Azerbaijan Journal of Mathematics
V. 6, No 1, 2016, January
ISSN 2218-6816

Homoclinic Orbits for a Class of Nonlinear Difference Equations

H. Shi*, X. Liu, Y. Zhang

Abstract. By using the critical point theory, the existence of a nontrivial homoclinic orbit for a class of nonlinear difference equations is obtained. The proof is based on the Mountain Pass Lemma in combination with periodic approximations. Our conditions on the nonlinear term are rather relaxed and we improve some existing results in the literature.

Key Words and Phrases: homoclinic orbits, nonlinear, difference equations, discrete variational theory.

2010 Mathematics Subject Classifications: 34C37, 37J45, 39A12

1. Introduction

Below **N**, **Z** and **R** denote the sets of all natural numbers, integers and real numbers, respectively. For any a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a < b$.

Difference equations have attracted the interest of many researchers in the past twenty years since they provided a natural description of several discrete models. Such discrete models are often investigated in various fields of science and technology such as computer science, economics, neural networks, ecology, cybernetics, biological systems, optimal control, and population dynamics. These studies cover many of the branches of difference equations, such as stability, attractivity, periodicity, homoclinic orbits, oscillation, and boundary value problems, see [2, 4, 5, 6, 7, 8, 9, 10, 21, 24, 29, 30, 31, 32] and the references therein. For the general background of difference equations, one can refer to [1].

The present paper considers the following forward and backward difference equation

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) - q_n u_n^{\delta} + f(n, u_{n+1}, u_n, u_{n-1}) = 0, \ n \in \mathbf{Z}, \tag{1}$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $\delta > 0$ is the ratio of odd positive integers, $\{p_n\}$ and $\{q_n\}$ are real sequences, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$, T is a given positive integer, $p_{n+T} = p_n > 0$, $q_{n+T} = q_n > 0$, $f(n+T, v_1, v_2, v_3) = f(n, v_1, v_2, v_3)$.

*Corresponding author.

Eq. (1) can be considered as a discrete analogue of the following second-order nonlinear functional differential equation

$$\big(p(t)\varphi(u')\big)' + q(t)u(t) + f(t, u(t+1), u(t), u(t-1)) = 0,\ \ t \in \mathbf{R}. \tag{2}$$

Eq. (2) includes the following equation

$$\big(p(t)\varphi(u')\big)' + f(t, u(t)) = 0,\ \ t \in \mathbf{R},$$

which has arisen in the study of fluid dynamics, combustion theory, gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, and adiabatic reactor [3, 11, 22]. Equations similar in structure to (2) arise in the study of homoclinic orbits [14, 16, 17, 18] of functional differential equations.

When $f(n, u_{n+1}, u_n, u_{n-1}) = 0,\ n \in \mathbf{Z}(0)$, (1) reduces to the following equation

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + q_n u_n^{\delta} = 0, \tag{3}$$

which has been studied in [21] for results on oscillation, asymptotic behavior and the existence of positive solutions.

In 2008, Cai and Yu [2] have obtained some sufficient conditions for the existence of periodic solutions of the following nonlinear difference equation:

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + q_n u_n^{\delta} = f(n, u_n),\ \ n \in \mathbf{Z}. \tag{4}$$

It is well known that critical point theory is an effective approach to study the behavior of differential equations [13, 14, 15, 16, 17, 18, 27, 28]. Only since 2003, critical point theory has been employed to establish sufficient conditions for the existence of periodic solutions for second order difference equations [19, 20]. Along this direction, Ma and Guo [25] (without periodicity assumption) and [26] (with periodicity assumption) applied variational methods to prove the existence of homoclinic orbits for the special form of (1) (with $\delta = 1$). Chen and Wang [9] studied the existence of infinitely many homoclinic orbits of the following equation:

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) - q_n u_n^{\delta} + f(n, u_n) = 0,\ \ n \in \mathbf{Z}, \tag{5}$$

by using critical point theory. A crucial role that the Ambrosetti-Rabinowitz condition plays is to ensure the boundedness of Palais-Smale sequences. This is very crucial in applying the critical point theory.

However, it seems that the results on homoclinic orbits of (1) are scarce in the literature. Since (1) contains both advance and retardation, there are very few manuscripts dealing with this subject, the traditional ways of establishing the functional in [2, 10, 19, 20, 23, 31, 32] are inapplicable to our case. The main purpose of this paper is to develop a new approach to above problem without the classical Ambrosetti-Rabinowitz condition. In particular, our conditions on the nonlinear term are rather relaxed and we improve some existing results in the known literature. In fact, one can see the following Remarks 2 and 3 for details. The motivation for the present work stems from the recent papers [4, 9, 18].

Let

$$\underline{p} = \min_{n\in\mathbf{Z}(1,T)}\{p_n\},\ \bar{p} = \max_{n\in\mathbf{Z}(1,T)}\{p_n\},\ \underline{q} = \min_{n\in\mathbf{Z}(1,T)}\{q_n\},\ \bar{q} = \max_{n\in\mathbf{Z}(1,T)}\{q_n\}.$$

Our main results are as follows.

