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On One Generalization of Banach Frame

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Abstract. This work is dedicated to the generalization of frames and Riesz bases in Banach spaces with respect to the Banach space of vector-valued sequences. The concepts of $\tilde{X}$-frame and $\tilde{X}$-Riesz basis generated by a bilinear mapping are introduced. Criteria for $\tilde{X}$-frameness and $\tilde{X}$-Riesz basicity are found.

Key Words and Phrases: $\tilde{X}$-frames, $\tilde{X}$-Riesz bases, b-biorthogonal systems.

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1. Introduction

The concept of a frame in a Hilbert space H was introduced by R.J. Duffin and A.C. Schaeffer [1]: a system of non-zero elements $\{f_n\}_{n\in N}\subset H$ is called a frame in H, if there exist the constants $A>0$ and $B>0$ such that the inequality

$$A\,\|f\|_H^2 \leq \sum_{n=1}^{\infty} |(f,f_n)|^2 \leq B\,\|f\|_H^2\,.$$

holds for every $f \in H$, where $\|\cdot\|_H$ is a norm on H generated by the scalar production $(\cdot,\cdot)$. The constants A and B are called the lower and upper frame bounds, respectively. Frame $\{f_n\}_{n\in N}$ in H defines the boundedly invertible frame operator S such that every $f\in H$ has a decomposition of the form

$$f=\sum_{n=1}^{\infty}(f,S^{-1}(f_n))f_n=\sum_{n=1}^{\infty}(f,f_n)S^{-1}(f_n),$$

with respect to the frames $\{f_n\}_{n\in N}$ and $\left\{S^{-1}(f_n)\right\}_{n\in N}$. Frames $\{f_n\}_{n\in N}$ and $\left\{S^{-1}(f_n)\right\}_{n\in N}$ are called conjugate frames. More facts about the theory of frames in a Hilbert space can be found in [2, 3]. Frames are widely used in many branches of natural science, such as signal processing, image processing, data compression, etc. Many works have been dedicated to frames (see, e.g., [4, 5, 6, 7]). A special case of frame is a Riesz basis [8], i.e. a system $\{\varphi_n\}_{n\in N}\subset H$ which is an image of an orthonormal basis for

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a bounded invertible operator. This is equivalent to saying that the system $\{\varphi_n\}_{n\in N}$ is complete in H and there exists the constants $A > 0$ and $B > 0$ such that

$$A\sum_k |c_k|^2 \le \left\|\sum_k c_k\varphi_k\right\|_H^2 \le B\sum_k |c_k|^2\,,$$

for every finite set of numbers $\{c_k\}$. Also, Paley-Wiener type theorems are true for frames. These matters have been considered in [9, 10]. g-frames and g-Riesz bases, introduced and studied in [11, 12, 13, 14], are the generalizations of frames and Riesz bases in Hilbert spaces. t-frame [15], obtained by tensor product, is another generalization of frame in a Hilbert space. Riesz bases have been extended to the Banach case in [12, 16, 17].

In [18], Gröchenig introduced the concepts of Banach frame and atomic decomposition in Banach spaces for the first time. Atomic decompositions are used largely in Gabor theory and wavelet theory. The problems related to Banach frames and atomic decompositions, including their stability, have been studied in [19, 20, 21, 22, 23, 24]. The generalizations of frames and Riesz bases in Banach spaces - p-frames and p-Riesz bases, as well as the relationship between them, have been studied in [25, 26]. The results of [25, 26] have been extended to X_d-frames and X_d-Riesz bases in [27, 28, 29], where X_d is a Banach space of numerical sequences (for more details see [30]). In [31], the concept of $\tilde{X}$-frame in a Banach space with respect to the Banach space $\tilde{X}$ of vector-valued sequences has been introduced, and a number of results for X_d-frames have been obtained. Banach frames and atomic decompositions for a bilinear mapping b in the sense of b-basis [12] have been considered in [32]. The works [33, 34] deal with the frame properties of degenerate trigonometric systems in Lebesgue spaces.

In this work, we introduce the concept of $\tilde{X}$-Riesz b-basis in a Banach space for some bilinear mapping b. Equivalent conditions for $\tilde{X}$-frameness and $\tilde{X}$-Riesz b-basicity in Banach spaces are found, the relationship between them is established. An example is given.

2. Some Notations and Auxiliaries

In this section, we give some notations, concepts and facts.

Let X, Y and Z be Banach spaces equipped with the norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. By $L(X,Y)$ we denote a space of all linear bounded operators from X to Y. The image of the operator $T \in L(X,Y)$ is denoted by ImT. The conjugate of the operator T is denoted by T^*. We will need the following fact [30].

Theorem 1. *Let X and Y be Banach spaces, and let $T \in L(X,Y)$. Then the conjugate operator $T^* \in L(Y^*,X^*)$ is surjective only when T has a bounded inverse on ImT.*

Consider a bilinear mapping $b(x,y) : X \times Y \to Z$ satisfying the condition

$$\exists M > 0 : \|b(x,y)\|_Z \le M\|x\|_X\|y\|_Y, \forall x \in X, y \in Y.$$

Let's recall some information from [12].

Definition 1. *Let* $\{y_n\}_{n\in N} \subset Y$ *and* $\{y_n^*\}_{n\in N} \subset L(Z, X)$.

1) The system $\{y_n\}_{n\in N} \subset Y$ *is called b-complete in* Z*, if the totality of all possible finite sums of the form* $\sum_k b(x_k, y_k)$, $x_k \in X$*, is dense in* Z.

2) The system $\{y_n\}_{n\in N}$ *is called a b-basis for* Z*, if every* Z *can be uniquely represented in the form* $z = \sum_{k=1}^{\infty} b(x_k, y_k)$, $x_n \in X$, $n \in N$*. In this case, the sequence* $\{x_n\}_{n\in N}$ *is called a sequence of coefficients of* $z \in Z$ *with respect to the b-basis* $\{y_n\}_{n\in N}$.

3) The system $\{y_n\}_{n\in N}$ *is called b-ω-linearly independent in* Z*, if* $\sum_k b(x_k, y_k) = 0$ *implies* $x_k = 0$, $\forall k \in N$.

