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Existence of Positive Solution for a Coupled Hybrid System of Quadratic Fractional Integral Equations

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Abstract. In this paper, we study the solvability of coupled hybrid systems of quadratic fractional integral equations. Applying hybrid fixed point theory, due to B. C. Dhage, the existence of at least one positive solution for mentioned systems is proved. At the end, illustrating obtained results, an example is given.

Key Words and Phrases: fractional integrals, integral equations, hybrid fixed point theory, positive solution.

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1. Introduction

The fractional calculus is the theory of arbitrary order integrations and differentiations that generalizes the integer order ones in classic differential calculus. Not only in theoretical manner, but also as a result of more accurate description of real world phenomena in comparison with classic differential calculus, we can observe the boom of developments of theory of fractional calculus in less than three decades in almost whole sciences related to mathematics such as biosciences, medicine, engineering, economy and so on. More details and applications can be found in the monographs and papers [25],[23],[1]-[12],[14]-[20].

On the other hand, the theory of integral equations by itself has been introduced as full applicable theory in mathematics and great theories such as geomagnetic theory, transport theory, mechanics and so forth. So we can conclude that the combination of fractional calculus and integral equations may provide more effective tool for analysis and description of topics mentioned above. In this way more interesting applications can be found in references [24],[26] and references cited therein.

In this paper, we study so called quadratic fractional integral equations, namely the integral equations of the form

$$x(t) = f(t) + \mathcal{A}(t) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s, x(s)) ds, \quad t \in J = [0,T],\ T,\ \alpha > 0,$$

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where $f : J \to \mathbb{R}$ and $u : J \times \mathbb{R} \to \mathbb{R}$ are given functions and $\mathcal{A} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is an appropriate operator. Let us now state our main motivations for preparing this paper. The authors in [22], by means of Dhage hybrid fixed point theory, obtained periodic solutions of integral equation

$$x(t) = \sum_{i=1}^{n} f_i(t, x(a_i(t))). \int_{\mathbb{R}} k_i(t,s) g_i(s, x(b_i(s)))ds.$$

The authors in [13] considered the fractional order integral equation

$$x(t) = f(t,x(t)) + g(t,x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s,x(s))ds, \quad t \in J = [0,1],\ \alpha > 0,$$

and via above mentioned theory proved the existence of at least one solution for this fractional integral equation.
In this paper, we consider the coupled hybrid system of fractional quadratic integral equations

$$\begin{cases} u(t) = \displaystyle\sum_{i=1}^{n} \left\{ k_{1,i}(t,u(t),v(t)) + g_{1,i}(t,u(t),v(t)). \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_{1,i}(s,u(s),v(s))ds \right\}, \\ v(t) = \displaystyle\sum_{j=1}^{n} \left\{ k_{2,j}(t,u(t),v(t)) + g_{2,j}(t,u(t),v(t)). \int_0^t \frac{(t-s)^{\beta_j - 1}}{\Gamma(\beta_j)} f_{2,j}(s,u(s),v(s))ds \right\}, \end{cases} \tag{1}$$

where $t \in \mathbb{J} = [0,T],\ T, \alpha_i, \beta_j \in \mathbb{R}^+,\ i,j = 1,2,...,n.$
Assume that the following necessary conditions are satisfied throughout this paper.

(A_1) $k_{1,i} \in C(\mathbb{J} \times \mathbb{R}^2, \mathbb{R}^+)$ for $i = 1,2,...,n$ and there exist positive constants $L_{1,k_{1,i}}, L_{2,k_{1,i}}$, $\rho_{k_{1,i}}$ with $L_{1,k_{1,i}} \leq L_{2,k_{1,i}}$ such that

$$|k_{1,i}(t,u_1,.) - k_{1,i}(t,u_2,.)| \leq \frac{1}{2n} \left(\frac{L_{1,k_{1,i}} |u_1 - u_2|}{L_{2,k_{1,i}} + |u_1 - u_2|} \right),\ t \in \mathbb{J},\ u_1,u_2 \in C(\mathbb{R}),$$

also assume that

$$\sup k_{1,i}(t,u,v) = \rho_{k_{1,i}}, \quad i = 1,2,...,n,\ t \in \mathbb{J},\ u,v \in C(\mathbb{R}).$$

