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Difference and Discrete Equations on a Half-axis and the Wiener–Hopf Method

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Abstract. In this paper we suggest to apply the Wiener–Hopf method for studying solvability of some classes of linear difference and discrete equations. We introduce general concepts, describe main classes of equations under consideration and correlation between discrete and difference equations.

Key Words and Phrases: difference equation, discrete equation, symbol, factorization, Fredholm property.

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1. Introduction

This short report is dedicated to studying some special difference equations of type

$$\sum_{-\infty}^{+\infty} a_k(x)u(x+\beta_k) = v(x), \quad x \in \mathbf{R}, \tag{1}$$

where $\{\beta_k\}_{-\infty}^{+\infty} \subset \mathbf{R}$, $a_k(x)$ are complex-valued given functions on $\mathbf{R}$. Such equations appear in many applied problems, for example, in a control theory and signal processing [15, 16], hence this study may be useful to develop these subjects. For the theory of such equations with constant coefficients we refer the reader to the classical books [7, 8, 9], but for the case of variable coefficients this theory is not complete.

"Differential-difference equation" means that there are two ways for considering the variable x.

We say the equation (1) is a difference equation if x is a continuous variable, and a discrete one if x is a discrete variable, and use $L_2(D)$ as appropriate functional space although such equations can be considered in more general $L_p(D)$-spaces.

A general linear difference equation of order n has the following from [8, 9]:

$$\sum_{k=0}^{n} a_k(x)u(x+k) = v(x), \quad x \in \mathbf{R}, \tag{2}$$

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where the functions $a_k(x), k = 1, \dots, n, v(x)$ are defined on $\mathbf{R}$.

More general type of a finite order difference equation is

$$\sum_{k=0}^{n} a_k(x)u(x+\beta_k) = v(x), \quad x \in \mathbf{R}, \tag{3}$$

where $\{\beta_k\}_{k=0}^n \subset \mathbf{R}$ (such equations were obtained in [10, 11]).

2. Linear difference operators

Definition 1. *The operator*

$$\mathcal{D}: \ u(x) \longmapsto \sum_{-\infty}^{+\infty} a_k(x)u(x+\beta_k), \quad x \in \mathbf{R},$$

is called a linear difference operator, and a function $\sigma(x,\xi)$ represented by the series

$$\sigma(x,\xi) = \sum_{-\infty}^{+\infty} a_k(x)e^{i\beta_k\xi},$$

is called its symbol.

We will assume that $a_k(x), \forall k \in \mathbf{Z}, \sigma(x,\xi)$ are continuous and bounded functions on $\mathbf{R}\times\mathbf{R}$, i.e. $\exists \lim\limits_{\xi\to\infty} \sigma(x,\xi) \in C(\mathbf{R})$, and $\exists\ \sigma(\pm\infty,\xi) = \lim\limits_{x\to\pm\infty} \sigma(x,\xi)$.

Together with the operator $\mathcal{D}$ we consider a family of operators $\mathcal{D}_{x_0}, x_0 \in \mathbf{R}$, where

$$\mathcal{D}_{x_0}: \ u(x) \longmapsto \sum_{-\infty}^{+\infty} a_k(x_0)u(x+\beta_k), \quad x_0 \in \mathbf{R}.$$

Lemma 1. *The operator $\mathcal{D}$ is a linear bounded operator $L_2(\mathbf{R}) \to L_2(\mathbf{R})$ if $\sum\limits_{-\infty}^{+\infty} a_k(x) \in L_\infty(\mathbf{R})$.*

Proof for this assertion easily follows from definition of the space $L_\infty(\mathbf{R})$ and can be obtained immediately.

If we consider the equation (1) with constant coefficients

$$\sum_{k=0}^{n} a_k u(x+\beta_k) = v(x), \quad x \in \mathbf{R}, \tag{4}$$

then it can be easily solved by the Fourier transform:

$$(Fu)(\xi) \equiv \widetilde{u}(\xi) = \int\limits_{-\infty}^{+\infty} e^{-ix\cdot\xi}u(x)dx.$$

Indeed, applying the Fourier transform to (4) we obtain

$$\widetilde{u}(\xi)\sum_{k=0}^{n} a_k e^{i\beta_k \xi} = \widetilde{v}(\xi),$$

or renaming

$$\widetilde{u}(\xi)p_n(\xi) = \widetilde{v}(\xi).$$

The function $p_n(\xi)$ is called a symbol of a difference operator in the left-hand side of (3) (cf. [3]). If $p_n(\xi) \neq 0,\ \forall \xi \in \mathbf{R}$, then the equation (3) can be easily solved:

$$u(x) = F^{-1}_{\xi \to x}\left(p_n^{-1}(\xi)\widetilde{v}(\xi)\right).$$

Let us introduce the following function defined on all operators $\mathcal{D}_{x_0}$ by the formula

$$f(x_0) = ||\mathcal{D}_{x_0}||, \quad x_0 \in \mathbf{R}.$$

Lemma 2. *$f(x)$ is a continuous function on $\mathbf{R}$ and has one-side limits $f(\pm\infty)$ if the function $ess \sup\limits_{\xi \in \mathbf{R}} |\sigma(x,\xi)|$ is the same.*