4) The systems $\{y_n\}_{n\in N}$ *and* $\{y_n^*\}_{n\in N}$ *are called b-biorthogonal in* Z*, if*

$$y_n^*(b(x, y_k)) = \delta_{nk}x, \forall k, n \in N, x \in X,$$

where δ_{nk} *is the Kronecker symbol. In this case, the system* $\{y_n^*\}_{n\in N}$ *is called b-biorthogonal to the system* $\{y_n\}_{n\in N}$.

The next theorem describes the structure of the space of sequences of coefficients of a b-basis.

Theorem 2. *Let the system* $\bar{y} = \{y_n\}_{n\in N} \subset Y$ *form a b-basis for* Z *and* $\|b(x, y_k)\|_Z \geq a_k \|x\|_X$, $a_k > 0$, $\forall k \in N$, $x \in X$*. Then the space* $\tilde{X}_{\bar{y}}$*, consisting of sequences* $\tilde{x} = \{x_k\}_{k\in N} \subset X$ *for which the series* $\sum_{k=1}^{\infty} b(x_k, y_k)$ *is convergent, is a Banach space equipped with the norm*

$$\left\|\{x_k\}_{k\in N}\right\| = \sup_n \left\|\sum_{k=1}^{n} b(x_k, y_k)\right\|.$$

The next theorem presents a criterion of b-basicity.

Theorem 3. *Let* $\bar{y} = \{y_n\}_{n\in N} \subset Y$ *and* $\|b(x, y_k)\|_Z \geq a_k \|x\|_X$, $a_k > 0$, $\forall k \in N$*. Then* $\{y_n\}_{n\in N}$ *forms a b-basis for* Z *only when the following conditions are satisfied:*

i) $\{y_n\}_{n\in N}$ *is b-complete in* Z;

ii) $\exists C \geq 1$, $m \geq n$: $\|\sum_{k=1}^{n} b(x_k, y_k)\|_Z \leq C \|\sum_{k=1}^{m} b(x_k, y_k)\|_Z$ *for any* $x_1, x_2, \ldots, x_m \in X$.

Proof. Let $\{y_n\}_{n\in N}$ form a b-basis for Z. Then $\forall z \in Z$ is uniquely represented in the form

$$z = \sum_{k=1}^{\infty} b(x_k, y_k), x_k \in X.$$

Consequently, $\{y_n\}_{n\in N}$ is b-complete in Z. Define the operator $F : \tilde{X}_{\bar{y}} \to Z$ by the formula $F(\tilde{x}) = \sum_{k=1}^{\infty} b(x_k, y_k)$. Evidently, F maps $\tilde{X}_{\bar{y}}$ isomorphically to Z. Consider the operator y_n^* defined by $y_n^*(z) = x_n$. It is clear that the operator y_n^* is linear and $y_n^*(b(x, y_k)) = \delta_{nk}x$. Moreover, we have

$$\|y_n^*(z)\| = \|x_n\| \leq \frac{1}{a_n} \|b(x_n, y_n)\| = \frac{1}{a_n} \left\|\sum_{k=1}^{n} b(x_k, y_k) - \sum_{k=1}^{n-1} b(x_k, y_k)\right\| \leq$$

$$\leq \frac{2}{a_n} \|\tilde{x}\|_{\tilde{X}_{\tilde{y}}} \leq \frac{2 \left\|F^{-1}\right\|}{a_n} \|z\|_Z ,$$

i.e. y_n^* is a bounded operator.

Let $S_n(z) = \sum_{k=1}^{n} b(y_k^*(z), y_k)$. Evidently, $S_n \in L(Z)$. In view of $\lim\limits_{n\to\infty} S_n(z) = z$, we have $C = \sup\limits_n \|S_n\| < +\infty$. As the relation

$$S_n \left(\sum_{k=1}^{m} b(x_k, y_k) \right) = \sum_{k=1}^{n} b(x_k, y_k),$$

holds for $m \geq n$, we have

$$\left\| \sum_{k=1}^{n} b(x_k, y_k) \right\|_Z = \left\| S_n \left(\sum_{k=1}^{m} b(x_k, y_k) \right) \right\|_Z \leq C \left\| \sum_{k=1}^{m} b(x_k, y_k) \right\|_Z .$$

It is clear that $C \geq 1$.

Conversely, let i) and ii) be satisfied. Consider an arbitrary $z \in Z$. By virtue of b-completeness of the system $\{y_k\}_{k\in N}$ in Z, there exists $x_k^{(n)} \in X$ such that

$$z = \lim_{n\to\infty} \sum_k b(x_k^{(n)}, y_k). \tag{1}$$

For every fixed $k \in N$, consider the sequence $\left\{x_k^{(n)}\right\}_{n\in N}$. For $m > n$ we have

$$\left\| x_k^{(n)} - x_k^{(m)} \right\|_X \leq \frac{1}{a_k} \left\| b(x_k^{(n)} - x_k^{(m)}, y_k) \right\|_Z \leq$$

$$\leq \frac{1}{a_k} \cdot C \left\| \sum_k b(x_k^{(n)}, y_k) - \sum_k b(x_k^{(m)}, y_k) \right\|_Z . \tag{2}$$

From (1) and (2) it follows that $\left\{x_k^{(n)}\right\}_{n\in N}$ is fundamental in X. Let $x_k = \lim\limits_{n\to\infty} x_k^{(n)}$. Then from (1) we obtain $z = \sum_{k=1}^{\infty} b(x_k, y_k)$. To prove the uniqueness of decomposition, it suffices to show that the system $\{y_n\}_{n\in N}$ is b-ω-linearly independent in Z. Let $\sum_{k=1}^{\infty} b(x_k, y_k) = 0$. In view of (1), $\forall i \in N$ we have

$$\|x_i\|_X \leq \frac{1}{a_i} \|b(x_i, y_i)\|_Z \leq \frac{1}{a_i} C \left\| \sum_{k=1}^{n} b(x_k, y_k) \right\|_Z \to 0, n \to \infty,$$

i.e. $x_i = 0$. ◄

Remark 1. *Let the system $\{y_n\}_{n\in N} \subset Y$ form a b-basis for Z and $\|b(x, y_k)\|_Z \geq a_k \|x\|_X$, $a_k > 0$, $\forall k \in N$, $x \in X$. Then $\{y_n\}_{n\in N}$ has a unique b-biorthogonal system $\{y_n^*\}_{n\in N} \subset L(Z, X)$.*

In fact, for $z \in Z$, y_n^* is defined by the formula $y_n^*(z) = x_n$, where $z = \sum_{k=1}^{\infty} b(x_k, y_k)$. The uniqueness of b-biorthogonal system is obvious.

