Azerbaijan Journal of Mathematics
V. 6, No 1, 2016, January
ISSN 2218-6816

Int-soft Hyper-MV-deductive Systems in Hyper-MV-algebras

G. Muhiuddin

Abstract. The concepts of int-soft hyper-MV-subalgebras, (weak) int-soft hyper-MV-deductive systems and previously weak int-soft hyper-MV-deductive systems are introduced, and investigate their relations/properties.

Key Words and Phrases: int-soft hyper MV-subalgebra, (weak) int-soft hyper MV-deductive system, previously weak int-soft hyper MV-deductive system.
2010 Mathematics Subject Classifications: 06F35, 03G25, 06D72

1. Introduction

MV-algebras introduced by C. C. Chang [1] in 1958 provide an algebraic proof of completeness theorem of infinite valued Lukasewicz propositional calculus. The hyper structure theory was introduced by F. Marty [6] at the 8th congress of Scandinavian Mathematicians in 1934. Since then, many research articles have been published in these areas. Recently in [3], Sh. Ghorbani, A. Hasankhni and E. Eslami applied the hyper structure to MV-algebras and introduced the concept of a hyper-MV-algebra which is a generalization of an MV-algebra and investigated some related results. Based on [3, 4], L. Torkzadeh and A. Ahadpanah [8] discussed hyper-MV-ideals in hyper MV-algebras. Present authors [5] introduced the notions of (weak) hyper-MV-deductive systems and (weak) implicative hyper-MV-deductive systems, and investigated several properties. They also discussed relations among hyper-MV-deductive systems, weak hyper MV-deductive systems, implicative hyper-MV-deductive systems and weak implicative hyper-MV-deductive systems.

In this paper, we introduce the concepts of int-soft hyper-MV-subalgebras, (weak) int-soft hyper-MV-deductive systems and previously weak int-soft hyper-MV-deductive systems, and investigate their relations/properties.

2. Preliminaries

A *hyper-MV-algebra* (see [2]) is a nonempty set M endowed with a hyper operation "$\oplus$", a unary operation "$*$" and a constant "0" satisfying the following axioms:

(a1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$, (a2) $x \oplus y = y \oplus x$,

(a3) $(x^*)^* = x$,

(a4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,

(a5) $0^* \in x \oplus 0^*$,

(a6) $0^* \in x \oplus x^*$,

(a7) $x \ll y,\ y \ll x \ \Rightarrow\ x = y$,

for all $x, y, z \in M$, where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

For every subsets A and B of M, we define

$$A \ll B \ \Leftrightarrow\ (\exists\, a \in A)\,(\exists\, b \in B)\,(a \ll b),$$

$$A \oplus B = \bigcup_{a \in A, b \in B} a \oplus b.$$

We also define $0^* = 1$ and $A^* = \{a^* \mid a \in A\}$.

Every hyper-MV-algebra M satisfies the following assertions (see [2]):

(b1) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$,

(b2) $0 \ll x, \quad x \ll 1$,

(b3) $x \ll x$,

(b4) $x \ll y \ \Rightarrow\ y^* \ll x^*$,

(b5) $A \ll B \ \Rightarrow\ B^* \ll A^*$,

(b6) $A \ll A$,

(b7) $A \subseteq B \ \Rightarrow\ A \ll B$,

(b8) $x \ll x \oplus y, \quad A \ll A \oplus B$,

(b9) $(A^*)^* = A$,

(b10) $0 \oplus 0 = \{0\}$,

(b11) $x \in x \oplus 0$,

(b12) $y \in x \oplus 0 \ \Rightarrow\ y \ll x$,

(b13) $y \oplus 0 = x \oplus 0 \ \Rightarrow\ x = y$, for all $x, y, z \in M$ and subsets A, B and C of M.

A nonempty subset S of a hyper-MV-algebra M is called a *hyper-MV-subalgebra* (see [2]) of M if S is a hyper-MV-algebra under the hyper operation "$\oplus$" and the unary operation "$*$" on M.