(A_2) $g_{1,i} \in C(\mathbb{J} \times \mathbb{R}^2, \mathbb{R}^+)$ for $i = 1,2,...,n$ and there exist positive constants $L_{1,g_{1,i}}, L_{2,g_{1,i}}$, $\rho_{g_{1,i}}$ with $L_{1,g_{1,i}} \leq L_{2,g_{1,i}}$ such that

$$|g_{1,i}(t,u_1,.) - g_{1,i}(t,u_2,.)| \leq \frac{1}{2nM_i} \left(\frac{L_{1,g_{1,i}} |u_1 - u_2|}{L_{2,g_{1,i}} + |u_1 - u_2|} \right),\ t \in \mathbb{J},\ u_1,u_2 \in C(\mathbb{R}),$$

where $M_i = \|I_{0^+}^{\alpha_i} f_{1,i}(.,u,v)\|$, also suppose that

$$\sup g_{1,i}(t,u,v) = \rho_{g_{1,i}}, \quad i = 1,2,...,n,\ t \in \mathbb{J},\ u,v \in C(\mathbb{R}).$$

(A_3) $f_{1,i}, f_{2,j} \in C(\mathbb{J} \times \mathbb{R}^2, \mathbb{R}^+)$ for $i, j = 1, 2, ..., n$ and

$$\sup f_{1,i}(t, u, v) = \theta_{1,i}, \quad \sup f_{2,j}(t, u, v) = \theta_{2,j}, \quad i, j = 1, 2, ..., n, \ t \in \mathbb{J}, u, v \in C(\mathbb{R}).$$

(A_4) If i, u and α in the conditions $(A_1), (A_2)$ are replaced by j, v and β, respectively, then the corresponding conditions will be satisfied for $k_{2,j}$, $g_{2,j}$ for $j = 1, 2, ..., n$ in hybrid system (1).

2. Preliminaries

This section contains two steps. First, we present some concepts from fractional calculus that will be needed in the sequel, and then, in preparatory manner we briefly overview the hybrid fixed point theory.

Definition 1 ([23])**.** *The Riemann-Liouville fractional integral of order $\alpha > 0$ for function $f : (0, \infty) \to \mathbb{R}$ is defined as*

$$I^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds, \tag{2}$$

provided that the right hand side is point-wise defined on $(0, \infty)$.

Definition 2 ([23])**.** *The Riemann-Liouville fractional derivative of order $\alpha > 0$ for function $f : (0, \infty) \to \mathbb{R}$ is given by*

$$D^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d^n}{dt^n}\right) \int_0^t (t - s)^{n - \alpha - 1} f(s) ds, \tag{3}$$

provided that the right hand side is point-wise defined on the positive half axis.

Remark 1 ([25])**.** *Fractional differentiation of power functions is given by*

$$D^{\alpha}_{0^+} t^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha}, \quad t > 0, \ \alpha > 0, \ \beta > -1. \tag{4}$$

Clearly, replacing α with $-\alpha$ gives us fractional integration of power functions.

In what follows we will use the Banach space $(\mathfrak{B}, \|.\|_{\mathfrak{B}})$:

$$\mathfrak{B} = E \times E, \quad E = \{u | \ u \in C(\mathbb{J}, \mathbb{R})\},$$
$$\|(u, v)\|_{\mathfrak{B}} = \|u\|_E + \|v\|_E, \quad \|u\|_E = \max_{t \in \mathbb{J}} \{u(t) | \ u \in E\}.$$