Let $\tilde{X}$ be a Banach space of sequences of vectors in X with coordinatewise linear operations such that the operator $P_k : X \to \tilde{X}$, $P_k(x) = \{\delta_{ik} x\}_{i \in N}$ is bounded and has bounded inverse on ImP_k. We call $\tilde{X}$ a CB-space, if the relation

$$\lim_{n\to\infty} \left\| \{x_k\}_{k\in N} - \sum_{k=1}^{n} \{\delta_{ik} x_k\}_{i\in N} \right\|_{\tilde{X}} = 0,$$

holds for every $\{x_k\}_{k\in N} \in \tilde{X}$. For a CB-space $\tilde{X}$, the conjugate space $\tilde{X}^*$ is isometrically isomorphic to the Banach space

$$\tilde{Y} = \left\{ \{x_k^*\}_{k\in N} \subset X^* : x_k^* = \tilde{x}^* P_k, \tilde{x}^* \in \tilde{X}^* \right\},$$

with the norm $\left\| \{x_k^*\}_{k\in N} \right\|_{\tilde{Y}} = \|\tilde{x}^*\|_{\tilde{X}^*}$, and every linear continuous functional $\tilde{x}^*$ on $\tilde{X}$ is defined by the formula $\tilde{x}^*(\{x_k\}_{k\in N}) = \sum_{k=1}^{\infty} x_k^*(x_k)$, where $x_k^* = \tilde{x}^* P_k$. Therefore, $\tilde{X}^*$ is identified with $\tilde{Y}$.

We call $\tilde{X}$ an RCB-space, if it is a reflexive CB-space.

Remark 2. *Let $\tilde{X}$ be a CB-space and the bilinear mapping $b : X : L(X, \tilde{X}) \to \tilde{X}$ be defined by the formula $b(x, P) = P(x)$ for $x \in X$ and $P \in L(X, \tilde{X})$. Then the system $\{P_k\}_{k\in N}$ forms a b-basis for $\tilde{X}$, and therefore, by Theorem 3, $\exists C \geq 1$ for $m \geq n$:*

$$\left\| \sum_{k=1}^{n} P_k(x_k) \right\|_Z \leq C \left\| \sum_{k=1}^{m} P_k(x_k) \right\|_Z, \tag{3}$$

for any $x_1, x_2, \ldots, x_m \in X$.

3. $\tilde{X}$-Frames in Banach Spaces

In this section, we introduce a concept of $\tilde{X}$-frame in a Banach space and study some of its properties.

Definition 2. *The system $\{g_k\}_{k\in N} \subset L(Z, X)$ is called an $\tilde{X}$-frame in Z, if there exist the constants $A > 0$ and $B > 0$ such that*

$$A \|z\|_Z \leq \left\| \{g_k(z)\}_{k\in N} \right\|_{\tilde{X}} \leq B \|z\|_Z, \forall z \in Z. \tag{4}$$

The constants A and B are called the $\tilde{X}$-frame bounds of $\{g_k\}_{k\in N}$. In case where $\{g_k\}_{k\in N}$ satisfies the inequality on the right of (4), the system $\{g_k\}_{k\in N}$ is called $\tilde{X}$-Besselian in Z with a bound B. If $\{g_k\}_{k\in N}$ is $\tilde{X}$-Besselian in Z, then there exists a bounded operator $U : Z \to \tilde{X}$:

$$U(z) = \{g_k(z)\}_{k\in N}. \tag{5}$$

Definition 3. *The system $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ is called an $\tilde{X}^*$-frame in Z^* with the bounds A and B, if $\{\Lambda_k^*\}_{k\in N}$ forms an $\tilde{X}^*$-frame in Z^* with the bounds A and B, i.e.*

$$A\|f\|_{Z^*} \le \left\|\{\Lambda_k^* f\}_{k\in N}\right\|_{\tilde{X}^*} \le B\|f\|_{Z^*}, \forall f \in Z^*. \tag{6}$$

If the inequality on the right of (6) is satisfied, then the system $\{\Lambda_k\}_{k\in N}$ is called $\tilde{X}^*$-Besselian in Z^* with a bound B. If $\{\Lambda_k\}_{k\in N}$ is $\tilde{X}^*$-Besselian in Z, then there exists a bounded operator $V : \tilde{X} \to Z$:

$$V(f) = \{\Lambda_k^*(f)\}_{k\in N}.$$

Let's state the criterion of $\tilde{X}$-Besselianness.

Theorem 4. *Let $\tilde{X}$ be a CB-space and $\{g_k\}_{k\in N} \subset L(Z,X)$. Then $\{g_k\}_{k\in N}$ is $\tilde{X}$-Besselian in Z with a bound B only when the bounded operator $T : \tilde{X}^* \to Z^*$ is defined:*

$$T(\tilde{x}^*) = \sum_{k=1}^{\infty} x_k^* g_k, \forall \tilde{x}^* = \{x_k^*\}_{k\in N} \in \tilde{X}^*, \tag{7}$$

and $\|T\| \le B$.

