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Approximation with an Arbitrary Order by Szász, Szász-Kantorovich and Baskakov Complex Operators in Compact Disks

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Abstract. By using a sequence $\lambda_n > 0$, $n \in \mathbb{N}$ with the property $\lambda_n \to 0$ as fast as we want, in this paper we obtain the approximation order $O(\lambda_n)$ for some generalized Szász, Szász-Kantorovich, and Baskakov complex operators attached to entire functions or to analytic functions of exponential growth in compact disks which do not involve the values on $[0, +\infty)$.

Key Words and Phrases: generalized Szász, Szász-Kantorovich and Baskakov complex operators, order of approximation, Voronovskaja-type results.
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1. Introduction

In [2], with the notations there for two sequences a_n and b_n, $n \in \mathbb{N}$, and denoting here $\lambda_n = \frac{b_n}{a_n}$, the authors introduced the generalized complex Szász operator by

$$S_n(f; \lambda_n)(z) = e^{-z/\lambda_n} \sum_{j=0}^{\infty} \frac{(z/\lambda_n)^j}{j!} f(j\lambda_n), \tag{1}$$

where $\lambda_n > 0$, $\lambda_n \to 0$.

For this operator, attached to functions $f : \overline{\mathbb{D}_R} \bigcup [R, +\infty) \to \mathbb{C}$ of exponential growth in $\overline{\mathbb{D}_R} \bigcup [R, +\infty)$, analytic in the disk $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$, $R > 1$ and continuous on $[0, +\infty)$, the exact order of approximation $O(\lambda_n)$ is obtained in [2]. Also, in the same paper a Voronovskaja-type result with an upper estimate of order $O(\lambda_n^2)$ is proved.

The first goal of the present paper is to extend, in Section 2, the results in [2] to the case of entire functions and then to a kind od Szász operator which does not involve the values of f on $[0, +\infty)$. Also, a complex operator of Szász-Kantorovich type is introduced, for which similar results are proved, essentially improving the order of approximation $O(1/n)$ obtained in [9].

The second goal is to introduce, in Section 3, generalized complex Baskakov operators, for which results similar to those obtained in Section 2 are proved.

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2. Generalized Szász and Szász-Kantorovich Complex Operators

In the case of complex Szász operator, we can prove the following result.

Theorem 1. *Let $\lambda_n > 0$, $n \in \mathbb{N}$ be such that $\lambda_n \to 0$ as fast as we want. Let $f : \mathbb{D}_R \to \mathbb{C}$, $1 < R \leq +\infty$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Suppose that there exist $M > 0$ and $A \in (1/R, 1)$, such that $|c_k| \leq M\frac{A^k}{k!}$, for all $k = 0, 1, ...,$ (which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in \mathbb{D}_R$). Consider $1 \leq r < \frac{1}{A}$.*

(i) If $R = +\infty$, ($1/R = 0$), i.e. f is an entire function, then $S_n(f;\lambda_n)(z)$ is an entire function, for all $z \in \mathbb{C}$, $n \in \mathbb{N}$ we have $S_n(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k S_n(e_k;\lambda_n)(z)$ and for all $|z| \leq r$ the following estimates hold

$$|S_n(f;\lambda_n)(z) - f(z)| \leq C_{r,M,A}\ \lambda_n,$$

$$|S_n^{(p)}(f;\lambda_n)(z) - f^{(p)}(z)| \leq \frac{p! r_1\, C_{r_1,M,A}}{(r_1 - r} \lambda_n,$$

$$\left|S_n(f;\lambda_n)(z) - f(z) - \frac{\lambda_n}{2} z f''(z)\right| \leq M_r(f)(z)\, \lambda_n^2 \leq C_r(f)\, \lambda_n^2,$$

$$\|S_n^{(p)}(f;\lambda_n) - f^{(p)}\|_r \sim \lambda_n,$$

the last equivalence holding if f is not a polynomial of degree $\leq p \in \mathbb{N}$ and the constants in the equivalence depend on f, r, p.