Proof. Indeed, the operator $\mathcal{D}_{x_0}$ is unitary equivalent to the operator

$$\widetilde{u}(\xi) \longmapsto \sigma(x_0,\xi)\cdot \widetilde{u}(\xi),$$

hence it is a miltiplicator with the function $\sigma(x_0,\xi)$. Further,

$$f(x_0) = ||\mathcal{D}_{x_0}|| = ess \sup_{\xi \in \mathbf{R}} |\sigma(x_0,\xi)|,$$

and thus we obtain the required property. ◄

Definition 2. *The operator $\mathcal{D}_{x_0}$ and its symbol $\sigma(x_0,\xi)$ are called elliptic in the point x_0 if*

$$\sigma(x_0,\xi) \neq 0, \quad \forall \xi \in \mathbf{R},$$

and elliptic ewerywhere if

$$\sigma(x_0,\xi) \neq 0, \quad \forall x_0, \xi \in \mathbf{R}.$$

Lemma 3. *If $\sigma(x_0,\xi) \neq 0, \quad \forall x_0, \xi \in \mathbf{R}$, then the operator $\mathcal{D}_{x_0}$ is invertible in the space $L_2(\mathbf{R})$.*

Proof. Under our assumptions, this property is equivalent to the invertibility of the corresponding multiplier operator. ◄

Definition 3. *An operator $\mathcal{D}$ has a Fredholm property if the dimensions of its kernel and co-kernel are finite, and the difference*

$$Ind\ \mathcal{D} = dim\ Ker\mathcal{D} - dim\ Coker\mathcal{D},$$

is called an index of the operator $\mathcal{D}$.

Lemma 4. *If the continuous family $\{D_{x_0}\}$ consists of invertible operators for all $x_0 \in \dot{\mathbf{R}}$, then the operator $\mathcal{D}$ has a Fredholm property in the space $L_2(\mathbf{R})$.*

Proof. The operators D_{x_0} are local representatives of the operator $\mathcal{D}$ at the point x_0, and although the operator $\mathcal{D}$ is not an operator of a local type, each operator D_{x_0} is an operator of a local type, [5]. It is easily seen in Fourier representation. According to the general theory [5] such operator family has an envelope which is an operator of a local type. Moreover, for operators of a local type the availability of a Fredholm property is equivalent to the availability of a Fredholm property for its local representatives. Since such an envelope is equivalent to the operator D_{x_0} at each point x_0, then our initial operator $\mathcal{D}$ and this envelope differ on a compact operator. Thus, Fredholm indices are the same for these operators.◀

These facts imply that to obtain a Fredholm property for the operator $\mathcal{D}$ we need an invertibility of its local representatives by I.B. Simonenko's terminology [5]. A crucial moment here is obtaining invertibility conditions at infinity. We will use the following notations: $\sigma(+\infty,\xi) \equiv \sigma_1(\xi), \sigma(-\infty,\xi) \equiv \sigma_2(\xi), P_\pm$ is a restriction operator on $\mathbf{R}_\pm, a$ is a multiplicator with the function $a(\xi)$.

The following operator is a local representative of the operator $\mathcal{D}$ at intinity:

$$\mathcal{D}_\infty = \mathcal{D}_{+\infty} \cdot P_+ + \mathcal{D}_{-\infty} \cdot P_-, \tag{5}$$

where $\mathcal{D}_{+\infty}, \mathcal{D}_{-\infty}$ are operators with symbols $\sigma_1(\xi), \sigma_2(\xi)$, and we need invertibility conditions for such operator.

If we apply the Fourier transform to the equation (5), we obtain the Fourier representation for the operator $\mathcal{D}_\infty$:

$$F\mathcal{D}_\infty = \sigma_1 \cdot \Pi_+ + \sigma_2 \cdot \Pi_-, \tag{6}$$

where $\Pi_\pm$ are the Fourier images of operators $P_\pm$ [6]. These operators are singular integral operators.

It is well known that an equation with the operator (6) one-to-one corresponds to the Riemann boundary value problem for upper and lower complex half-planes [1, 2, 3, 4]. Let's remind the statement of that problem. It is required to find a pair of functions $\Phi^\pm(\xi)$ which admit an analytic continuation into upper ($\mathbf{C}_+$) and lower ($\mathbf{C}_-$) half-planes in the complex plane $\mathbf{C}$ and whose boundary values on $\mathbf{R}$ satisfy the following linear relation:

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \tag{7}$$

where $G(\xi), g(\xi)$ are given functions on $\mathbf{R}$.