Proof. Let $\{g_k\}_{k\in N}$ be $\tilde{X}$-Besselian in Z with a bound B. Let's prove the convergence of the series $\sum_{k=1}^{\infty} x_k^* g_k$, for $\forall \tilde{x}^* = \{x_k^*\}_{k\in N} \in \tilde{X}^*$. For $n > m$ we have

$$\left\|\sum_{k=m}^{n} x_k^* g_k\right\|_{Z^*} = \sup_{\|z\|=1}\left|\sum_{k=m}^{n} x_k^* g_k(z)\right| =$$

$$= \sup_{\|z\|=1}\left|\{x_k^*\}_{k=m}^{n}\left(\{g_k(z)\}_{k\in N}\right)\right| \le B\left\|\sum_{k=m}^{n}\{\delta_{ik}x_k^*\}_{i\in N}\right\|_{\tilde{X}}.$$

Consequently, the series $\sum_{k=1}^{\infty} x_k^* g_k$ is convergent. Therefore, the operator T is defined and $\|T\| \le B$.

Conversely, let $T \in L(\tilde{X}^*, Z^*)$ and $\|T\| \le B$. Then

$$\left\|\{g_k(z)\}_{k\in N}\right\|_{\tilde{X}} = \sup_{\|\{x_k^*\}\|=1}\left|\sum_{k=1}^{\infty} x_k^* g_k(z)\right| \le \|T\|\,\|z\|_Z.$$

◄

The criterion for $\tilde{X}^*$-Besselianness of the system $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ in Z^* is proved in a similar way.

Theorem 5. *Let $\tilde{X}$ be a CB-space and $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$. Then the system $\{\Lambda_k\}_{k\in N}$ is $\tilde{X}^*$-Besselian in Z^* with a bound B only when the bounded operator $D : \tilde{X} \to Z$ is defined:*

$$D(\tilde{x}) = \sum_{k=1}^{\infty} \Lambda_k(x_k), \forall \tilde{x} = \{x_k\}_{k\in N} \in \tilde{X}, \tag{8}$$

and $\|D\| \le B$.

Remark 3. *Let $\tilde{X}$ be a CB-space, $\{g_k\}_{k\in N} \subset L(Z,X)$ be $\tilde{X}$-Besselian in Z and the operators U, T be defined by* (5) *and* (7)*, respectively. Then $T = U^*$. Hence, if Z is reflexive, then $T^* = U$.*

In fact, $\forall \tilde{x}^* \in \tilde{X}^*$ and $\forall z \in Z$ we have

$$\tilde{x}^*(U(z)) = \sum_{k=1}^{\infty} x_k^* g_k(z) = T(\tilde{x}^*)(z),$$

i.e. $T = U^*$.

From Theorems 4 and 5 we immediately obtain

Corollary 1. *Let $\tilde{X}$ be a CB-space and $\{g_k\}_{k\in N} \subset L(Z,X)$. Then $\{g_k\}_{k\in N}$ is $\tilde{X}$-Besselian in Z with a bound B only when the inequality*

$$\left\| \sum_k x_k^* g_k \right\|_{Z^*} \le B \left\| \{x_k^*\} \right\|_{\tilde{X}^*},$$

holds for every finite sequence $\{x_k^*\} \subset X^*$.

Corollary 2. *Let $\tilde{X}$ be a CB-space and $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$. Then the system $\{\Lambda_k\}_{k\in N}$ is $\tilde{X}^*$-Besselian in Z^* with a bound B only when the inequality*

$$\left\| \sum_k \Lambda_k(x_k) \right\|_{Z} \le B \left\| \{x_k\} \right\|_{\tilde{X}},$$

holds for every finite sequence $\{x_k\} \subset X$.

The next theorem presents the criterion of $\tilde{X}$-frameness.

Theorem 6. *Let $\tilde{X}$ be an RCB-space, and $\{g_k\}_{k\in N} \subset L(Z,X)$. Then the system $\{g_k\}_{k\in N}$ forms an $\tilde{X}$-frame in Z only when the operator $T : \tilde{X}^* \to Z^*$, defined by the formula* (7)*, is a bounded, surjective linear operator.*

Proof. Let $\{g_k\}_{k\in N}$ form an $\tilde{X}$-frame in Z with the bounds A and B. Then it is clear that the operator $T : \tilde{X}^* \to Z^*$ is defined and $\|T\| \le B$. As $\{g_k\}_{k\in N}$ is an $\tilde{X}$-frame in Z, the operator U defined by the formula (5) maps Z isomorphically to ImU. On the other hand, ImU, as a subspace of the reflexive space $\tilde{X}$, is reflexive. Therefore, Z is also reflexive. By Remark 3, we obtain $U^* = T$. Then, by Theorem 1, the operator T maps $\tilde{X}^*$ into the whole of Z^*.

Conversely, let the operator T map $\tilde{X}^*$ boundedly to Z^*. Then $\{g_k\}_{k\in N}$ is $\tilde{X}$-Besselian in Z. As $U^* = T$, U^* maps $\tilde{X}^*$ to Z^* and $\exists A > 0$: $\|U(z)\|_{\tilde{X}} \geq A\,\|z\|_Z$, i.e. $\{g_k\}_{k\in N}$ forms an $\tilde{X}$-frame in Z. ◄

The following criterion of $\tilde{X}^*$-frameness of the system $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ in Z^* is proved in a similar way.

Theorem 7. *Let $\tilde{X}$ be an RCB-space, and $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$. Then the system $\{\Lambda_k\}_{k\in N}$ forms an $\tilde{X}^*$-frame in Z^* only when the operator $D : \tilde{X} \to Z$, defined by the formula* (8)*, is a bounded, surjective linear operator.*

4. $\tilde{X}$-Riesz Bases in Banach Spaces

In this section, we consider a bilinear mapping $b(\cdot,\cdot)$, used to introduce the concept of $\tilde{X}$-Riesz b-basis in a Banach space. We also study the conditions of $\tilde{X}$-Riesz b-basicity.