Above, $C_{r,M,A} = \frac{M}{2r}\sum_{k=2}^{\infty}(k+1)(rA)^k < \infty$, $p \in \mathbb{N}$, $1 \leq r < r_1 < \frac{1}{A}$, $M_r(f)(z) = \frac{3MA|z|}{r^2}\sum_{k=2}^{\infty}(k+1)(rA)^{k-1} < \infty$, $C_r(f) = \frac{3MA}{r}\sum_{k=2}^{\infty}(k+1)(rA)^{k-1}$ and $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$.

(ii) If $R < +\infty$, then the complex approximation operator

$$S_n^*(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k\, S_n(e_k;\lambda_n)(z), z \in \overline{\mathbb{D}_r},$$

is well-defined and $S_n^*(f;\lambda_n)(z)$ satisfies all the estimates in (i), for all $1 \leq r < \frac{1}{A} < R$.

Proof. (i) We have

$$|S_n(f;\lambda_n)(z)| \leq |e^{-z/\lambda_n}| \sum_{j=0}^{\infty} \frac{(|z|/\lambda_n)^j}{j!} |f(j\lambda_n)| \leq$$

$$\leq M\, |e^{-z/\lambda_n}| \sum_{j=0}^{\infty} \frac{(|z|/\lambda_n)^j}{j!} e^{Aj\lambda_n} = M\, |e^{-z/\lambda_n}| \sum_{j=0}^{\infty} \frac{(e^{A\lambda_n}|z|/\lambda_n)^j}{j!} e^{Aj\lambda_n} =$$

$$= M\, |e^{-z/\lambda_n}|\, e^{e^{A\lambda_n|z|/\lambda_n}} < +\infty,$$

for all $z \in \mathbb{C}$, which shows that $S_n(f;\lambda_n)$ is an entire function.

We can write

$$S_n(f;\lambda_n)(z) = e^{-z/\lambda_n} \sum_{j=0}^{\infty} \frac{(z/\lambda_n)^j}{j!} \left[\sum_{k=0}^{\infty} c_k\,(j\lambda_n)^k\right].$$

If the above two infinite sums would commute, then we would obtain

$$S_n(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k \left[e^{-z/\lambda_n} \sum_{j=0}^{\infty} \frac{(z/\lambda_n)^j}{j!}\,(j\lambda_n)^k\right] = \sum_{k=0}^{\infty} c_k S_n(e_k;\lambda_n)(z).$$

It is a well-known Fubini type result that a sufficient condition for the commutativity of two infinite sums, i.e for $\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{k,j} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{k,j}$, is that $\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |a_{k,j}| < +\infty$.

Applied to our case, the last condition becomes

$$|e^{-z/\lambda_n}| \sum_{j=0}^{\infty} \frac{(|z|/\lambda_n)^j}{j!} \sum_{k=0}^{\infty} |c_k|\,(j\lambda_n)^k \le M|e^{-z/\lambda_n}| \sum_{j=0}^{\infty} \frac{(|z|/\lambda_n)^j}{j!} \sum_{k=0}^{\infty} \frac{(Aj\lambda_n)^k}{k!} =$$

$$= M|e^{-z/\lambda_n}| \sum_{j=0}^{\infty} \frac{(|z|/\lambda_n)^j}{j!} e^{Aj\lambda_n} = M|e^{-z/\lambda_n}|\, e^{e^{A\lambda_n}|z|/\lambda_n} < \infty,$$

for all $z \in \mathbb{C}$, which shows

$$S_n(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k S_n(e_k;\lambda_n)(z), z \in \mathbb{C}.$$

Now, the first estimate is immediate from Theorem 3, (i) in [2], the second estimate is immediate from Theorem 3, (ii) in [2], the third estimate is immediate from Theorem 4 in [2] and the fourth one is immediate from Theorem 7 in [2], by taking in all these results $\frac{b_n}{a_n} := \lambda_n$.