Theorem 1. *Suppose that the following hypotheses are satisfied:*
(F_1) *there exists a functional* $F(n,v_1,v_2) \in C^1(\boldsymbol{Z}\times\boldsymbol{R}^2,\boldsymbol{R})$ *with* $F(n+T,v_1,v_2) = F(n,v_1,v_2)$ *and it satisfies*

$$\frac{\partial F(n-1,v_2,v_3)}{\partial v_2} + \frac{\partial F(n,v_1,v_2)}{\partial v_2} = f(n,v_1,v_2,v_3);$$

(F_2) *there exist positive constants* ϱ *and* $a < \frac{\underline{q}}{2(\delta+1)}\left(\frac{\kappa_1}{\kappa_2}\right)^{\delta+1}$ *such that*

$$|F(n,v_1,v_2)| \le a\left(|v_1|^{\delta+1}+|v_2|^{\delta+1}\right) \text{ for all } n\in\boldsymbol{Z} \text{ and } \sqrt{v_1^2+v_2^2}\le\varrho;$$

(F_3) *there exist constants* $\rho, c > \frac{1}{2(\delta+1)}\left(\frac{\kappa_2}{\kappa_1}\right)^{\delta+1}\left(2^{\delta+1}\bar{p}+\bar{q}\right)$ *and* b *such that*

$$F(n,v_1,v_2) \ge c\left(|v_1|^{\delta+1}+|v_2|^{\delta+1}\right)+b \text{ for all } n\in\boldsymbol{Z} \text{ and } \sqrt{v_1^2+v_2^2}\ge\rho;$$

(F_4) $\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1+\frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2-(\delta+1)F(n,v_1,v_2) > 0$, *for all* $(n,v_1,v_2)\in\boldsymbol{Z}\times\boldsymbol{R}^2\backslash\{(0,0)\}$;
(F_5) $\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1+\frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2-(\delta+1)F(n,v_1,v_2)\to+\infty$ *as* $\sqrt{v_1^2+v_2^2}\to+\infty$.
Then (1) has a nontrivial homoclinic orbit.

Remark 1. *By* (F_3), *it is easy to see that there exists a constant* $\zeta>0$ *such that*

(F_3') $F(n,v_1,v_2)\ge c\left(|v_1|^{\delta+1}+|v_2|^{\delta+1}\right)+b-\zeta,\ \forall(n,v_1,v_2)\in\boldsymbol{Z}\times\boldsymbol{R}^2$.

As a matter of fact, letting

$$\zeta = \max\left\{\left|F(n,v_1,v_2)-c\left(|v_1|^{\delta+1}+|v_2|^{\delta+1}\right)-b\right| : n\in\mathbf{Z}, \sqrt{v_1^2+v_2^2}\le\rho\right\},$$

we can easily get the desired result.

Remark 2. *Theorem 1 extends Theorem 1.1 in* [26] *which is the special case of our Theorem 1 by letting* $\delta=1$.

Remark 3. *In many studies (see e.g.* [2, 10, 19, 20, 25, 26]*) of second order difference equations, the following classical Ambrosetti-Rabinowitz condition is assumed.*
(AR) *there exists a constant* $\beta>2$ *such that*

$$0<\beta F(n,u)\le uf(n,u) \text{ for all } n\in\boldsymbol{Z} \text{ and } u\in\boldsymbol{R}\setminus\{0\}.$$

Note that $(F_3)-(F_5)$ *are much weaker than* $(\boldsymbol{AR})$. *Thus our result improves the existing ones.*

Theorem 2. *Suppose that* $(F_1)-(F_5)$ *and the following hypothesis are satisfied:*
(F_6) $p_{-n}=p_n,\ q_{-n}=q_n,\ F(-n,v_1,v_2)=F(n,v_1,v_2)$.
Then (1) has a nontrivial even homoclinic orbit.

For the basic knowledge of variational methods, the reader is referred to [17, 27, 28].

2. Preliminaries

In this section, we present some definitions and lemmas that will be used in the proof of our results.

Let S be the set of sequences $u=(\cdots,u_{-n},\cdots,u_{-1},u_0,u_1,\cdots,u_n,\cdots)=\{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S=\{\{u_n\}|u_n\in\mathbf{R},\ n\in\mathbf{Z}\}.$$

For any $u,v\in S$, $a,b\in\mathbf{R}$, $au+bv$ is defined by

$$au+bv=\{au_n+bv_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, E_m is defined as a subspace of S by

$$E_m=\{u\in S|u_{n+2mT}=u_n,\ \forall n\in\mathbf{Z}\}.$$

Clearly, E_m is isomorphic to $\mathbf{R}^{2mT}$. E_m can be equipped with the inner product

$$\langle u,v\rangle=\sum_{j=-mT}^{mT-1}u_jv_j,\ \forall u,v\in E_m, \tag{6}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\|=\left(\sum_{j=-mT}^{mT-1}u_j^2\right)^{\frac{1}{2}},\ \forall u\in E_m. \tag{7}$$

It is obvious that E_m with the inner product (6) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{2mT}$.

On the other hand, we define the norm $\|\cdot\|_s$ on E_m as follows:

$$\|u\|_s=\left(\sum_{j=-mT}^{mT-1}|u_j|^s\right)^{\frac{1}{s}}, \tag{8}$$

for all $u\in E_m$ and $s>1$. Denote by l^s the set of all functions $u:\mathbf{Z}\to\mathbf{R}$ such that

$$\|u\|_s^s=\sum_{j\in\mathbf{Z}}|u_j|^s<+\infty.$$

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants $\kappa_1,\ \kappa_2$ such that $\kappa_2 \geq \kappa_1 > 0$, and

$$\kappa_1\|u\|_2 \leq \|u\|_s \leq \kappa_2\|u\|_2,\ \forall u \in E_m. \tag{9}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_m$, define the functional J on E_m as follows:

$$J(u) = \frac{1}{\delta+1}\sum_{n=-mT}^{mT-1} p_n\left(\Delta u_{n-1}\right)^{\delta+1} + \frac{1}{\delta+1}\sum_{n=-mT}^{mT-1} q_n u_n^{\delta+1} - \sum_{n=-mT}^{mT-1} F(n, u_{n+1}, u_n). \tag{10}$$