Define the set $S \subset \mathfrak{B}$ as

$$\begin{aligned} S &= \{(u, v) \in \mathfrak{B} | \ u(t), v(t) \geq 0, \ t \in \mathbb{J}, \ \|(u, v)\|_{\mathfrak{B}} \leq r\} \\ &= \{u, v \in E | \ u(t), v(t) \geq 0, \ t \in \mathbb{J}, \ \|u\|_E + \|v\|_E \leq r\}. \end{aligned} \tag{5}$$

Definition 3. *We define the fractional integral operators $T_1, T_2 : E \to E$ by*

$$T_{1,u}(t) = \sum_{i=1}^{n} \left\{ k_{1,i}(t, u(t), v(t)) + g_{1,i}(t, u(t), v(t)). \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_{1,i}(s, u(s), v(s)) ds \right\},$$
$$T_{2,v}(t) = \sum_{j=1}^{n} \left\{ k_{2,j}(t, u(t), v(t)) + g_{2,j}(t, u(t), v(t)). \int_0^t \frac{(t-s)^{\beta_j - 1}}{\Gamma(\beta_j)} f_{2,j}(s, u(s), v(s)) ds \right\}. \tag{6}$$

Now we can define the operator $\mathfrak{T} : \mathfrak{B} \to \mathfrak{B}$ as follows

$$\mathfrak{T}(u, v) = (T_{1,u}, T_{2,v}). \tag{7}$$

Definition 4 ([1])**.** *Let X be a normed vector space. A mapping $T : X \to X$ is said to be D-Lipschitzian, provided that there exists a continuous and nondecreasing function $\psi_T : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $x, y \in X$*

$$\big\|Tx - Ty\big\| \leq \psi_T\Big(\|x - y\|\Big), \quad \psi_T(0) = 0.$$

Remark 2 ([11])**.** *Every Lipschitzian mapping is D-Lipschitzian and if $\psi_T(r) < r$, then T is called nonlinear D-contraction on X with D-function ψ_T.*

Remark 3 ([11])**.** *Every nonlinear D-contraction is D-Lipschitzian while the reverse may not hold.*

Definition 5 ([21])**.** *Let X be a normed space and suppose $S \subset X$. A finite set of N balls $B(x_n, \epsilon)$ with $x_n \in X$ and $\epsilon > 0$ is said to be a finite $\epsilon-$covering of S, provided that every element of S lies inside one of the balls $B(x_n, \epsilon)$, i.e.*

$$S \subset \bigcup_{n=1}^{N} B(x_n, \epsilon).$$

The set of centers $\{x_n\}$ of a finite ϵ-covering is called a finite $\epsilon-$net for S.

Definition 6 ([21])**.** *Let X be a normed space. A set $S \subset X$ is said to be totally bounded if and only if it has a finite $\epsilon-$covering for every $\epsilon > 0$.*

Theorem 1 ([21])**.** *[Hausdorff compactness criterion] Assume that X is a normed space. A set $S \subset X$ is compact if and only if it is closed and totally bounded.*

Theorem 2 ([1])**.** *[Dhage fixed point theorem] Assume that S is a nonempty closed convex and bounded subset of Banach algebra X. Let $A, C : X \to X$ and $B : S \to X$ be three operators with the following properties:*

(i) A, C are D-Lipschitzian with D-functions ϕ_A and ϕ_C, respectively.

(ii) B is completely continuous.

(iii) $x = Ax + CxBy \Rightarrow x \in S$, for all $y \in S$.

(iv) $\phi_A(r) + M\phi_C(r) < r$, for $r > 0$, where $M = \big\|B(S)\big\|$.

Then the operator $Ax + CxBx$ has a fixed point in S.

3. Main Results

Theorem 3. *Suppose that the conditions* $(A_1) - (A_4)$ *hold. Then the fractional hybrid system (1) has at least one positive solution in* S.