Definition 1 ([5]). *A nonempty subset D of M is called a weak hyper-MV-deductive system of M if it satisfies:*

(d1) $0 \in D$,

(d2) $(\forall x, y \in M)$ $((x^* \oplus y)^* \subseteq D,\ y \in D \ \Rightarrow\ x \in D)$.

Definition 2 ([5]). *A nonempty subset D of M is called a hyper-MV-deductive system of M if it satisfies (d1) and*

(d3) $(\forall x, y \in M)((x^* \oplus y)^* \ll D,\ y \in D \ \Rightarrow\ x \in D)$.

Note that every hyper-MV-deductive system is a weak hyper-MV-deductive system, but the converse is not true (see [5, Theorem 3.10 and Example 3.11]).

In what follows, let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 3 ([7]). *A soft set $\left(\tilde{f}, A\right)$ of E (over U) is defined to be the set of ordered pairs*

$$\left(\tilde{f}, A\right) := \left\{\left(x, \tilde{f}(x)\right) : x \in E,\ \tilde{f}(x) \in P(U)\right\},$$

where $\tilde{f} : E \to P(U)$ such that $\tilde{f}(x) = \emptyset$ if $x \notin A$.

For a soft set $\left(\tilde{f}, M\right)$ of M (over U), the set

$$i_M\left(\tilde{f}; \gamma\right) = \left\{x \in M \mid \gamma \subseteq \tilde{f}(x)\right\},$$

is called the γ-*inclusive set* of $\left(\tilde{f}, M\right)$.

For any soft sets $\left(\tilde{f}, M\right)$ and $(\tilde{g}, M)$ of M, we define

$\left(\tilde{f}, M\right) \tilde{\subseteq} (\tilde{g}, M)$ if $\tilde{f}(x) \subseteq \tilde{g}(x)$ for all $x \in S$.

The *soft union* of $\left(\tilde{f}, M\right)$ and $(\tilde{g}, M)$, denoted by $\left(\tilde{f}, M\right) \tilde{\cup} (\tilde{g}, M)$, is defined to be the soft set $\left(\tilde{f} \tilde{\cup} \tilde{g}, M\right)$ of M (over U) in which $\tilde{f} \tilde{\cup} \tilde{g}$ is defined by

$$\left(\tilde{f} \tilde{\cup} \tilde{g}\right)(x) = \tilde{f}(x) \cup \tilde{g}(x) \text{ for all } x \in M.$$

The *soft intersection* of $\left(\tilde{f}, M\right)$ and $(\tilde{g}, M)$, denoted by $\left(\tilde{f}, M\right) \tilde{\cap} (\tilde{g}, M)$, is defined to be the soft set $\left(\tilde{f} \tilde{\cap} \tilde{g}, M\right)$ of M (over U) in which $\tilde{f} \tilde{\cap} \tilde{g}$ is defined by

$$\left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x) = \tilde{f}(x) \cap \tilde{g}(x) \text{ for all } x \in S.$$

A soft set $\left(\tilde{f}, M\right)$ of M is said to satisfy the *intersection property* if for any subset T of M there exists $x_0 \in T$ such that $\tilde{f}(x_0) = \bigcap_{x \in T} \tilde{f}(x)$.

3. Int-soft hyper-MV-subalgebras and int-soft (weak) hyper-MV-deductive systems

In what follows let M denote a hyper-MV-algebra unless otherwise specified.