Let X, Y and Z be B-spaces with the corresponding norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$. Define the bilinear mappings $b : X \times L(X,Z) \to Z$ and $b^* : X^* \times L(Z,X) \to Z^*$ as follows:

$$b(x,\Lambda) = \Lambda(x), \forall x \in X, \forall \Lambda \in L(X,Z),$$

$$b^*(x^*, g) = x^* g, \forall x^* \in X^*, \forall g \in L(Z,X).$$

Definition 4. *The system $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ is called an $\tilde{X}$-Riesz b-basis for Z, if $\{\Lambda_k\}_{k\in N}$ is b-complete in Z and there exist the constants $A > 0$ and $B > 0$ such that*

$$A\,\|\tilde{x}\|_{\tilde{X}} \leq \left\|\sum_{k=1}^{\infty} \Lambda_k(x_k)\right\|_Z \leq B\,\|\tilde{x}\|_{\tilde{X}}\,, \forall \tilde{x} \in \tilde{X}. \tag{9}$$

The constants A and B are called the lower and the upper bound of an $\tilde{X}$-Riesz b-basis $\{\Lambda_k\}_{k\in N}$, respectively.

Definition 5. *The system $\{g_k\}_{k\in N} \subset L(Z,X)$ is called an $\tilde{X}^*$-Riesz b^*-basis for Z^*, if $\{g_k\}_{k\in N}$ is b^*-complete in Z^* and there exist the constants $A > 0$ and $B > 0$ such that*

$$A\,\|\tilde{x}^*\|_{\tilde{X}^*} \leq \left\|\sum_{k=1}^{\infty} x_k^* g_k\right\|_{Z^*} \leq B\,\|\tilde{x}^*\|_{\tilde{X}^*}\,, \forall \tilde{x}^* \in \tilde{X}^*. \tag{10}$$

The constants A and B are called the lower and the upper bound of an $\tilde{X}^*$-Riesz b^*-basis $\{g_k\}_{k\in N}$, respectively.

The following theorem of b-basicity for $\tilde{X}$-Riesz b-bases is true.

Theorem 8. *Let $\tilde{X}$ be a CB-space, the system $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ form an $\tilde{X}$-Riesz b-basis for Z and $\|\Lambda_k(x)\|_Z \geq a_k \|x\|_X$, $a_k > 0$, $\forall x \in X$. Then $\{\Lambda_k\}_{k\in N}$ forms a b-basis for Z.*

Proof. Let $\{\Lambda_k\}_{k\in N}$ be an $\tilde{X}$-Riesz b-basis for Z. Using (9) and (3), for $m \geq n$ we obtain

$$\left\|\sum_{k=1}^{n} \Lambda_k(x_k)\right\|_Z \leq B \left\|\{x_k\}_1^n\right\|_{\tilde{X}} = B \left\|\sum_{k=1}^{n} P_k(x_k)\right\|_{\tilde{X}} \leq BC \left\|\sum_{k=1}^{m} P_k(x_k)\right\|_{\tilde{X}} =$$

$$= BC \left\|\{x_k\}_1^m\right\|_{\tilde{X}} \leq \frac{BC}{A} \left\|\sum_{k=1}^{m} \Lambda_k(x_k)\right\|_Z .$$

As $\{\Lambda_k\}_{k\in N}$ is b-complete in Z, by Theorem 3 ,$\{\Lambda_k\}_{k\in N}$ forms a b-basis for Z. ◀

The following necessary condition for $\tilde{X}^*$-Riesz b^*-basicity in Z^* is proved in a similar way.

Theorem 9. *Let $\tilde{X}$ be an RCB-space, the system $\{g_k\}_{k\in N} \subset L(Z,X)$ form an $\tilde{X}^*$-Riesz b^*-basis for Z^* and $\|g_k(z)\|_X \geq a_k \|z\|_Z$, $a_k > 0$, $\forall z \in Z$. Then $\{g_k\}_{k\in N}$ forms a b^*-basis for Z^*.*

Let's state a criterion of $\tilde{X}$-Riesz b-basicity.

Theorem 10. *Let $\tilde{X}$ be a CB-space, $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$, $\|\Lambda_k(x)\|_Z \geq a_k \|x\|_X$, $a_k > 0$, $\forall x \in X$. Then the system $\{\Lambda_k\}_{k\in N}$ is an $\tilde{X}$-Riesz b-basis for Z only when the operator $D : \tilde{X} \to Z$, defined by the formula (8), maps $\tilde{X}$ isomorphically to Z.*

Proof. Let $\{\Lambda_k\}_{k\in N}$ form an $\tilde{X}$-Riesz b-basis for Z. From (9) it follows that $D \in L(\tilde{X}, Z)$ and $A\|\tilde{x}\|_{\tilde{X}} \leq \|D(\tilde{x})\|_Z \leq B\|\tilde{x}\|_{\tilde{X}}$. By Theorem 8, the system $\{\Lambda_k\}_{k\in N}$ is a b-basis for Z, and therefore $ImD = Z$. Thus, the operator D is boundedly invertible.

Conversely, let the operator D map $\tilde{X}$ isomorphically to Z. Then it is clear that the relation (9) is fulfilled. $\forall x \in X$ we have

$$DP_k(x) = D(\{\delta_{ik}x\}_{i\in N}) = \Lambda_k(x).$$

Let's show that $\{\Lambda_k\}_{k\in N}$ is b-complete. Assume the contrary. Then $\exists f \in Z^*$, $f \neq 0$ and $f(\sum_{k=}^{\infty} \Lambda_k(x_k)) = 0$, $\forall \tilde{x} \in \tilde{X}$. Consequently

$$0 = f(\sum_{k=1}^{\infty} DP_k(x_k)) = fD(\sum_{k=1}^{\infty} P_k(x_k)) = fD(\tilde{x}),$$

and therefore, due to the arbitrariness of $\tilde{x}$, we obtain $fD = 0$. Hence, the invertibility of D implies $f = 0$, which contradicts our assumption. So $\{\Lambda_k\}_{k\in N}$ is b-complete in Z. ◀

The following criterion of $\tilde{X}^*$-Riesz b^*-basicity in Z^* is proved in a similar way.

Theorem 11. *Let $\tilde{X}$ be a RCB-space, $\{g_k\}_{k\in N} \subset L(Z,X)$ and $\|g_k(z)\|_X \geq a_k \|z\|_Z$, $a_k > 0$, $\forall z \in Z$. Then the system $\{g_k\}_{k\in N}$ forms an $\tilde{X}^*$-Riesz b-basis for Z^* only when the operator $T : \tilde{X}^* \to Z^*$, defined by the formula (7), maps $\tilde{X}^*$ isomorphically to Z^*.*

The next theorem presents the $\tilde{X}$-frameness of an $\tilde{X}^*$-Riesz b^*-basis.