Proof. We are going to carry out the proof in three steps as follows:

(S_1) Suppose that

$$\begin{aligned} A_{1,i}u(t) &= \sum_{i=1}^{n} k_{1,i}(t, u(t), v(t)), \\ C_{1,i}u(t) &= g_{1,i}(t, u(t), v(t)), \; i = 1, 2, ..., n, \\ B_{1,i}u(t) &= \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_{1,i}(s, u(s), v(s))ds, \; i = 1, 2, ..., n, \end{aligned} \tag{8}$$

$$\begin{aligned} A_{2,j}v(t) &= \sum_{j=1}^{n} k_{2,j}(t, u(t), v(t)), \\ C_{2,j}v(t) &= g_{2,j}(t, u(t), v(t)), \; j = 1, 2, ..., n, \\ B_{2,j}v(t) &= \int_0^t \frac{(t-s)^{\beta_j - 1}}{\Gamma(\beta_j)} f_{2,j}(s, u(s), v(s))ds, \; j = 1, 2, ..., n. \end{aligned} \tag{9}$$

Define

$$A_{1,i,2,j}(u, v)(t) = \begin{pmatrix} \sum_{i=1}^{n} k_{1,i}(t, u(t), v(t)) \\ \sum_{j=1}^{n} k_{2,j}(t, u(t), v(t)) \end{pmatrix}, \tag{10}$$

$$C_{1,i,2,j}(u, v)(t) = \begin{pmatrix} g_{1,i}(t, u(t), v(t)) & 0 \\ 0 & g_{2,j}(t, u(t), v(t)) \end{pmatrix}, \quad i, j = 1, 2, ..., n, \tag{11}$$

$$B_{1,i,2,j}(u, v)(t) = \begin{pmatrix} \int_0^t \frac{(t-s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f_{1,i}(s, u(s), v(s))ds \\ \int_0^t \frac{(t-s)^{\beta_j - 1}}{\Gamma(\beta_j)} f_{2,j}(s, u(s), v(s))ds \end{pmatrix}, \quad i, j = 1, 2, ..., n. \tag{12}$$

Now we can transform the operator $\mathfrak{T}(u, v)$ defined by (7) to the operator

$$\mathfrak{T}(u, v)(t) = A_{1,i,2,j}(u, v)(t) + \sum_i \sum_j C_{1,i,2,j}(u, v)(t) B_{1,i,2,j}(u, v)(t), \tag{13}$$

such that the above double summation acts on $(i, j) = (1, 1), (2, 2), ..., (n, n)$.
We shall show in this step that $\mathfrak{T}(u, v)$ is a nonlinear D-contraction with D-function

$$\psi_{u,v} = \sum_{i=1}^{n} \psi_{k_{1,i}} + \sum_{j=1}^{n} \psi_{k_{2,j}} + \sum_i \sum_j \{\psi_{g_{1,i}} M_i + \psi_{g_{2,j}} M_j\}, \tag{14}$$

where $(i,j)=(1,1),(2,2),...,(n,n)$ and $\psi_{k_{1,i}},\ \psi_{k_{2,j}},\ \psi_{g_{1,i}},\ \psi_{g_{2,j}}$ are D-functions corresponding to the nonlinear D-contractions $k_{1,i},\ k_{2,j},\ g_{1,i},\ g_{2,j}$. By the conditions $(A_1),(A_4)$ we have

$$\begin{aligned}
&\|A_{1,i,2,j}(u_1,v_1)-A_{1,i,2,j}(u_2,v_2)\|_{\mathfrak{B}}\leq\\
&\leq\sum_{i=1}^{n}\|k_{1,i}(.,u_1,v_1)-k_{1,i}(.,u_2,v_2)\|_E+\sum_{j=1}^{n}\|k_{2,j}(.,u_1,v_1)-k_{2,j}(.,u_2,v_2)\|_E\leq\\
&\leq\sum_{i=1}^{n}\frac{L_{1,k_{1,i}}\|u_1-u_2\|_E}{2n\left(L_{1,k_{1,i}}+\|u_1-u_2\|_E\right)}+\sum_{j=1}^{n}\frac{L_{2,k_{2,j}}\|v_1-v_2\|_E}{2n\left(L_{2,k_{2,j}}+\|v_1-v_2\|_E\right)}<\\
&<\frac{\|u_1-u_2\|_E}{2}+\frac{\|v_1-v_2\|_E}{2}=\frac{\|(u_1,v_1)-(u_2,v_2)\|_{\mathfrak{B}}}{2}.
\end{aligned}\tag{15}$$