(ii) $S_n^*(f;\lambda_n)(z)$ is well defined for all $z \in \mathbb{D}_R$ (i.e. for all $|z| \le r$ with $r < R$), $n \in \mathbb{N}$, because

$$|S_n^*(f;\lambda_n)(z)| \le \sum_{k=0}^{\infty} |c_k|\,|S_n(e_k;\lambda_n)(z)| \le M \sum_{k=0}^{\infty} \frac{A^k}{k!}\,|S_n(e_k;\lambda_n)(z)|.$$

On the other hand, by the inequality (6) in the proof of Theorem 3, (i) in [2] (denoting there $\frac{b_n}{a_n} = \lambda_n$), we obtain

$$|S_n(e_k;\lambda_n)(z)| \le |S_n(e_k;\lambda_n)(z) - e_k(z)| + |e_k(z)| \le \frac{(k+1)!}{2} r^{k-1}\lambda_n + r^k,$$

which, taken into account above, leads to

$$|S_n^*(f;\lambda_n)(z)| \le M \sum_{k=0}^{\infty} \frac{A^k}{k!}\,\frac{(k+1)!}{2}\,r^{k-1}\,\lambda_n + M \sum_{k=0}^{\infty} \frac{A^k}{k!} r^k =$$

$$= \frac{M}{2r}\lambda_n \sum_{k=0}^{\infty}(k+1)(Ar)^k + Me^{Ar} < \infty.$$

Finally, the estimates in this case follow immediately from the same theorems mentioned in the proof of (i). ◄

Define the generalized complex Szász-Kantorovich operator by the formula

$$K_n(f;\lambda_n)(z) = e^{-z/\lambda_n} \sum_{j=0}^{\infty} \frac{(z/\lambda_n)^j}{j!} \frac{1}{\lambda_n} \int_{j\lambda_n}^{(j+1)\lambda_n} f(v)dv =$$

$$= e^{-z/\lambda_n} \sum_{j=0}^{\infty} \frac{(z/\lambda_n)^j}{j!} \int_0^1 f\left((t+j)\lambda_n\right) dt.$$

Denote $F(z) = \int_0^z f(t)dt$. Then simple calculations provide the formula (under the hypothesis that the series $S_n(F;\lambda_n)(z)$ is uniformly convergent)

$$K_n(f;\lambda_n)(z) = S_n'(F;\lambda_n)(z). \tag{2}$$

We can prove the following results.

Theorem 2. *Let $\lambda_n > 0$, $n \in \mathbb{N}$ be such that $\lambda_n \to 0$ as fast as we want. Let $f : \mathbb{D}_R \to \mathbb{C}$, $1 < R \leq +\infty$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Suppose that there exist $M > 0$ and $A \in (1/R, 1)$, such that $|c_k| \leq M\frac{A^k}{k!}$, for all $k = 0, 1, ...,$ (which implies $|f(z)| \leq Me^{A|z|}$ for all $z \in \mathbb{D}_R$). Also, consider $1 \leq r < 1/A$.*

(i) If $R = +\infty$, $(1/R = 0)$, i.e. f is an entire function, then, $K_n(f;\lambda_n)(z)$ is an entire function, for all $z \in \mathbb{C}$, $n \in \mathbb{N}$ we have $K_n(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k K_n(e_k;\lambda_n)(z)$ and for all $|z| \leq r$ the following estimates hold :

$$\left| K_n(f;\lambda_n)(z) - f(z) - \frac{\lambda_n}{2}[f'(z) + zf''(z)] \right| \leq C_r'(f)\,\lambda_n^2,$$

$$\|K_n^{(p)}(f;\lambda_n) - f^{(p)}\|_r \sim \lambda_n,$$

the last equivalence holding if f is not a polynomial of degree $\leq p$ and the constants in the equivalence depend on f, r, p.

Above $p \in \mathbb{N}\bigcup\{0\}$, $C_r'(f) < \infty$ is a constant independent of n and z and $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$.

(ii) If $R < +\infty$, then the complex approximation operator

$$K_n^*(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k\, K_n(e_k;\lambda_n)(z), z \in \overline{\mathbb{D}_r},$$

is well-defined and $K_n^*(f;\lambda_n)(z)$ satisfies all the estimates in (i), for all $1 \leq r < \frac{1}{A} < R$.