Theorem 12. *Let $\tilde{X}$ be an RCB-space, the system $\{g_k\}_{k\in N} \subset L(Z,X)$ form an $\tilde{X}^*$-Riesz b^*-basis for Z^* with the bounds A, B and $\|g_k(z)\|_X \geq a_k \|z\|_Z$, $a_k > 0$, $\forall z \in Z$. Then $\{g_k\}_{k\in N}$ is an $\tilde{X}$-frame in Z with the bounds A, B and $ImU = \tilde{X}$.*

Proof. From Theorem 11 it follows that the operator T maps $\tilde{X}^*$ isomorphically to Z^*. Then, by Theorem 6, the system $\{g_k\}_{k\in N}$ is an $\tilde{X}$-frame in Z. Due to the reflexivity of Z and $\tilde{X}$, we have $T^* = U$. As $ImT^* = \tilde{X}$, we have $ImU = \tilde{X}$. Let's show that $\{g_k\}_{k\in N}$ has bounds A and B. Obviously, $B = \|T\|$ and $A = \left\|T^{-1}\right\|^{-1}$. As $\{g_k\}_{k\in N}$ has bounds $\left\|U^{-1}\right\|^{-1}$ and $\|U\|$, we have

$$\left\|U^{-1}\right\|^{-1} = \left\|(T^*)^{-1}\right\|^{-1} = \left\|T^{-1}\right\|^{-1} = A,$$

$$\|U\| = \|T^*\| = \|T\| = B.$$

◄

The next theorem can be proved in a manner similar to the way Theorem 12 was proved.

Theorem 13. *Let $\tilde{X}$ be an RCB-space, the system $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ form an $\tilde{X}$-Riesz b-basis for Z with the bounds A, B and $\|\Lambda_k(x)\|_Z \geq a_k \|x\|_X$, $a_k > 0$, $\forall x \in X$. Then $\{\Lambda_k\}_{k\in N}$ is an $\tilde{X}^*$-frame in Z^* with the bounds A, B and $ImV = \tilde{X}^*$.*

Now let's find the conditions under which an $\tilde{X}^*$-frame is an $\tilde{X}$-Riesz b-basis.

Theorem 14. *Let $\tilde{X}$ be an RCB-space, the system $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ form an $\tilde{X}^*$-frame in Z^* and $\|\Lambda_k(x)\|_Z \geq a_k \|x\|_X$, $a_k > 0$, $\forall x \in X$. Then the following conditions are equivalent:*

i) $\{\Lambda_k\}_{k\in N}$ forms an $\tilde{X}$-Riesz b-basis for Z;

ii) $\{\Lambda_k\}_{k\in N}$ forms a b-basis for Z;

iii) $\{\Lambda_k\}_{k\in N}$ is b-ω-linearly independent with respect to $\tilde{X}$;

iv) $ImV = \tilde{X}^*$;

v) $\{\Lambda_k\}_{k\in N}$ has a unique b-biorthogonal system $\{g_k\}_{k\in N} \subset L(Z,X)$.

Proof. i) $\Rightarrow$ ii) follows from Theorem 8.

i) $\Rightarrow$ iii) follows from the inequality on the left of (9).

i) $\Rightarrow$ iv) follows from Theorem 13.

ii), iii) $\Rightarrow$ i). As $\{\Lambda_k\}_{k\in N}$ is an $\tilde{X}^*$-frame in Z^*, by Theorem 7 the operator D maps $\tilde{X}$ into the whole of Z. From ii) or iii) it follows that D maps $\tilde{X}$ isomorphically to Z. Then, by Theorem 10, the condition i) is fulfilled.

iv)$\Rightarrow$ i). Then V maps Z^* isomorphically to $\tilde{X}^*$. As $D = V^*$, the operator D maps $\tilde{X}$ isomorphically to Z. Consequently, by Theorem 10, the condition i) is fulfilled.

ii) $\Rightarrow$ v). By Remark 1, the system $\{\Lambda_k\}_{k\in N}$ has a unique b-biorthogonal system $\{g_k\}_{k\in N} \subset L(Z, X)$:

$$g_k(\Lambda_i(x)) = \delta_{ki}x, \forall x \in X.$$

v) $\Rightarrow$ iv). Let $\{g_k\}_{k\in N} \subset L(Z, X)$ be a system b-biorthogonal to $\{\Lambda_k\}_{k\in N}$. It is easy to show that

$$\Lambda_k^*(x^* g_k)) = \delta_{ki}x^*, \forall x^* \in X^*.$$

As $\{\Lambda_k\}_{k\in N}$ is an $\tilde{X}^*$-frame in Z, ImV is a subspace of $\tilde{X}^*$. $\forall \tilde{x}^* \in \tilde{X}^*$ we have

$$\tilde{x}^* = \sum_{k=1}^{\infty} \{\delta_{ik}x_k^*\}_{i\in N} = \sum_{k=1}^{\infty} \{\Lambda_i^*(x_k^* g_k)\}_{i\in N} = \sum_{k=1}^{\infty} V(x_k^* g_k) = VT(\tilde{x}^*),$$

i.e. $\tilde{x}^* \in ImV$.◄

Let's prove the same assertion for an $\tilde{X}$-frame.

Theorem 15. *Let $\tilde{X}$ be an RCB-space, the system $\{g_k\}_{k\in N} \subset L(Z, X)$ form an $\tilde{X}$-frame in Z and $\|g_k(z)\|_X \geq a_k \|z\|_Z$, $a_k > 0$, $\forall z \in Z$. Then the following conditions are equivalent:*

i) $\{g_k\}_{k\in N}$ forms an $\tilde{X}^*$-Riesz b^*-basis for Z^*;

ii) $\{g_k\}_{k\in N}$ forms a b^*-basis for Z^*;

iii) $\{g_k\}_{k\in N}$ is b^*-ω-linearly independent with respect to $\tilde{X}^*$;

iv) $ImU = \tilde{X}$;

v) $\{g_k\}_{k\in N}$ has a unique b^*-biorthogonal system $\{\Lambda_k\}_{k\in N} \subset L(X, Z)$.