Clearly, $J \in C^1(E_m, \mathbf{R})$ and for any $u = \{u_n\}_{n\in\mathbf{Z}} \in E_m$, by the periodicity of $\{u_n\}_{n\in\mathbf{Z}}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n} = -\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) + q_n u_n^{\delta} - f(n, u_{n+1}, u_n, u_{n-1}),\ \forall n \in \mathbf{Z}(-mT, mT-1). \tag{11}$$

Thus, u is a critical point of J on E_m if and only if

$$\Delta\left(p_n(\Delta u_{n-1})^{\delta}\right) - q_n u_n^{\delta} + f(n, u_{n+1}, u_n, u_{n-1}) = 0,\ \forall n \in \mathbf{Z}(-mT, mT-1).$$

Due to the periodicity of $u = \{u_n\}_{n\in\mathbf{Z}} \in E_m$ and $f(n, v_1, v_2, v_3)$ in the first variable n, we reduce the existence of periodic solutions of (1) to the existence of critical points of J on E_m. That is, the functional J is just the variational framework of (1).

In what follows, we define a norm $\|\cdot\|_\infty$ in E_m by

$$\|u\|_\infty = \max_{j\in\mathbf{Z}(-mT, mT-1)} |u_j|\,,\ \forall u \in E_m.$$

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{u_n\} \subset E$ for which $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0\ (n \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 1. *(Mountain Pass Lemma [28]). Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*
(J_1) there exist constants $\rho,\ \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$, and
(J_2) there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$,
then J possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]} J(g(s)), \tag{12}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}. \tag{13}$$

Lemma 2. *The following inequality is true:*

$$\frac{1}{\delta+1}\sum_{n=-mT}^{mT-1} p_n\left(\Delta u_{n-1}\right)^{\delta+1} \leq \frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1}\|u\|^{\delta+1}. \tag{14}$$

Proof. $\frac{1}{\delta+1}\sum_{n=-mT}^{mT-1} p_n\left(\Delta u_{n-1}\right)^{\delta+1} \leq \frac{\bar{p}}{\delta+1}\sum_{n=-mT}^{mT-1}\left|\Delta u_n\right|^{\delta+1}$

$$= \frac{\bar{p}}{\delta+1}\left[\left(\sum_{n=-mT}^{mT-1}\left|\Delta u_n\right|^{\delta+1}\right)^{\frac{1}{\delta+1}}\right]^{\delta+1}$$

$$\leq \frac{\bar{p}}{\delta+1}\left[\kappa_2\left(\sum_{n=-mT}^{mT-1}\left|\Delta u_n\right|^{2}\right)^{\frac{1}{2}}\right]^{\delta+1}$$

$$\leq \frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}\left[\sum_{n=-mT}^{mT-1} 2\left(u_{n+1}^2+u_n^2\right)\right]^{\frac{\delta+1}{2}}$$

$$= \frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1}\|u\|^{\delta+1}.$$ ◀

3. Proof of theorems

In this section, we prove our main results by using the critical point method.

Lemma 3. *Suppose that* $(F_1)-(F_5)$ *are satisfied. Then* J *satisfies the P.S. condition.*

Proof. Assume that $\left\{u^{(i)}\right\}_{i\in\mathbf{N}}$ in E_m is a sequence such that $\left\{J\left(u^{(i)}\right)\right\}_{i\in\mathbf{N}}$ is bounded. Then there exists a constant $K>0$ such that $-K\leq J\left(u^{(i)}\right)$. By (14) and (F_3'), we have

$$-K \leq J\left(u^{(i)}\right) \leq \frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1}\left\|u^{(i)}\right\|^{\delta+1} + \frac{\bar{q}}{\delta+1}\left[\left(\sum_{n=-mT}^{mT-1}\left|u_n^{(i)}\right|^{\delta+1}\right)^{\frac{1}{\delta+1}}\right]^{\delta+1}$$
$$-\sum_{n=-mT}^{mT-1}\left[c\left(\left|u_{n+1}^{(i)}\right|^{\delta+1}+\left|u_n^{(i)}\right|^{\delta+1}\right)+b-\zeta\right]$$
$$\leq \left(\frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1}+\frac{\bar{q}}{\delta+1}\kappa_2^{\delta+1}-2c\kappa_1^{\delta+1}\right)\left\|u^{(i)}\right\|^{\delta+1}+2mT\left(\zeta-b\right).$$

Therefore,

$$\left(2c\kappa_1^{\delta+1}-\frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1}-\frac{\bar{q}}{\delta+1}\kappa_2^{\delta+1}\right)\left\|u^{(i)}\right\|^{\delta+1} \leq 2mT\left(\zeta-b\right)+K. \tag{15}$$

Since $c>\frac{1}{2(\delta+1)}\left(\frac{\kappa_2}{\kappa_1}\right)^{\delta+1}\left(2^{\delta+1}\bar{p}+\bar{q}\right)$, (15) implies that $\left\{u^{(i)}\right\}_{i\in\mathbf{N}}$ is bounded in E_m. Thus, $\left\{u^{(i)}\right\}_{i\in\mathbf{N}}$ possesses a convergent subsequence in E_m. The desired result follows. ◀

Lemma 4. *Suppose that* $(F_1)-(F_5)$ *are satisfied. Then for any given positive integer* m, *(1) possesses a* $2mT$*-periodic solution* $u^{(m)}\in E_m$.