The correspondence between the Riemann boundary value problem (7) and the singular integral equation with the operator (6)

$$(\sigma_1 \cdot \Pi_+ + \sigma_2 \cdot \Pi_-)U = V,$$

is the following:

$$G(\xi) = \sigma_1^{-1}(\xi)\sigma_2(\xi), \quad V(\xi) = \sigma_1^{-1}(\xi)g(\xi).$$

3. A Fredholm property

To formulate our main result, we introduce the following

Definition 4. *By the factorization of the elliptic symbol* $\sigma(\xi)$ *we mean its representation in the form*

$$\sigma(\xi) = \sigma_+(\xi) \cdot \sigma_-(\xi),$$

where factors $\sigma_\pm(\xi)$ *admit an analytical continuation into upper and lower complex half-planes* $\mathbf{C}_\pm$, *and* $\sigma_\pm(\xi) \in L_\infty(\mathbf{R})$.

Note that such factorization can be constructed effectively with the help of the Cauchy type integral [1, 2, 3, 4]. Let us remind we denote $\sigma(\xi) = \sigma_1^{-1}(\xi)\sigma_2(\xi)$ and use the theory of classical Riemann boundary value problem [1, 2, 3, 4].

If the so-called transmission property

$$\sigma(-\infty) = \sigma(+\infty),$$

holds, then one can use the classical theory of Riemann boundary value problems for a closed curve.

If it is not so, we consider a variation of the argument of $\sigma(\xi)$ when ξ varies from $-\infty$ to $+\infty$, and the function

$$\omega(\xi) = (\xi + i)^{-\delta} \cdot (\xi - i)^{\delta},$$

where

$$\delta = \frac{1}{\pi i} \ln \frac{\sigma(-\infty)}{\sigma(+\infty)}, \quad \alpha = Re\ \delta, \quad \alpha = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} d \arg \sigma(\xi).$$

Thus, the function $\omega(\xi) \cdot \sigma(\xi)$ has vanishing variation of its argument along the real line, and such function has a transmission property. So, we write $\sigma(\xi) = \omega^{-1}(\xi)\omega(\xi) \cdot \sigma(\xi)$, for $\omega^{-1}(\xi)$ we have a special factorization as two natural factors, but for the $\omega(\xi) \cdot \sigma(\xi)$ we use a construction above.

Further, we denote by γ a closed curve in a complex plane $\mathbf{C}$, which is obtained by the following way. We take the image of the function $\sigma(\xi) \equiv \sigma_1^{-1}(\xi)\sigma_2(\xi), \xi \in \mathbf{R}$, and join two points $\sigma(-\infty)$ and $\sigma(+\infty)$ by a line segment.

Definition 5. *The curve* γ *is called non-singular if* $0 \notin \gamma$.

Remark 1. *It means that* $-1/2 < \alpha < 1/2$.

Definition 6. *Winding number* æ *of such non-singular curve* γ *is called an index of the Riemann boundary value problem* (7).

Lemma 5. *The operator* (5) *is invertible in the space* $L_2(\mathbf{R})$ *iff* $\sigma(\xi) \neq 0, \forall \xi \in \dot{\mathbf{R}}$ *and the winding number* æ *is equal to* 0.

Proof. The operators (5) and (6) are unitary equivalent, and the operator (6) is a classical one-dimensional singular integral operator [1, 2, 3, 4], its invertibility conditions can be obtained by different methods. We use the functional-theoretic approach of [4] and corresponding terminology. ◀

Theorem 1. *The operator $\mathcal{D}$ has a Fredholm property and $Ind\ \mathcal{D}$ is vanishing in the space $L_2(\mathbf{R})$ iff $\sigma(x,\xi) \neq 0, \forall x, \xi \in \dot{\mathbf{R}}$ and the winding number æ for $\sigma(\xi)$ is equal to* 0*.*

Proof. Since a Fredholm property for the operator $\mathcal{D}$ is equivalent to a Fredholm property of its local representatives, the conditions of the theorem guarantee the invertibility of all its local representatives. It remains for us to consider the index. The index will be zero because the operator $\mathcal{D}$ is homotopic to any of its local representatives. Indeed, such homotopy can be constructed by the following way. Fix the point $x_0 \in \mathbf{R}$ and consider the following symbol family:

$$\sigma_t(x,\xi) = \sigma((1-t)x_0 + tx_0, \xi), \quad t \in [0,1].$$

Obviously $\sigma_0(x,\xi) = \sigma(x_0,\xi), \sigma_1(x,\xi) = \sigma(x,\xi)$, and all intermediate operators with symbols $\sigma_t(x,\xi)$ have a Fredholm property. ◀

4. Possible Generalizations

Some cases are left unconsidered here. Hopefully we treat them later.

- The first is the one where the winding number æ is not zero.
- It seems multi-dimensional situations for difference-discrete equations are very interesting, there are many surprises in a multi-dimensional space.
- The case of discrete equations should also be considered, we have already some works in this direction [12, 13].

5. Conclusion

The theory presented in this work may be useful for applications and description of real processes. Moreover, perhaps this is the first time when such a theory, related to the factorization technique [17], is applied to difference equations. For properly discrete equations, the corresponding theory of periodic Riemann boundary value problem was developed by the authors in [12, 14].

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