Proof. i) $\Rightarrow$ ii) follows from Theorem 9.

i) $\Rightarrow$ iii) follows from the inequality on the left of (10).

i) $\Rightarrow$ iv) follows from Theorem 12.

Now suppose that ii) or iii) is fulfilled. As $\{g_k\}_{k\in N}$ is an $\tilde{X}$-frame in Z, by Theorem 6 the operator T maps $\tilde{X}^*$ into the whole of Z^*. From ii) or iii) it follows that T maps $\tilde{X}^*$ isomorphically to Z^*. Then, by Theorem 11, the condition i) is fulfilled.

Let's show the validity of iv)$\Rightarrow$ i). Then U maps Z isomorphically to $\tilde{X}$. As $T = U^*$, the operator T maps $\tilde{X}^*$ isomorphically to Z^*. Consequently, by Theorem 11, the condition i) is fulfilled.

ii)$\Rightarrow$v). Then the system $\{g_k\}_{k\in N}$ has a unique b^*-biorthogonal system $\{\Lambda_k\}_{k\in N} \subset L(X, Z)$:

$$g_k(\Lambda_i(x)) = \delta_{ki}x, \forall x \in X.$$

Let's show the validity of v) $\Rightarrow$ iv). As $\{g_k\}_{k\in N}$ is an $\tilde{X}$-frame in Z, ImU is a subspace of $\tilde{X}$. $\forall \tilde{x} \in \tilde{X}$ we have

$$\tilde{x} = \sum_{k=1}^{\infty} \{\delta_{ik} x_k\}_{i\in N} = \sum_{k=1}^{\infty} \{g_i(\Lambda_k(x_k))\}_{i\in N} = \sum_{k=1}^{\infty} U(\Lambda_k(x_k)) = UD(\tilde{x}),$$

i.e. $\tilde{x} \in ImU$.◄

Now we proceed to study the matter of decomposition in the spaces Z and Z^*.

Theorem 16. *Let $\tilde{X}$ be an RCB-space, the system $\{g_k\}_{k\in N} \subset L(Z, X)$ form an $\tilde{X}^*$-Riesz b^*-basis for Z^* with the bounds A, B and $\|g_k(z)\|_X \geq a_k \|z\|_Z$, $a_k > 0$, $\forall z \in Z$. Then there exists a unique $\tilde{X}$-Riesz b-basis $\{\Lambda_k\}_{k\in N} \subset L(X, Z)$ for Z with the bounds $\frac{1}{B}$ and $\frac{1}{A}$ such that*

$$z = \sum_{k=1}^{\infty} \Lambda_k(g_k(z)) \text{ for every } z \in Z, \tag{11}$$

$$f = \sum_{k=1}^{\infty} f\Lambda_k g_k, \forall f \in Z^*, \tag{12}$$

where $\{\Lambda_k\}_{k\in N}$ is a unique system b-biorthogonal to $\{g_k\}_{k\in N}$.

Proof. By Theorem 12, the system $\{g_k\}_{k\in N}$ is an $\tilde{X}$-frame in Z with the bounds A and B, the operator U is boundedly invertible and $\|U^{-1}\| \leq \frac{1}{A}$. Let $\Lambda_k = U^{-1}P_k$. Obviously, $\{\Lambda_k\}_{k\in N} \subset L(X, Z)$. We have

$$g_k(\Lambda_i(x)) = \delta_{ki}x,$$

because if $\Lambda_i(x) = z$, then $P_k(x) = U(z)$. Hence $g_k(z) = \delta_{ki}x$. Further,

$$z = U^{-1}U(z) = U^{-1}(\sum_{k=1}^{\infty} P_k g_k(z)) = \sum_{k=1}^{\infty} \Lambda_k(g_k(z)).$$

The validity of (12) follows from (11). As $ImD = Z$, by Theorem 7 the system $\{\Lambda_k\}_{k\in N}$ is an $\tilde{X}^*$-frame in Z^*. We have

$$\left\|\{\Lambda_k^*(f)\}_{k\in N}\right\|_{\tilde{X}^*} = \sup_{\|\tilde{x}\|=1} \left|\sum_{k=1}^{\infty} f\Lambda_k(x_k)\right| = \sup_{\|\tilde{x}\|=1} \left|f(U^{-1}(\tilde{x}))\right| = \left\|(U^*)^{-1}f\right\|_{\tilde{X}^*}.$$

Then

$$\left\|\{\Lambda_k^* f\}_{k\in N}\right\|_{\tilde{X}^*} \leq \left\|U^{-1}\right\| \|f\|_{Z^*} \leq \frac{1}{A} \|f\|,$$

$$\left\|\{\Lambda_k^* f\}_{k\in N}\right\|_{\tilde{X}^*} \geq \|U\|^{-1} \|f\| \geq \frac{1}{B} \|f\|.$$

The system $\{\Lambda_k\}_{k\in N}$ is b-ω-linearly independent with respect to $\tilde{X}$. In fact, if $\sum_{k=1}^{\infty}\Lambda_k(x_k)=0$, then $\exists z$: $U(z)=\tilde{x}$, i.e. $g_k(z)=x_k$. Therefore, from (11) we have

$$z=\sum_{k=1}^{\infty}\Lambda_k(g_k(z))=0,$$

hence $\tilde{x}=0$. By Theorem 14, the system $\{\Lambda_k\}_{k\in N}$ forms an $\tilde{X}$-Riesz b-basis for Z with the bounds $\frac{1}{B}$ and $\frac{1}{A}$. As $\{\Lambda_k\}_{k\in N}$ forms a b-basis for Z, it is a unique system b-biorthogonal to $\{g_k\}_{k\in N}$. ◄

The next theorem presents a characteristic property of $\tilde{X}^*$-Riesz b^*-bases.