Proof. (i) From (2) we have

$$\left|K_n(f;\lambda_n)(z) - f(z) - \frac{\lambda_n}{2}[f'(z) + zf''(z)]\right| =$$

$$= \left|S_n'(F;\lambda_n)(z) - F'(z) - \frac{\lambda_n}{2}[F''(z) + zF'''(z)]\right| =$$

$$= \left|\left[S_n(F;\lambda_n)(z) - F(z) - \frac{\lambda_n}{2}zF''(z)\right]'\right|.$$

Let Γ be the circle of radius r_1 and center 0, with $\frac{1}{A} > r_1 > r$. For any $|z| \le r$ and $v \in \Gamma$, we have $|v - z| \ge r_1 - r$.

Denoting $E_n(F)(z) = S_n(F;\lambda_n)(z) - F(z) - \frac{\lambda_n}{2}zF''(z)$, from the Cauchy's formula and from the above Theorem 1, we get

$$|E_n'(F)(z)| = \frac{1}{2\pi}\left|\int_\Gamma \frac{E_n(F)(z)}{(v-z)^2}dv\right| \le C_r(F)\,\frac{1}{2\pi}\,\frac{2\pi r_1}{(r_1 - r)^2}\,\lambda_n^2 := C_r'(f)\,\lambda_n^2,$$

where $C_r'(f) = C_r(F)\,\frac{r_1}{(r_1-r)^2}$ is a constant independent of n and z.

On the other hand, from (2) and from the equivalence in the above Theorem 1, (i), we get

$$\|K_n^{(p)}(f;\lambda_n) - f^{(p)}\|_r = \|S_n^{(p+1)}(F;\lambda_n) - F^{(p+1)}\|_r \sim \lambda_n,$$

if F is not a polynomial of degree $\le p+1$, i.e. if f is not a polynomial of degree $\le p$.

(ii) Firstly, we show that $K_n^*(f;\lambda_n)(z)$ is well defined for all $z \in \mathbb{D}_R$. Indeed, we have

$$|K_n^*(f;\lambda_n)(z)| \le \sum_{k=0}^{\infty}|c_k|\,|K_n(e_k;\lambda_n)(z)| \le M\sum_{k=0}^{\infty}\frac{A^k}{k!}\,|K_n(e_k;\lambda_n)(z)| =$$

$$= M\sum_{k=0}^{\infty}\frac{A^k}{k!}\,\frac{1}{k+1}\,|S_n'(e_{k+1};\lambda_n)(z)|.$$

By the Cauchy's formula, taking into account the estimate for $S_n(e_{k+1};\lambda_n)(z)$ in the section (ii) of the above Theorem 1, it follows

$$|S_n'(e_{k+1};\lambda_n)(z)| \le \left|\frac{1}{2\pi}\int_\Gamma \frac{S_n(e_{k+1};\lambda_n)(v)}{(v-z)^2}dv\right| \le \frac{r_1}{(r_1-r)^2}\left[\frac{(k+2)!}{2}r^k\lambda_n + r^{k+1}\right].$$

Above, Γ is a disk of radius r_1 with $r < r_1 < R$ and center 0.

Now it follows

$$|K_n^*(f;\lambda_n)(z)| \le M\frac{r_1}{(r_1-r)^2}\sum_{k=0}^{\infty}\frac{A^k}{k!}\,\frac{1}{k+1}\left[\frac{(k+2)!}{2}r^k\lambda_n + r^{k+1}\right] =$$

$$= \frac{Mr_1\lambda_n}{2(r_1-r)^2}\sum_{k=0}^{\infty}(k+2)(Ar)^k + \frac{Mr_1 r}{(r_1-r)^2}\,e^{Ar} < \infty.$$

Finally, the estimates in this case follow immediately from the same theorems mentioned in the proof of (i). ◄

