\|f_k\|\Big)\Big];$$

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[‖] In the original version, there was a misprint that the $\mathcal{R}$-boundedness is claimed in the angle, but for sufficiently large λ. In fact, one can see from the proof of [5, Theorem 6], that by replacing A and $A_1(x)$ with $A + M_0I$ and $A_1(x) - M_0I$, respectively, for some sufficiently large $M_0 > 0$, the latter can also be treated.

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