Definition 4. *A soft set $\left(\tilde{f}, M\right)$ of M is called an int-soft hyper-MV-subalgebra of M if it satisfies:*

$$(\forall x \in M)\left(\tilde{f}(x) \subseteq \tilde{f}(x^*)\right), \tag{1}$$

$$(\forall x, y \in M)\left(\bigcap_{a \in x \oplus y} \tilde{f}(a) \supseteq \tilde{f}(x) \cap \tilde{f}(y)\right). \tag{2}$$

Example 1. *Let $M = \{0, a, 1\}$ be a set with the hyper operation "$\oplus$" and the unary operation "$*$" which are given by Table 1.*

Table 1: $\oplus$-multiplication and unary operation

$\oplus$	0	a	1
0	$\{0\}$	$\{a\}$	$\{1\}$
a	$\{a\}$	$\{0, a, 1\}$	$\{0, a, 1\}$
1	$\{1\}$	$\{0, a, 1\}$	$\{1\}$

x	x^*
0	1
a	a
1	0

Then $(M, \oplus, ^*, 0)$ is a hyper-MV-algebra (see [2]). Let $\left(\tilde{f}, M\right)$ be a soft set of M in which

$$\tilde{f}(x) := \begin{cases} \alpha & \text{if } x \in \{0, 1\}, \\ \beta & \text{if } x = a, \end{cases}$$

where $\alpha \supsetneq \beta$ in $P(U)$. It is easy to check that $\left(\tilde{f}, M\right)$ is an int-soft hyper-MV-subalgebra of M.

Proposition 1. *Every int-soft hyper-MV-subalgebra* $\left(\tilde{f}, M\right)$ *of* M *satisfies the following inclusion:*

$$(\forall x \in M)\left(\tilde{f}(x) \subseteq \tilde{f}(1)\right). \tag{3}$$

Proof. Since $1 = 0^* \in x^* \oplus x$ for all $x \in M$, it follows from (1) and (2) that

$$\tilde{f}(1) \supseteq \bigcap_{a \in x^* \oplus x} \tilde{f}(a) \supseteq \tilde{f}(x^*) \cap \tilde{f}(x) \supseteq \tilde{f}(x) \cap \tilde{f}(x) = \tilde{f}(x),$$

for all $x \in M$. This completes the proof. ◀

We provide a characterization of an int-soft hyper-MV-subalgebra.

Theorem 1. *Let* $\left(\tilde{f}, M\right)$ *be a soft set of* M. *Then the following are equivalent.*

(1) $\left(\tilde{f}, M\right)$ *is an int-soft hyper-MV-subalgebra of* M.

(2) $(\forall \gamma \in P(U))\left(i_M\left(\tilde{f}; \gamma\right) \neq \emptyset \ \Rightarrow \ i_M\left(\tilde{f}; \gamma\right) \text{ is a hyper-MV-subalgebra of } M\right)$.

We say that $i_M\left(\tilde{f}; \gamma\right)$ is an *inclusive hyper-MV-subalgebra* of $\left(\tilde{f}, M\right)$ in M.

Proof. Assume that $\left(\tilde{f}, M\right)$ is an int-soft hyper-MV-subalgebra of M. Let $\gamma \in P(U)$ be such that $i_M\left(\tilde{f}; \gamma\right) \neq \emptyset$. Let $x, y \in i_M\left(\tilde{f}; \gamma\right)$. Then $\gamma \subseteq \tilde{f}(x)$ and $\gamma \subseteq \tilde{f}(y)$. Using (1), we have $\tilde{f}(x^*) \supseteq \tilde{f}(x) \supseteq \gamma$ and so $x^* \in i_M\left(\tilde{f}; \gamma\right)$. Let $a \in x \oplus y$. Using (2), we get

$$\tilde{f}(a) \supseteq \bigcap_{b \in x \oplus y} \tilde{f}(b) \supseteq \tilde{f}(x) \cap \tilde{f}(y) \supseteq \gamma,$$

and so $a \in i_M\left(\tilde{f}; \gamma\right)$. Hence $x \oplus y \subseteq i_M\left(\tilde{f}; \gamma\right)$. Therefore $i_M\left(\tilde{f}; \gamma\right)$ is a hyper-MV-subalgebra of M.