Thus, $k_{1,i},\ k_{2,j}$ are two nonlinear D-contractions with corresponding D-functions

$$\psi_{k_{1,i}}(r)=\frac{L_{1,k_{1,i}}r}{2n(L_{1,k_{1,i}}+r)},\quad \psi_{k_{2,j}}(r)=\frac{L_{2,k_{2,j}}r}{2n(L_{2,k_{2,j}}+r)},\ i,j=1,2,...,n.\tag{16}$$

It follows that $A_{1,i,2,j}(u,v)$ is a nonlinear D-contraction with corresponding D-function

$$\psi_{A_{1,i,2,j}}(r)=\left(\sum_{i=1}^{n}\psi_{k_{1,i}}+\sum_{j=1}^{n}\psi_{k_{2,j}}\right)r.\tag{17}$$

Similarly, by means of conditions $(A_2),(A_4)$ we conclude that

$$\begin{aligned}
&\sum_i\sum_j\|C_{1,i,2,j}B_{1,i,2,j}(u_1,v_1)-C_{1,i,2,j}B_{1,i,2,j}(u_2,v_2)\|_{\mathfrak{B}}\leq\\
&\leq\sum_i\sum_j\Big\{\|g_{1,i}(.,u_1,v_1)-g_{1,i}(.,u_2,v_2)\|_E\|I_{0^+}^{\alpha_i}f_{1,i}\|_E+\\
&\quad+\|g_{2,j}(.,u_1,v_1)-g_{2,j}(.,u_2,v_2)\|_E\|I_{0^+}^{\beta_j}f_{2,j}\|_E\Big\}\leq\\
&\leq\sum_i\sum_j\left\{\frac{L_{1,g_{1,i}}\|u_1-u_2\|_EM_i}{2nM_i\left(L_{1,g_{1,i}}+\|u_1-u_2\|_E\right)}+\frac{L_{2,g_{2,j}}\|v_1-v_2\|_EM_j}{2nM_j\left(L_{2,g_{2,j}}+\|v_1-v_2\|_E\right)}\right\}<\\
&<\frac{\|u_1-u_2\|_E}{2}+\frac{\|v_1-v_2\|_E}{2}=\frac{\|(u_1,v_1)-(u_2,v_2)\|_{\mathfrak{B}}}{2},
\end{aligned}\tag{18}$$

for $(i,j)=(1,1),(2,2),...,(n,n)$. Hence we deduce that both of operators $g_{1,i},\ g_{2,j}$ are nonlinear D-contractions with corresponding D-functions

$$\psi_{g_{1,i}}(r)=\frac{L_{1,g_{1,i}}r}{2nM_i(L_{1,g_{1,i}}+r)},\quad \psi_{g_{2,j}}(r)=\frac{L_{2,g_{2,j}}r}{2nM_j(L_{2,g_{2,j}}+r)},\ i,j=1,2,...,n.\tag{19}$$

Therefore we conclude that $\sum_i \sum_j C_{1,i,2,j} B_{1,i,2,j}(u,v)$ is a nonlinear D-contraction with corresponding D-function

$$\psi_{C_{1,i,2,j}}(r) = \sum_i \sum_j \left[\psi_{g_{1,i}} M_i + \psi_{g_{2,j}} M_j\right] r, \ (i,j) = (1,1),(2,2),...,(n,n). \tag{20}$$

At last, by means of (15)-(20) one can observe that $\mathfrak{T}(u,v)$ defined by (13) is a nonlinear D-contraction with corresponding D-function $\psi_{u,v}$ defined by (14). So (S_1) is complete now.