Theorem 17. *Let $\tilde{X}$ be an RCB-space, and $\{g_k\}_{k\in N}\subset L(Z,X)$, $\|g_k(z)\|_X\geq a_k\|z\|_Z$, $a_k>0$, $\forall z\in Z$. Then the following conditions are equivalent:*
i) $\{g_k\}_{k\in N}$ forms an $\tilde{X}^*$-Riesz b^*-basis for Z^*;
ii) $\{g_k\}_{k\in N}$ is b^*-complete in Z^* and $\tilde{X}$-Besselian in Z, there exists a system $\{\Lambda_k\}_{k\in N}$ which is $\tilde{X}^*$-Besselian in Z^* and b-biorthogonal to $\{g_k\}_{k\in N}$.

Proof. Let's show the validity of i)⇒ii). By Theorem 12, the system $\{g_k\}_{k\in N}$ is an $\tilde{X}$-frame in Z. Then, according to Theorems 15 and 16, there exists an $\tilde{X}$-Riesz b-basis $\{\Lambda_k\}_{k\in N}$, which is biorthogonal to $\{g_k\}_{k\in N}$. Applying Theorem 12, we obtain that the system $\{\Lambda_k\}_{k\in N}$ is $\tilde{X}^*$-Besselian in Z^*.

Conversely, suppose that ii) is true. Then, by Theorems 4 and 5, the following bounded operators are defined by (7) and (8). We have

$$T(\tilde{x}^*)(D(\tilde{x}))=\sum_{k=1}^{\infty}x_k^*\left(\sum_{i=1}^{\infty}g_k(\Lambda_i x_i)\right)=\sum_{k=1}^{\infty}x_k^*(x_k)=\tilde{x}^*(\tilde{x}).$$

Then

$$\|\tilde{x}^*\|_{\tilde{X}}=\sup_{\|\tilde{x}\|=1}|\tilde{x}^*(\tilde{x})|=\sup_{\|\tilde{x}\|=1}|T(\tilde{x}^*)(D(\tilde{x}))|\leq\|T(\tilde{x}^*)\|\,\|D\|.$$

Thus

$$\|D\|^{-1}\|\tilde{x}^*\|_{\tilde{X}^*}\leq\|T(\tilde{x}^*)\|_{Z^*}\leq\|T\|\,\|\tilde{x}^*\|_{\tilde{X}^*},$$

i.e. the relation (10) is true. ◄

Now let's give a non-trivial example related to the matters discussed in this work.

Example 1. *Let $p>1$, the system $\{\varphi_n(t)\}_{n\in N}$ be such that $|\varphi_n(t)|\leq M$ a.e. in $[0;1]$ and $\lambda_{n_0}\neq 0$, where $\lambda_n=\inf\limits_{t\in[0;1]}vrai\,|\varphi_n(t)|$. Denote by $l_{p,2}(0;1)$ the set of sequences $a(t)=\{a_n(t)\}_{n\in N}\subset L_p(0;1)$ such that*

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1|a_n(t)|^p\,dt<+\infty.$$

Equipped with the norm $\|a\|_{l_{p,2}(0,1)} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 |a_n(t)|^p \, dt\right)^{\frac{1}{p}}$*, the set* $l_{p,2}(0;1)$ *becomes a Banach space. A space conjugate to* $l_{p,2}(0;1)$ *is isometric and isomorphic to* $l_{q,2}(0,1)$*, with* $\frac{1}{p} + \frac{1}{q} = 1$*. Besides,*

$$b(a) = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 a_n(t) b_n(t) dt, b(t) = \{b_n(t)\}_{n \in N} \in l_{q,2}(0;1),$$

is a linear continuous functional on $l_{p,2}(0;1)$*. Consider* $\Lambda_n : L_p(0;1) \to L_p(0;1)$ *defined by the formula* $\Lambda_n(x)(t) = x(t)\varphi_n(t)$*. It is clear that*

$$\lambda_n \|x\|_{L_p} \leq \|\Lambda_n(x)\|_{L_p} \leq M \|x\|_{L_p}$$

and

$$\Lambda_n^*(y)(t) = y(t)\varphi_n(t), y(t) \in L_q(0;1).$$

Then $\{\Lambda_n\}_{n \in N}$ *forms an* $l_{q,2}(0;1)$*-frame and*

$$\left(\sum_{n=1}^{\infty} \frac{\lambda_n^q}{n^2}\right)^{\frac{1}{q}} \|y\|_{L_q(0;1)} \leq \left\|\{\Lambda_n^*(y)\}_{n \in N}\right\|_{l_{p,2}(0;1)} \leq M \left(\frac{\pi^2}{6}\right)^{\frac{1}{q}} \|y\|_{L_q(0;1)}, \forall y(t) \in L_q(0;1).$$

In fact

$$\left\|\{\Lambda_n^*(y)\}_{n \in N}\right\|_{l_{q,2}(0;1)} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \|\Lambda_n^*(y)\|_{L_q(0;1)}^q\right)^{\frac{1}{q}},$$

hence

$$\left(\sum_{n=1}^{\infty} \frac{\lambda_n^q}{n^2}\right)^{\frac{1}{q}} \|y\|_{L_q(0;1)} \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \|\Lambda_n^*(y)\|_{L_q(0;1)}^q\right)^{\frac{1}{q}} \leq M \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{\frac{1}{q}} \|y\|_{L_q(0;1)}.$$

If $\lambda_n > 0$*, then Theorem 14 implies the equivalence of the following conditions:*

a) $\forall x(t) \in L_p(0;1)$ *is uniquely representable in the form* $x(t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(t)$*;*

b) the totality of all possible finite sums of the form $\sum_n a_n(t)\varphi_n(t), a_n(t) \in L_p(0;1)$*, is dense in* $L_p(0;1)$ *and* $\exists A > 0, B > 0:$

$$A\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 |a_n(t)|^p \, dt\right)^{\frac{1}{p}} \leq \left(\int_0^1 \left|\sum_{n=1}^{\infty} a_n(t)\varphi_n(t)\right|^p dt\right)^{\frac{1}{p}} \leq B\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 |a_n(t)|^p \, dt\right)^{\frac{1}{p}}.$$

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