Remark 1. *It is worth noting that in the case of real variable, the generalized Szász operators defined by (1) were considered in [5], where, denoting* $\frac{b_n}{a_n} := \lambda_n$*, the approximation order* $\omega_1(f;\sqrt{\lambda_n})$ *was obtained, with* ω_1 *denoting the modulus of continuity of* f *on* $[0,+\infty)$*. The results in the real case in [5] and those in the complex case in Theorems 1 and 2, seem to be of definitive type, in the sense that they exhibit operators which can approximate the functions with an arbitrary chosen order.*

Remark 2. *The first estimate in the statement of Theorem 1, (i), was extended (with a different constant, of course) in [4] to the approximation by generalized Szász-Faber polynomials in compact sets in* $\mathbb{C}$*.*

3. Generalized Complex Baskakov Operators

For x real and ≥ 0, the original formula of the classical now Baskakov operator is given by (see [1])

$$Z_n(f)(x) = (1+x)^{-n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k f(k/n).$$

Many approximation results for these operators have been published.

According to [8], Theorem 2, under the same hypothesis on f, $Z_n(f)(x)$ is well defined, and denoting by $[0,1/n,...,j/n;f]$ the divided difference of f on the knots 0, ..., j/n, for $x \geq 0$ we can write $Z_n(f)(x) = W_n(f)(x)$, $x \geq 0$, where

$$W_n(f)(x) := \sum_{j=0}^{\infty}\left(1+\frac{1}{n}\right)\ ...\ \left(1+\frac{j-1}{n}\right)[0,1/n,...,j/n;f]x^j, x \geq 0, \tag{3}$$

(here for $j=0$ and $j=1$ we take $(1+1/n)\ ...\ (1+(j-1)/n)=1$).

For an arbitrary $\lambda_n \to 0$, by formula (1) in the paper [6] (particularizing there $\varphi_n(\lambda_n;x) = (1+x)^{-1/\lambda_n}$), $Z_n(f)(x)$ can be generalized to

$$Z_n(f;\lambda_n)(x) =$$

$$= (1+x)^{-1/\lambda_n}\sum_{j=0}^{\infty}\frac{1}{j!}\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\ ...\ \left(j-1+\frac{1}{\lambda_n}\right)\left(\frac{x}{1+x}\right)^j f\left(j\lambda_n\right), x \geq 0,$$

where it is assumed that $\frac{1}{\lambda_n}\left(1+\frac{1}{\lambda_n}\right)\ ...\ \left(j-1+\frac{1}{\lambda_n}\right)=1$ for $j=0$.

For this generalization, in [6] the order of approximation $\omega_1(f;\sqrt{\lambda_n}\sqrt{x(1+x)})$ was obtained.

Similarly, $W_n(f)(x)$ given by (3), can be generalized to

$$W_n(f;\lambda_n)(x) = \sum_{j=0}^{\infty}(1+\lambda_n)\dots(1+(j-1)\lambda_n)\ [0,\lambda_n,...,j\lambda_n;f]x^j, x \geq 0,$$

where it is assumed that $(1+\lambda_n)\dots(1+(j-1)\lambda_n)=1$ for $j=0$.

It is clear that $Z_n(f;\lambda_n)(x) = W_n(f;\lambda_n)(x)$ for all $x \geq 0$, but, as noted in [3], p. 124, in the special case $\lambda_n = \frac{1}{n}$, if $|x| < 1$ is not positive, then $W_n(f;\lambda_n)(x)$ and $Z_n(f;\lambda_n)(x)$ do not necessarily coincide and because of this reason they were studied separately in Section 1.8 of [3], pp. 124-138, under different hypotheses on f and $z \in \mathbb{C}$.

The first goal of this section is to study the approximation properties of the complex generalized Baskakov operators $W_n(f;\lambda_n)(z)$ attached to analytic functions satisfying some exponential-type growth condition.