Conversely, let $\gamma \in P(U)$ be such that $i_M\left(\tilde{f}; \gamma\right) \neq \emptyset$ and $i_M\left(\tilde{f}; \gamma\right)$ is a hyper-MV-subalgebra of M. For any $x \in M$, let $\tilde{f}(x) = \gamma$. Then $x \in i_M\left(\tilde{f}; \gamma\right)$, and so $x^* \in i_M\left(\tilde{f}; \gamma\right)$ since $i_M\left(\tilde{f}; \gamma\right)$ is a hyper-MV-subalgebra of M. Hence $\tilde{f}(x^*) \supseteq \gamma = \tilde{f}(x)$. For any $x, y \in M$, let $\tilde{f}(x) \cap \tilde{f}(y) = \gamma$. Then $x, y \in i_M\left(\tilde{f}; \gamma\right)$, and thus $x \oplus y \subseteq i_M\left(\tilde{f}; \gamma\right)$ because $i_M\left(\tilde{f}; \gamma\right)$ is a hyper-MV-subalgebra of M. It follows that $a \in i_M\left(\tilde{f}; \gamma\right)$ for any $a \in x \oplus y$ and so that $\tilde{f}(a) \supseteq \gamma = \tilde{f}(x) \cap \tilde{f}(y)$ for all $a \in x \oplus y$. Therefore

$$\bigcap_{a \in x \oplus y} \tilde{f}(a) \supseteq \tilde{f}(x) \cap \tilde{f}(y),$$

and consequently $\left(\tilde{f}, M\right)$ is an int-soft hyper-MV-subalgebra of M. ◀

Theorem 2. *Any hyper-MV-subalgebra of M can be realized as an inclusive hyper-MV-subalgebra of some int-soft hyper-MV-subalgebra of* $\left(\tilde{f}, M\right)$.

Proof. Straightforward. ◄

Theorem 3. *If* $\left(\tilde{f}, M\right)$ *and* $(\tilde{g}, M)$ *are int-soft hyper-MV-subalgebras of M, then so is* $\left(\tilde{f}, M\right) \tilde{\cap} (\tilde{g}, M)$.

Proof. For any $x, y \in M$, we have

$$\left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x^*) = \tilde{f}(x^*) \cap \tilde{g}(x^*) \supseteq \tilde{f}(x) \cap \tilde{g}(x) = \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x),$$

and

$$\begin{aligned}
\bigcap_{a \in x \oplus y} \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(a) &= \bigcap_{a \in x \oplus y} \left(\tilde{f}(a) \cap \tilde{g}(a)\right) \\
&= \left(\bigcap_{a \in x \oplus y} \tilde{f}(a)\right) \cap \left(\bigcap_{a \in x \oplus y} \tilde{g}(a)\right) \\
&\supseteq \left(\left(\tilde{f}(x) \cap \tilde{f}(y)\right) \cap (\tilde{g}(x) \cap \tilde{g}(y))\right) \\
&= \left(\left(\tilde{f}(x) \cap \tilde{g}(x)\right) \cap \left(\tilde{f}(y) \cap \tilde{g}(y)\right)\right) \\
&= \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x) \cap \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(y).
\end{aligned}$$

Therefore $\left(\tilde{f}, M\right) \tilde{\cap} (\tilde{g}, M)$ is an int-soft hyper-MV-subalgebra of M. ◄

Definition 5. *A soft set* $\left(\tilde{f}, M\right)$ *of M is called a weak int-soft hyper-MV-deductive system of M if it satisfies the following conditions*

$$(\forall x \in M)\left(\tilde{f}(x) \subseteq \tilde{f}(0)\right). \tag{4}$$

$$(\forall x, y \in M)\left(\tilde{f}(x) \supseteq \left(\bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a)\right) \cap \tilde{f}(y)\right). \tag{5}$$