(S_2) In this step we must prove that the operator $B_{1,i,2,j}(u,v)$ defined by (12) is completely continuous on S defined by (5). First, applying the Hausdorff compactness criterion given by Theorem 1, we will prove that S is a compact subset of Banach space $\mathfrak{B}$. It is clear that $S \subset \mathfrak{B}$ is a cone in $\mathfrak{B}$. Let us define the bounded subset $S_u \subset \mathfrak{B}$ as follows

$$S_u = \{u \in E| \ u(t) \geq 0, \ \|u\|_E \leq r, \ t \in \mathbb{J}\}. \tag{21}$$

Clearly, S_u is closed. As we know, each closed subset of a complete space is complete. Thus, as a result of equicontinuity of $u(t)$, the Arzela-Ascoli theorem implies that S_u is relatively compact. Hence Theorem 1 ensures that S_u is totally bounded. Thus, by Definition 6 we conclude that there exists a finite $\epsilon-$covering

$$\mathfrak{U}_\epsilon(u_i), \quad i = 1,2,3,...,l_1,$$

such that

$$S_u \subset \bigcup_{i=1}^{l_1} \mathfrak{U}_\epsilon(u_i), \tag{22}$$

where

$$\mathfrak{U}_\epsilon(u_i) = \{u \in S_u| \ \|u - u_i\|_E < \epsilon\}. \tag{23}$$

Define

$$S_i = \{(u,v) \in S_u \times S_u| \ u,v \in \mathfrak{U}_\epsilon(u_i)\}.$$

It is easy to see that $S \subset S_u \times S_u \subset \bigcup_i S_i, \ 1 \leq i \leq l_1$.
In fact, if we take $(u_i, v_i) \in S_i$, then $S_u \times S_u$ can be covered by finite $4\epsilon-$covering

$$\mathfrak{U}_{4\epsilon}(u_i, v_i) = \left\{(u,v) \in S_u \times S_u \middle| \ \|(u,v) - (u_i,v_i)\|_{\mathfrak{B}} < 4\epsilon\right\}.$$

In other words, for every $(u,v) \in S_u \times S_u$, there exists an index k such that

$$u, v \in \mathfrak{U}_\epsilon(u_k).$$

Therefore

$$
\begin{aligned}
|u-u_i| &\le |u-u_k|+|u_k-u_i| < \epsilon+\epsilon = 2\epsilon,\\
|v-v_i| &\le |v-u_k|+|u_k-v_i| < \epsilon+\epsilon = 2\epsilon.
\end{aligned}
\tag{24}
$$

(24) implies that $\|(u,v)-(u_i,v_i)\|_{\mathfrak{B}} < 4\epsilon$. Hence S has a finite $4\epsilon-$covering. Therefore, using Theorem 1 we conclude that S is compact.
Turning back to the definition of $B_{1,i,2,j}(u,v)$ given by (12) and considering the condition (A_3), we conclude that $B_{1,i,2,j}(u,v)$ is continuous on S. Thus, $B_{1,i,2,j}(S)$ is completely continuous on S. This completes the (S_2).

(S_3) In the last step we are going to show that if

$$
\begin{aligned}
u_* &= \sum_{i=1}^{n}\left\{k_{1,i}(.,u_*,v)+g_{1,i}(.,u_*,v).I_{0^+}^{\alpha_i} f_{1,i}(.,u_*,v)\right\},\\
v_* &= \sum_{j=1}^{n}\left\{k_{2,j}(.,u,v_*)+g_{2,j}(.,u,v_*).I_{0^+}^{\beta_j} f_{2,j}(.,u,v_*)\right\},
\end{aligned}
\tag{25}
$$

then $(u_*,v_*)\in S$ for all $(u,v)\in S$. By means of conditions $(A_1)-(A_4)$, it is easy to check that