Proof. In our case, it is clear that $J(0)=0$. By Lemma 3, J satisfies the P.S. condition. By (F_2), we have

$$J(u) \geq \frac{p}{\delta+1}\sum_{n=-mT}^{mT-1}|\Delta u_n|^{\delta+1} + \frac{q}{\delta+1}\sum_{n=-mT}^{mT-1}|u_n|^{\delta+1}$$

$$-a\sum_{n=-mT}^{mT-1}\left(|u_{n+1}|^{\delta+1}+|u_n|^{\delta+1}\right)$$

$$\geq \frac{q}{\delta+1}\kappa_1^{\delta+1}\|u\|^{\delta+1} - 2a\kappa_2^{\delta+1}\|u\|^{\delta+1}$$

$$= \left(\frac{q}{\delta+1}\kappa_1^{\delta+1} - 2a\kappa_2^{\delta+1}\right)\|u\|^{\delta+1}.$$

Taking $\alpha = \left(\frac{q}{\delta+1}\kappa_1^{\delta+1} - 2a\kappa_2^{\delta+1}\right)\varrho^{\delta+1} > 0$, we obtain

$$J(u)|_{\partial B_\varrho} \geq \alpha > 0,$$

which implies that J satisfies the condition (J_1) of the Mountain Pass Lemma.

Next, we shall verify the condition (J_2).

There exists a sufficiently large number $\varepsilon > \max\{\varrho, \rho\}$ such that

$$\left(2c\kappa_1^{\delta+1} - \frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1} - \frac{\bar{q}}{\delta+1}\kappa_2^{\delta+1}\right)\varepsilon^{\delta+1} \geq |b|. \tag{16}$$

Let $e \in E_m$ and

$$e_n = \begin{cases} \varepsilon, & \text{if } n = 0, \\ 0, & \text{if } n \in \{j \in \mathbf{Z} : -mT \leq j \leq mT-1 \text{ and } j \neq 0\}, \end{cases}$$

$$e_{n+1} = \begin{cases} \varepsilon, & \text{if } n = 0, \\ 0, & \text{if } n \in \{j \in \mathbf{Z} : -mT \leq j \leq mT-1 \text{ and } j \neq 0\}. \end{cases}$$

Then

$$F(n, e_{n+1}, e_n) = \begin{cases} F(0,\varepsilon,\varepsilon), & \text{if } n = 0, \\ 0, & \text{if } n \in \{j \in \mathbf{Z} : -mT \leq j \leq mT-1 \text{ and } j \neq 0\}. \end{cases}$$

With (16) and (F_3), we have

$$J(e) = \frac{1}{\delta+1}\sum_{n=-mT}^{mT-1} p_n(\Delta e_{n-1})^{\delta+1} + \frac{1}{\delta+1}\sum_{n=-mT}^{mT-1} q_n e_n^{\delta+1} - \sum_{n=-mT}^{mT-1} F(n, e_{n+1}, e_n)$$

$$\leq \frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1}\|e\|^{\delta+1} + \frac{\bar{q}}{\delta+1}\kappa_2^{\delta+1}\|e\|^{\delta+1} - 2c\kappa_1^{\delta+1}\|e\|^{\delta+1} - b$$

$$= -\left(2c\kappa_1^{\delta+1} - \frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1} - \frac{\bar{q}}{\delta+1}\kappa_2^{\delta+1}\right)\varepsilon^{\delta+1} - b \leq 0. \tag{17}$$

All the assumptions of the Mountain Pass Lemma have been verified. Consequently, J possesses a critical value c_m given by (12) and (13) with $E = E_m$ and $\Gamma = \Gamma_m$, where $\Gamma_m = \{g_m \in C([0,1], E_m) | g_m(0) = 0,\ g_m(1) = e,\ e \in E_m \backslash B_\varepsilon\}$. Let $u^{(m)}$ denote the corresponding critical point of J on E_m. Note that $\|u^{(m)}\| \neq 0$ since $c_m > 0$. ◀

Lemma 5. *Suppose that* $(F_1) - (F_5)$ *are satisfied. Then there exist positive constants* ϱ *and* η *independent of* m *such that*

$$\varrho \leq \left\|u^{(m)}\right\|_\infty \leq \eta. \tag{18}$$

Proof. The continuity of $F(0, v_1, v_2)$ with respect to the second and third variables implies that there exists a constant $\tau > 0$ such that $|F(0, v_1, v_2)| \leq \tau$ for $\sqrt{v_1^2 + v_2^2} \leq \varrho$. It is clear that

$$J\left(u^{(m)}\right) \leq \max_{0\leq s\leq 1}\left\{\frac{1}{\delta+1}\sum_{n=-mT}^{mT-1}\left|p_n\left(\Delta(se)_{n-1}\right)^{\delta+1} + q_n(se)_n^{\delta+1}\right| - \sum_{n=-mT}^{mT-1} F\left(n, (se)_{n+1}, (se)_n\right)\right\}$$

$$\leq \left(\frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1} + \frac{\bar{q}}{\delta+1}\kappa_2^{\delta+1}\right)\|e\|^{\delta+1} + \tau$$

$$= \left(\frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1} + \frac{\bar{q}}{\delta+1}\kappa_2^{\delta+1}\right)\varepsilon^{\delta+1} + \tau.$$