Example 2. *Consider a hyper-MV-algebra* $(M = \{0, a, 1\}, \oplus, *, 0)$ *with the hyper operation "$\oplus$" and the unary operation "$*$" which are given by Table 2.*

Let $\left(\tilde{f}, M\right)$ *be a soft set of M in which*

$$\tilde{f}(x) := \begin{cases} \gamma_1 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = a, \\ \gamma_3 & \text{if } x = 1, \end{cases}$$

where $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3$ *in* $P(U)$. *Then* $\left(\tilde{f}, M\right)$ *is a weak int-soft hyper-MV-deductive system of M.*

Table 2: $\oplus$-multiplication and unary operation

$\oplus$	0	a	1
0	$\{0\}$	$\{0,a\}$	$\{0,1\}$
a	$\{0,a\}$	$\{0,a,1\}$	$\{0,a,1\}$
1	$\{0,1\}$	$\{0,a,1\}$	$\{0,1\}$

x	x^*
0	1
a	a
1	0

Example 3. *Let $X = \{0, a, 1\}$ be a hyper-MV-algebra which is given in Example 1. Let $\left(\tilde{f}, M\right)$ be a soft set of M in which*

$$\tilde{f}(x) := \begin{cases} \gamma_1 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = a, \\ \gamma_3 & \text{if } x = 1, \end{cases}$$

where $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3$ in $P(U)$. Then $\left(\tilde{f}, M\right)$ is not a weak int-soft hyper-MV-deductive system of M since

$$\tilde{f}(1) = \gamma_3 \subsetneq \gamma_2 = \left(\bigcap_{b\in(1^*\oplus a)^*} \tilde{f}(b)\right) \cap \tilde{f}(a).$$

Proposition 2. *Every weak int-soft hyper-MV-deductive system $\left(\tilde{f}, M\right)$ of M satisfies the following assertion.*

$$(\forall x, y \in M) \left(\tilde{f}(y^*) \supseteq \left(\bigcap_{a\in(x^*\oplus y)^*} \tilde{f}(a)\right) \cap \tilde{f}(x^*)\right). \tag{6}$$

Proof. It follows from (a2), (a3) and (5). ◄

Theorem 4. *If $\left(\tilde{f}, M\right)$ and $(\tilde{g}, M)$ are weak int-soft hyper-MV-deductive systems of M, then so is $\left(\tilde{f}, M\right) \tilde{\cap} (\tilde{g}, M)$.*

Proof. For any $x, y \in M$, we have

$$(\tilde{f} \tilde{\cap} \tilde{g})(x) = \tilde{f}(x) \cap \tilde{g}(x) \subseteq \tilde{f}(0) \cap \tilde{g}(0) = \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(0),$$

and

$$\begin{aligned}\left(\tilde{f}\,\tilde{\cap}\,\tilde{g}\right)(x) &= \tilde{f}(x)\cap\tilde{g}(x)\\ &\supseteq \left(\left(\bigcap_{a\in(x^*\oplus y)^*}\tilde{f}(a)\right)\cap\tilde{f}(y)\right)\cap\left(\left(\bigcap_{a\in(x^*\oplus y)^*}\tilde{g}(a)\right)\cap\tilde{g}(y)\right)\\ &= \left(\left(\bigcap_{a\in(x^*\oplus y)^*}\tilde{f}(a)\right)\cap\left(\bigcap_{a\in(x^*\oplus y)^*}\tilde{g}(a)\right)\right)\cap\left(\tilde{f}(y)\cap\tilde{g}(y)\right)\\ &= \left(\bigcap_{a\in(x^*\oplus y)^*}\left(\tilde{f}(a)\cap\tilde{g}(a)\right)\right)\cap\left(\tilde{f}\,\tilde{\cap}\,\tilde{g}\right)(y)\\ &= \left(\bigcap_{a\in(x^*\oplus y)^*}\left(\tilde{f}\,\tilde{\cap}\,\tilde{g}\right)(a)\right)\cap\left(\tilde{f}\,\tilde{\cap}\,\tilde{g}\right)(y).\end{aligned}$$