For this aim, we prove the following

Theorem 3. *Let $0 < \lambda_n \leq \frac{1}{2}$, $n \in \mathbb{N}$ be such that $\lambda_n \to 0$ as fast as we want. Let $f : \mathbb{D}_R \to \mathbb{C}$, $1 < R \leq +\infty$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Suppose that there exist $M > 0$ and $A \in (1/R, 1)$ such that $|c_k| \leq M \frac{A^k}{k!}$, for all $k = 0, 1, ...$, (which implies $|f(z)| \leq M e^{A|z|}$ for all $z \in \mathbb{D}_R$). Consider $1 \leq r < \frac{1}{A}$.*

(i) If $R = +\infty$, ($1/R = 0$), i.e. f is an entire function, then for $|z| \leq r$, $W_n(f;\lambda_n)(z)$ is analytic, we have $W_n(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k W_n(e_k;\lambda_n)(z)$ and the following estimates hold

$$|W_n(f;\lambda_n)(z) - f(z)| \leq C_{r,M,A}\, \lambda_n,$$

$$|W_n^{(p)}(f;\lambda_n)(z) - f^{(p)}(z)| \leq \frac{p! r_1\, C_{r_1,M,A}}{(r_1 - r} \lambda_n,$$

$$\left| W_n(f;\lambda_n)(z) - f(z) - \frac{\lambda_n}{2} z f''(z) \right| \leq M_r(f)\, \lambda_n^2,$$

$$\|W_n^{(p)}(f;\lambda_n) - f^{(p)}\|_r \sim \lambda_n,$$

the last equivalence holding if f is not a polynomial of degree $\leq p \in \mathbb{N}$ and the constants in the equivalence depend on f, r, p.

Above, $C_{r,M,A} = 6M \sum_{k=2}^{\infty} (k+1)(k-1)(rA)^k < \infty$, $p \in \mathbb{N}$, $1 \leq r < r_1 < \frac{1}{A}$, $M_r(f) = 16M \sum_{k=3}^{\infty} (k-1)(k-2)(rA)^k < \infty$ and $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$.

(ii) If $R < +\infty$, then the complex approximation operator

$$W_n^*(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k\, W_n(e_k;\lambda_n)(z), z \in \overline{\mathbb{D}_r},$$

is well-defined and $W_n^*(f;\lambda_n)(z)$ satisfies all the estimates in (i), for all $1 \leq r < \frac{1}{A} < R$.

Proof. (i) Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$, $z \in \mathbb{C}$. We (formally) can write

$$W_n(f;\lambda_n)(z) = \sum_{j=0}^{\infty} (1+\lambda_n)\ ...\ (1+(j-1)\lambda_n)\, [0, \lambda_n, ..., j\lambda_n; \sum_{k=0}^{\infty} c_k e_k]\, z^j.$$

Since, in general, for a divided difference we have the formula $[x_0, ..., x_{j-1} : F] = \sum_{p=0}^{j-1} \frac{F(x_p)}{u_p(x_p)}$, where $u_p(x) = (x - x_0)\ ...\ (x - x_{p-1})(x - x_{p+1}) ... (x - x_{j-1})$ and since a finite sum commutes with an infinite sum, we obtain

$$W_n(f;\lambda_n)(z) = \sum_{j=0}^{\infty} (1+\lambda_n)\ ...\ (1+(j-1)\lambda_n) \sum_{k=0}^{\infty} c_k\, [0, \lambda_n, ..., j\lambda_n; e_k]\, z^j.$$

Now, if the above two infinite sums would commute, then we would obtain

$$W_n(f;\lambda_n)(z) =$$

$$= \sum_{k=0}^{\infty} c_k \sum_{j=0}^{\infty} (1+\lambda_n) \ ... \ (1+(j-1)\lambda_n) \, [0, \lambda_n, ..., j\lambda_n; e_k] \, z^j = \sum_{k=0}^{\infty} c_k W_n(e_k;\lambda_n)(z).$$

It is a well-known Fubini type result that a sufficient condition for the commutativity of two infinite sums, i.e for $\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_{k,j} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{k,j}$, is that $\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} |a_{k,j}| < +\infty$.