Let $\xi = \left(\frac{\bar{p}}{\delta+1}\kappa_2^{\delta+1}2^{\delta+1} + \frac{\bar{q}}{\delta+1}\kappa_2^{\delta+1}\right)\varepsilon^{\delta+1} + \tau$. Then we have $J\left(u^{(m)}\right) \leq \xi$, which is independent of m. From (10) and (11), we have

$$J\left(u^{(m)}\right) = \frac{1}{\delta+1}\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n-1, u_n^{(m)}, u_{n-1}^{(m)})}{\partial v_2}u_n^{(m)} + \frac{\partial F(n, u_{n+1}^{(m)}, u_n^{(m)})}{\partial v_2}u_n^{(m)}\right] - \sum_{n=-mT}^{mT-1} F(n, u_{n+1}^{(m)}, u_n^{(m)})$$

$$= \frac{1}{\delta+1}\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n, u_{n+1}^{(m)}, u_n^{(m)})}{\partial v_1}u_{n+1}^{(m)} + \frac{\partial F(n, u_{n+1}^{(m)}, u_n^{(m)})}{\partial v_2}u_n^{(m)}\right] - \sum_{n=-mT}^{mT-1} F(n, u_{n+1}^{(m)}, u_n^{(m)}) \leq \xi.$$

By (F_4) and (F_5), there exists a constant $\eta > 0$ such that
$\frac{1}{\delta+1}\left(\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2\right) - F(n,v_1,v_2) > \xi$, for all $n \in \mathbf{Z}$ and $\sqrt{v_1^2+v_2^2} \geq \eta$,

which implies that $\left|u_n^{(m)}\right| \leq \eta$ for all $n \in \mathbf{Z}$, that is $\left\|u^{(m)}\right\|_\infty \leq \eta$.

From the definition of J, we have

$$0 = \left\langle J'(u^{(m)}), u^{(m)}\right\rangle \geq \underline{q}\sum_{n=-mT}^{mT-1}\left|u_n^{(m)}\right|^{\delta+1}$$

$$-\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n-1,u_n^{(m)},u_{n-1}^{(m)})}{\partial v_2}u_n^{(m)} + \frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}u_n^{(m)}\right]$$

$$\geq \underline{q}\kappa_1^{\delta+1}\|u^{(m)}\|^{\delta+1} - \sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_1}u_{n+1}^{(m)} + \frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}u_n^{(m)}\right].$$

Therefore, combined with (F_2), we get

$$\underline{q}\kappa_1^{\delta+1}\|u^{(m)}\|^{\delta+1} \leq \sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_1}u_{n+1}^{(m)} + \frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}u_n^{(m)}\right]$$

$$\leq \left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_1}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\|u^{(m)}\|_{\delta+1}$$

$$+\left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\|u^{(m)}\|_{\delta+1}$$

$$\leq \kappa_2\|u^{(m)}\|\left\{\left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_1}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\right.$$

$$\left.+\left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\right\}.$$

That is

$$\frac{\underline{q}\kappa_1^{\delta+1}}{\kappa_2}\|u^{(m)}\|^{\delta} \leq \left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_1}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}} +$$

$$+\left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}.$$

Thus

$$\frac{\underline{q}^{\frac{\delta+1}{\delta}}\kappa_1^{\frac{(\delta+1)^2}{\delta}}}{\kappa_2^{\frac{\delta+1}{\delta}}}\|u^{(m)}\|^{\delta+1}$$

$$\leq\left\{\left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_1}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}+\left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\right\}^{\frac{\delta+1}{\delta}}. \tag{19}$$

Combined with (F_2), we get

$$\frac{\underline{q}^{\delta+1}\kappa_1^{\frac{(\delta+1)^2}{\delta}}}{\kappa_2^{\frac{\delta+1}{\delta}}}\|u^{(m)}\|^{\delta+1}$$

$$\leq\left\{\left\{\sum_{n=-mT}^{mT-1}\left[(\delta+1)a\left|u_{n+1}^{(m)}\right|^{\delta}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}+\left\{\sum_{n=-mT}^{mT-1}\left[(\delta+1)a\left|u_n^{(m)}\right|^{\delta}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\right\}^{\frac{\delta+1}{\delta}}$$

$$\leq 2^{\frac{\delta+1}{\delta}}[a(\delta+1)]^{\frac{\delta+1}{\delta}}\kappa_2^{\delta+1}\|u^{(m)}\|^{\delta+1}.$$

Thus, we have $u^{(m)}=0$. But this contradicts $\|u^{(m)}\|\neq 0$, which shows that

$$\|u^{(m)}\|_\infty\geq\varrho,$$

and the proof of Lemma 5 is finished. ◀

Proof of Theorem 1. In the following, we shall give the existence of a nontrivial homoclinic orbit.

Consider the sequence $\left\{u_n^{(m)}\right\}_{n\in\mathbf{Z}}$ of $2mT$-periodic solutions found in Lemma 4. First, by (18), for any $m\in\mathbf{N}$, there exists a constant $n_m\in\mathbf{Z}$ independent of m such that

$$\left|u_{n_m}^{(m)}\right|\geq\varrho. \tag{20}$$

Since p_n, q_n and $f(n,v_1,v_2,v_3)$ are all T-periodic in n, $\left\{u_{n+jT}^{(m)}\right\}$ $(\forall j\in\mathbf{N})$ is also $2mT$-periodic solution of (1). Hence, making such shifts, we can assume that $n_m\in\mathbf{Z}(0,T-1)$ in (20). Moreover, passing to a subsequence of ms, we can even assume that $n_m=n_0$ is independent of m.

Next, we extract a subsequence, still denoted by $u^{(m)}$, such that

$$u_n^{(m)} \to u_n,\ m \to \infty,\ \forall n \in \mathbf{Z}.$$

Inequality (20) implies that $|u_{n_0}| \geq \xi$ and, hence, $u = \{u_n\}$ is a nonzero sequence. Moreover,

$$\begin{aligned}
&\Delta\left(p_n(\Delta u_{n-1})^\delta\right) - q_n u_n^\delta + f(n, u_{n+1}, u_n, u_{n-1}) \\
&= \lim_{n\to\infty}\left[\Delta\left(p_n\left(\Delta\left(u_{n-1}^{(m)}\right)\right)^\delta\right) - q_n\left(u_n^{(m)}\right)^\delta + f\left(n, u_{n+1}^{(m)}, u_n^{(m)}, u_{n-1}^{(m)}\right)\right] = 0.
\end{aligned}$$

So $u = \{u_n\}$ is a solution of (1).