$$
\begin{aligned}
T_{1,u}(t) &\le \sum_{i=1}^{n}\frac{\Gamma(\alpha_i+1)\rho_{k_{1,i}}+\rho_{g_{1,i}}\theta_{1,i}T^{\alpha_i}}{\Gamma(\alpha_i+1)} = r_1,\\
T_{2,v}(t) &\le \sum_{j=1}^{n}\frac{\Gamma(\beta_j+1)\rho_{k_{2,j}}+\rho_{g_{2,j}}\theta_{2,j}T^{\beta_j}}{\Gamma(\beta_j+1)} = r_2.
\end{aligned}
\tag{26}
$$

On the other hand, $\|\mathfrak{T}(u,v)\|_{\mathfrak{B}} = \|T_{1,u}\|_E + \|T_{2,v}\|_E$. So we have

$$\|\mathfrak{T}(u,v)\|_{\mathfrak{B}} \le r = 2\max\{r_1,r_2\}.$$

Equivalently, the recent inequality shows that $\mathfrak{T}(S)\subset S$, that is if (25) is satisfied, then $(u_*,v_*)\in S$ for all $(u,v)\in S$. So (S_3) is completed.

Since all of the conditions $(i)-(iv)$ in Theorem 2 hold, the coupled hybrid system of fractional quadratic integral equations (1) has at least one positive solution in S.◀

4. An Example

Let us consider the coupled hybrid system of FQIEs

$$
\begin{cases}
u(t) = \displaystyle\sum_{i=1}^{n}\left\{k_{1,i}(t,u(t),v(t))+g_{1,i}(t,u(t),v(t))\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} f_{1,i}(s,u(s),v(s))ds\right\},\\
v(t) = \displaystyle\sum_{j=1}^{n}\left\{k_{2,j}(t,u(t),v(t))+g_{2,j}(t,u(t),v(t))\int_0^t \frac{(t-s)^{\beta_j-1}}{\Gamma(\beta_j)} f_{2,j}(s,u(s),v(s))ds\right\}.
\end{cases}
\tag{27}
$$

Take $n=1,\ \mathbb{J}=[0,1]$ and $\alpha=\beta=\frac{3}{2}$. Setting

$$\begin{cases} k_{1,1}(t,u,v)=\frac{t^2[u(t)+v(t)]}{6}, \quad g_{1,1}(t,u,v)=\frac{t^2u(t)}{6}, \quad f_{1,1}(t,u,v)=\sin(t^2+u(t)+v(t)), \\ k_{2,1}(t,u,v)=\frac{t^3v(t)}{4}, \quad g_{2,1}(t,u,v)=\frac{t^3u(t)v(t)}{1+4u(t)}, \quad f_{2,1}(t,u,v)=\sin(t^3+u(t)+v(t)), \end{cases} \tag{28}$$

by direct calculation we have

$$M_1=M_2=\frac{1}{\Gamma(\frac{3}{2})}=\frac{2}{\sqrt{\pi}}.$$

Thus we conclude that

$$\begin{aligned} |k_{1,1}(t,u_1,v)-k_{1,1}(t,u_2,v)| &\le \frac{|u_1-u_2|}{2(1+|u_1-u_2|)}, \\ |g_{1,1}(t,u_1,v)-g_{1,1}(t,u_2,v)| &\le \frac{|u_1-u_2|}{2M_1(1+|u_1-u_2|)}, \\ |k_{2,1}(t,u,v_1)-k_{2,1}(t,u,v_2)| &\le \frac{|v_1-v_2|}{2(1+|v_1-v_2|)}, \\ |g_{2,1}(t,u,v_1)-g_{2,1}(t,u,v_2)| &\le \frac{|v_1-v_2|}{2M_2(1+|v_1-v_2|)}. \end{aligned}$$

Since all of the conditions $(A_1)-(A_4)$ hold, according to Theorem 3 we deduce that the coupled hybrid system of FQIEs (27) has at least one positive solution in S.

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