For $|z| \le r$, denote $a_{k,j} := c_k(1+\lambda_n) \ ... \ (1+(j-1)\lambda_n)[0, \lambda_n, ..., j\lambda_n; e_k]z^j$. Then the last condition becomes

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} |c_k|(1+\lambda_n) \ ... \ (1+(j-1)\lambda_n)[0, \lambda_n, ..., (j-1)\lambda_n; e_k]|z|^j =$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k} |c_k|(1+\lambda_n) \ ... \ (1+(j-1)\lambda_n)[0, \lambda_n, ..., (j-1)\lambda_n; e_k]|z|^j \le$$

$$\le M \sum_{k=0}^{\infty} \frac{(Ar)^k}{k!} \sum_{j=0}^{k} (1+\lambda_n) \ ... \ (1+(j-1)\lambda_n)[0, \lambda_n, ..., (j-1)\lambda_n; e_k] \le$$

$$\le M \sum_{k=0}^{\infty} \frac{(Ar)^k}{k!}(k+1)! = M\sum_{k=0}^{\infty}(k+1)(Ar)^k < \infty.$$

Above we used the inequality in Lemma 3.2 in [7]:

$$\sum_{j=0}^{k} (1+\lambda_n) \ ... \ (1+(j-1)\lambda_n)[0, \lambda_n, ..., (j-1)\lambda_n; e_k] \le (k+1)!, z \in \mathbb{C}.$$

Therefore, we obtain

$$W_n(f;\lambda_n)(z) = \sum_{k=0}^{\infty} c_k W_n(e_k;\lambda_n)(z), z \in \mathbb{C}, |z| \le r.$$

This relationship also proves that $W_n(f;\lambda_n)(z)$ is analytic in $|z| < r$, because, as above, we have

$$|W_n(f;\lambda_n)(z)| \le M \sum_{k=0}^{\infty} \frac{(A|z|)^k}{k!} \, (k+1)! = M \sum_{k=0}^{\infty}(k+1)(A|z|)^k < \infty, |z| < r.$$

Now, let us denote

$$T_{n,k}(z) = W_n(e_k;\lambda_n)(z) = \sum_{j=0}^{k} (1+\lambda_n) \ ... \ (1+(j-1)\lambda_n)[0, \lambda_n, ..., j\lambda_n; e_k]z^j.$$

Using the same reasoning as in the proof of Theorem 1.9.1, page 126 in [3], we obtain the recurrence formula

$$T_{n,p+1}(z) = z(1+z)\,\lambda_n T'_{n,p}(z) + zT_{n,p}(z),$$

and, following the reasoning of the proof of Theorem 1.9.1 (which actually means the replacing of $\frac{1}{n}$ by λ_n in all the formulas there), we easily arrive at the estimate

$$\|T_{n,p} - e_p\|_r \leq 6r^p(p+1)!(p-1)\,\lambda_n,\ p = 2, 3, ...,$$

which finally leads to the estimate

$$|W_n(f;\lambda_n)(z) - f(z)| \leq \left[6M\sum_{k=2}^{\infty}(k+1)(k-1((rA)^k\right]\lambda_n,$$

for all $|z| \leq r$, $n \in \mathbb{N}$.

Denote by γ the circle of radius $r_1 > r$ and center 0. Using the same reasoning as in [3], section (ii), page 128, we easily arrive at the second estimate in the theorem.

Similarly, replacing $\frac{1}{n}$ by λ_n in all the formulas in the statement and proof of Theorem 1.9.3 in [3], pages 130-131, we get the third estimate in the present theorem.

Finally, the forth estimate follows from Corollary 1.9.4, in [3], p. 132, by replacing $\frac{1}{n}$ in all the formulas in its proof by λ_n.

(ii) From the last estimate in the proof of the above section (i), it directly follows that $W^*_n(f;\lambda_n)(z)$ is well defined for all $|z| \leq r$.

Finally, the estimates in this case follow immediately from the same theorems mentioned in the proof of (i). ◄

Remark 3. *Due to the results in the real case in [6] and to those in the complex case in Theorem 3, we can say that they seem to be of definitive type, in the sense that they exhibit Baskakov type operators which can approximate the functions with an arbitrary chosen order.*

Remark 4. *The first estimate in the statement of Theorem 3, (i), was extended (with a different constant, of course) in [7] to the approximation by generalized Baskakov-Faber polynomials in compact sets in $\mathbb{C}$.*

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