Finally, we show that $u \in l^{\delta+1}$. For $u_m \in E_m$, let

$$P_m = \left\{n \in \mathbf{Z} : \left|u_n^{(m)}\right| < \frac{\sqrt{2}}{2}\varrho, -mT \leq n \leq mT-1\right\},$$

$$Q_m = \left\{n \in \mathbf{Z} : \left|u_n^{(m)}\right| \geq \frac{\sqrt{2}}{2}\varrho, -mT \leq n \leq mT-1\right\}.$$

Since $F(n, v_1, v_2) \in C^1(\mathbf{Z}\times\mathbf{R}^2, \mathbf{R})$, there exist constants $\bar{\xi} > 0$, $\underline{\xi} > 0$ such that

$$\begin{aligned}
\max&\left\{\left\{\left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,v_1,v_2)}{\partial v_1}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\right.\right. \\
&\left.\left. + \left\{\sum_{n=-mT}^{mT-1}\left[\frac{\partial F(n,v_1,v_2)}{\partial v_2}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\right\}^{\frac{\delta+1}{\delta}} : \varrho \leq \sqrt{v_1^2+v_2^2} \leq \eta, n \in \mathbf{Z}\right\} \leq \bar{\xi},
\end{aligned}$$

$$\begin{aligned}
\min&\left\{\frac{1}{\delta+1}\left[\frac{\partial F(n,v_1,v_2)}{\partial v_1}v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2}v_2\right] - \right. \\
&\left. -F(n,v_1,v_2) : \varrho \leq \sqrt{v_1^2+v_2^2} \leq \eta, n \in \mathbf{Z}\right\} \geq \underline{\xi}.
\end{aligned}$$

For $n \in Q_m$,

$$\begin{aligned}
&\left\{\left[\frac{\partial F(n, u_{n+1}^{(m)}, u_n^{(m)})}{\partial v_1}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}} + \left\{\left[\frac{\partial F(n, u_{n+1}^{(m)}, u_n^{(m)})}{\partial v_2}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}} \\
&\leq \frac{\bar{\xi}}{\underline{\xi}}\left\{\frac{1}{\delta+1}\left[\frac{\partial F(n, u_{n+1}^{(m)}, u_n^{(m)})}{\partial v_1}u_{n+1}^{(m)} + \frac{\partial F(n, u_{n+1}^{(m)}, u_n^{(m)})}{\partial v_2}u_n^{(m)})\right] - F(n, u_{n+1}^{(m)}, u_n^{(m)})\right\}.
\end{aligned}$$

By (19), we have

$$\frac{\underline{q}^{\frac{\delta+1}{\delta}}\kappa_1^{\frac{(\delta+1)^2}{\delta}}}{\kappa_2^{\frac{\delta+1}{\delta}}}\|u^{(m)}\|^{\delta+1}$$

$$\leq \left\{\left\{\sum_{n\in P_m}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_1}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}+\left\{\sum_{n\in P_m}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\right\}^{\frac{\delta+1}{\delta}}$$

$$+\left\{\left\{\sum_{n\in Q_m}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_1}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}+\left\{\sum_{n\in Q_m}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\right\}^{\frac{\delta+1}{\delta}}$$

$$\leq \left\{\left\{\sum_{n\in P_m}\left[(\delta+1)a\left|u_{n+1}^{(m)}\right|^{\delta}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}+\left\{\sum_{n\in P_m}\left[(\delta+1)a\left|u_{n}^{(m)}\right|^{\delta}\right]^{\frac{\delta+1}{\delta}}\right\}^{\frac{\delta}{\delta+1}}\right\}^{\frac{\delta+1}{\delta}}$$

$$+\frac{\bar{\xi}}{\underline{\xi}}\left\{\frac{1}{\delta+1}\sum_{n\in Q_m}\left[\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_1}u_{n+1}^{(m)}+\frac{\partial F(n,u_{n+1}^{(m)},u_n^{(m)})}{\partial v_2}u_n^{(m)})\right]-F(n,u_{n+1}^{(m)},u_n^{(m)})\right\}$$

$$\leq 2^{\frac{\delta+1}{\delta}}[a(\delta+1)]^{\frac{\delta+1}{\delta}}\kappa_2^{\delta+1}\|u^{(m)}\|^{\delta+1}+\frac{\bar{\xi}\xi}{\underline{\xi}}.$$

Thus,

$$\left\|u^{(m)}\right\|^{\delta+1}\leq\frac{\bar{\xi}\xi\kappa_2^{\frac{\delta+1}{\delta}}}{\underline{\xi}\left\{\underline{q}^{\frac{\delta+1}{\delta}}\kappa_1^{\frac{(\delta+1)^2}{\delta}}-[2a(\delta+1)]^{\frac{\delta+1}{\delta}}\kappa_2^{\frac{(\delta+1)^2}{\delta}}\right\}}.$$

For any fixed $D\in\mathbf{Z}$ and m large enough, we have

$$\sum_{n=-D}^{D}\left|u_n^{(m)}\right|^{\delta+1}\leq\|u^{(m)}\|^{\delta+1}\leq\frac{\bar{\xi}\xi\kappa_2^{\frac{\delta+1}{\delta}}}{\underline{\xi}\left\{\underline{q}^{\frac{\delta+1}{\delta}}\kappa_1^{\frac{(\delta+1)^2}{\delta}}-[2a(\delta+1)]^{\frac{\delta+1}{\delta}}\kappa_2^{\frac{(\delta+1)^2}{\delta}}\right\}}.$$

Since $\bar{\xi}$, $\underline{\xi}$, ξ, $\underline{q}$, a, δ, κ_1 and κ_2 are constants independent of m, passing to the limit, we have

$$\sum_{n=-D}^{D}|u_n|^{\delta+1}\leq\frac{\bar{\xi}\xi\kappa_2^{\frac{\delta+1}{\delta}}}{\underline{\xi}\left\{\underline{q}^{\frac{\delta+1}{\delta}}\kappa_1^{\frac{(\delta+1)^2}{\delta}}-[2a(\delta+1)]^{\frac{\delta+1}{\delta}}\kappa_2^{\frac{(\delta+1)^2}{\delta}}\right\}}.$$

Due to the arbitrariness of D, $u\in l^{\delta+1}$. Therefore, u satisfies $u_n\to 0$ as $|n|\to\infty$. The existence of a nontrivial homoclinic orbit is obtained. ◀

Proof of Theorem 2. Consider the following boundary problem:

$$\begin{cases} \Delta\left(p_n(\Delta u_{n-1})^\delta\right) - q_n u_n^\delta + f(n, u_{n+1}, u_n, u_{n-1}) = 0, & n \in \mathbf{Z}(-mT, mT), \\ p_{-mT} = p_{mT} = 0,\ q_{-mT} = q_{mT} = 0, \\ p_{-n} = p_n,\ q_{-n} = q_n, & n \in \mathbf{Z}(-mT, mT). \end{cases}$$

Let S be the set of sequences $u = (\cdots, u_{-n}, \cdots, u_{-1}, u_0, u_1, \cdots, u_n, \cdots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \{\{u_n\} | u_n \in \mathbf{R},\ n \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, $\tilde{E}_m$ is defined as a subspace of S by

$$\tilde{E}_m = \{u \in S | u_{-n} = u_n,\ \forall n \in \mathbf{Z}\}.$$

Clearly, $\tilde{E}_m$ is isomorphic to $\mathbf{R}^{2mT+1}$. $\tilde{E}_m$ can be equipped with the inner product

$$\langle u, v\rangle = \sum_{j=-mT}^{mT} u_j v_j,\ \forall u, v \in \tilde{E}_m,$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=-mT}^{mT} u_j^2\right)^{\frac{1}{2}},\ \forall u \in \tilde{E}_m.$$

It is obvious that $\tilde{E}_m$ is a Hilbert space with $2mT+1$-periodicity and linearly homeomorphic to $\mathbf{R}^{2mT+1}$.

Similarly to the proof of Theorem 1, we can also prove Theorem 2. For simplicity, we omit its proof. ◀

4. Example

In this section, we give an example to illustrate our results.

Example 1. *Let*

$$f(n, v_1, v_2, v_3) = \begin{cases} \gamma\,|v_2|^\delta \frac{v_2}{|v_2|}\left(\frac{|v_1|^{\delta+1}+|v_2|^{\delta+1}}{|v_1|^{\delta+1}+|v_2|^{\delta+1}+1} + \frac{|v_2|^{\delta+1}+|v_3|^{\delta+1}}{|v_2|^{\delta+1}+|v_3|^{\delta+1}+1}\right), & \text{if } v_2 \neq 0, \\ 0, & \text{if } v_2 = 0, \end{cases}$$

and

$$F(n, v_1, v_2) = \frac{\gamma}{\delta+1}\left[|v_1|^{\delta+1} + |v_2|^{\delta+1} - \ln\left(|v_1|^{\delta+1} + |v_2|^{\delta+1} + 1\right)\right],$$

where $\gamma > 2^{\delta+1}\bar{p} + \bar{q}$. *It is easy to verify that all the assumptions of Theorem 1 are satisfied. Consequently, a nontrivial homoclinic orbit is obtained.*

Acknowledgements

This project is supported by the National Natural Science Foundation of China (No. 11401121), Natural Science Foundation of Guangdong Province (No. S2013010014460) and Hunan Provincial Natural Science Foundation of China (No. 2015JJ2075)

References

[1] C.D. Ahlbrandt, A.C. Peterson, *Discrete Hamiltonian Systems: Difference Equations, Continued Fraction and Riccati Equations*, Kluwer Academic Publishers, Dordrecht, 1996.

[2] X.C. Cai, J.S. Yu, *Existence theorems of periodic solutions for second-order nonlinear difference equations*, Adv. Difference Equ., **2008**, 2008, 1–11.

[3] A. Castro, R. Shivaji, *Nonnegative solutions to a semilinear Dirichlet problem in a ball are positive and radially symmetric*, Comm. Partial Differential Equations, **14(8-9)**, 1989, 1091-1100.

[4] P. Chen, H. Fang, *Existence of periodic and subharmonic solutions for second-order p-Laplacian difference equations,* Adv. Difference Equ., **2007**, 2007, 1–9.

[5] P. Chen, X.H. Tang, *Existence of infinitely many homoclinic orbits for fourth-order difference systems containing both advance and retardation*, Appl. Math. Comput., **217(9)**, 2011, 4408–4415.

[6] P. Chen, X.H. Tang, *Existence and multiplicity of homoclinic orbits for 2nth-order nonlinear difference equations containing both many advances and retardations*, J. Math. Anal. Appl., **381(2)**, 2011, 485–505.

[7] P. Chen, X.H. Tang, *Infinitely many homoclinic solutions for the second-order discrete p-Laplacian systems*, Bull. Belg. Math. Soc., **20(2)**, 2013, 193–212.

[8] P. Chen, X.H. Tang, *Existence of homoclinic solutions for some second-order discrete Hamiltonian systems*, J. Difference Equ. Appl., **19(4)**, 2013, 633–648.

[9] P. Chen, Z.M. Wang, *Infinitely many homoclinic solutions for a class of nonlinear difference equations*, Electron. J. Qual. Theory Differ. Equ., **47**, 2012, 1–18.

[10] X.Q. Deng, G. Cheng, *Homoclinic orbits for second order discrete Hamiltonian systems with potential changing sign*, Acta Appl. Math., **103(3)**, 2008, 301–314.

[11] J.R. Esteban, J.L. Vázquez, *On the equation of turbulent filtration in one-dimensional porous media*, Nonlinear Anal., **10(11)**, 1986, 1303–1325.

[12] H. Fang, D.P. Zhao, *Existence of nontrivial homoclinic orbits for fourth-order difference equations,* Appl. Math. Comput., **214(1)**, 2009, 163–170.

[13] C.J. Guo, R.P. Agarwal, C.J. Wang, D. O'Regan, *The existence of homoclinic orbits for a class of first order superquadratic Hamiltonian systems*, Mem. Differential Equations Math. Phys., **61**, 2014, 83–102.

[14] C.J. Guo, D. O'Regan, R.P. Agarwal, *Existence of homoclinic solutions for a class of the second-order neutral differential equations with multiple deviating arguments*, Adv. Dyn. Syst. Appl., **5(1)**, 2010, 75–85.

[15] C.J. Guo, D. O'Regan, Y.T. Xu, R.P. Agarwal, *Existence of subharmonic solutions and homoclinic orbits for a class of high-order differential equations*, Appl. Anal., **90(7)**, 2011, 1169–1183.

[16] C.J. Guo, D. O'Regan, Y.T. Xu, R.P. Agarwal, *Homoclinic orbits for a singular second-order neutral differential equation*, J. Math. Anal. Appl., **366(2)**, 2010, 550–560.

[17] C.J. Guo, D. O'Regan, Y.T. Xu, R.P. Agarwal, *Existence and multiplicity of homoclinic orbits of a second-order differential difference equation via variational methods.* Appl. Math. Inform. Mech., **4(1)**, 2012, 1–15.

[18] C.J. Guo, D. O'Regan, Y.T. Xu, R.P. Agarwal, *Existence of homoclinic orbits of a class of second order differential difference equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, **20(6)**, 2013, 675–690.

[19] Z.M. Guo, J.S. Yu, *Existence of periodic and subharmonic solutions for second-order superlinear difference equations*, Sci. China Math., **46(4)**, 2003, 506–515.

[20] Z.M. Guo, J.S. Yu, *The existence of periodic and subharmonic solutions of subquadratic second order difference equations*, J. London Math. Soc., **68(2)**, 2003, 419–430.

[21] X.Z. He, *Oscillatory and asymptotic behavior of second order nonlinear difference equations*, J. Math. Anal. Appl., **175(2)**, 1993, 482–498.

[22] H.G. Kaper, M. Knaap, M.K. Kwong, *Existence theorems for second order boundary value problems.* Differential Integral Equations, **4(3)**, 1991, 543-554.

[23] Y.H. Long, *Homoclinic solutions of some second-order non-periodic discrete systems*, Adv. Difference Equ., **2011**, 2011, 1–12.

[24] Y.H. Long, *Homoclinic orbits for a class of noncoercive discrete Hamiltonian systems*, J. Appl. Math., **2012**, 2012, 1–21.

[25] M.J. Ma, Z.M. Guo, *Homoclinic orbits for second order self-adjoint difference equations*, J. Math. Anal. Appl., **323(1)**, 2006, 513–521.

[26] M.J. Ma, Z.M. Guo, *Homoclinic orbits and subharmonics for nonlinear second order difference equations*, Nonlinear Anal., **67(6)**, 2007, 1737–1745.

[27] J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.

[28] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Amer. Math. Soc., Providence, RI, New York, 1986.

[29] W.M. Tan, Z. Zhou, *Existence of multiple solutions for a class of n-dimensional discrete boundary value problems*, Int. J. Math. Math. Sci., **2010**, 2010, 1-14.

[30] Q. Wang, Z. Zhou, *Solutions of the boundary value problem for a $2n$th-order nonlinear difference equation containing both advance and retardation*, Adv. Difference Equ., **2013**, 2013, 1–9.

[31] Z. Zhou, J.S. Yu, *Homoclinic solutions in periodic nonlinear difference equations with superlinear nonlinearity*, Acta Math. Sin. Engl. Ser., **29(9)**, 2013, 1809-1822.

[32] Z. Zhou, J.S. Yu, *On the existence of homoclinic solutions of a class of discrete nonlinear periodic systems*, J. Differential Equations, **249(5)**, 2010, 1199–1212.

Haiping Shi
Modern Business and Management Department, Guangdong Construction Vocational Technology Institute, Guangzhou 510440, China
E-mail: shp7971@163.com

Xia Liu
Oriental Science and Technology College, Hunan Agricultural University, Changsha 410128, China; Science College, Hunan Agricultural University, Changsha 410128, China
E-mail: xia991002@163.comu

Yuanbiao Zhang
Packaging Engineering Institute, Jinan University, Zhuhai 519070, China
E-mail: abiaoa@163.com

Received 13 June 2015
Accepted 1 September 2015