Therefore $\left(\tilde{f}, M\right)\,\tilde{\cap}\,(\tilde{g}, M)$ is a weak int-soft hyper-MV-deductive system of M. ◀

The following example shows that the soft union $\left(\tilde{f}, M\right)\,\tilde{\cup}\,(\tilde{g}, M)$ of two weak int-soft hyper-MV-deductive systems $\left(\tilde{f}, M\right)$ and $(\tilde{g}, M)$ may not be a weak int-soft hyper-MV-deductive system.

Example 4. *Let $M = \{0, a, b, 1\}$ be a set with the hyper operation "$\oplus$" and the unary operation "$*$" which are given by Table 3.*

Table 3: $\oplus$-multiplication and unary operation

$\oplus$	0	a	b	1
0	$\{0\}$	$\{0, a, b\}$	$\{0, b\}$	$\{0, a, b, 1\}$
a	$\{0, a, b\}$	$\{0, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$
b	$\{0, b\}$	$\{0, a, b, 1\}$	$\{b\}$	$\{0, a, b, 1\}$
1	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$	$\{0, a, b, 1\}$

x	x^*
0	1
a	b
b	a
1	0

Then $(M, \oplus, ^*, 0)$ is a hyper-MV-algebra. For any $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3 \supsetneq \gamma_4$ in $P(U)$, define two soft sets $\left(\tilde{f}, M\right)$ and $(\tilde{g}, M)$ of M by

$$\tilde{f} = \begin{pmatrix} 0 & a & b & 1 \\ \gamma_1 & \gamma_4 & \gamma_4 & \gamma_3 \end{pmatrix},$$

and

$$\tilde{g} = \begin{pmatrix} 0 & a & b & 1 \\ \gamma_2 & \gamma_2 & \gamma_4 & \gamma_4 \end{pmatrix},$$

respectively. Then $\left(\tilde{f}, M\right)$ *and* $(\tilde{g}, M)$ *are weak int-soft hyper-MV-deductive systems of* M*. The soft union* $\left(\tilde{f}, M\right) \tilde{\cup} (\tilde{g}, M)$ *of* $\left(\tilde{f}, M\right)$ *and* $(\tilde{g}, M)$ *is represented by*

$$\tilde{f} \cup \tilde{g} = \begin{pmatrix} 0 & a & b & 1 \\ \gamma_1 & \gamma_2 & \gamma_4 & \gamma_3 \end{pmatrix},$$

which is not a weak int-soft hyper-MV-deductive system of M *since*

$$(\tilde{f} \tilde{\cup} \tilde{g})(b) = \gamma_4 \not\supseteq \gamma_3 = \left(\bigcap_{z \in (b^* \oplus a)^*} \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(z) \right) \cap \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(a).$$

We provide a characterization of a weak int-soft hyper-MV-deductive system.

Theorem 5. *Let* $\left(\tilde{f}, M\right)$ *be a soft set of* M*. Then* $\left(\tilde{f}, M\right)$ *is a weak int-soft hyper-MV-deductive system of* M *if and only if* $i_M\left(\tilde{f}; \gamma\right)$ *is a weak hyper-MV-deductive system of* M *whenever* $i_M\left(\tilde{f}; \gamma\right) \neq \emptyset$ *for* $\gamma \in P(U)$*.*

We say that $i_M\left(\tilde{f}; \gamma\right)$ is an *inclusive weak hyper-MV-deductive system* of $\left(\tilde{f}, M\right)$ in M.

Proof. Assume that $\left(\tilde{f}, M\right)$ is a weak int-soft hyper-MV-deductive system of M. Let $\gamma \in P(U)$ be such that $i_M\left(\tilde{f}; \gamma\right) \neq \emptyset$. Obviously $0 \in i_M\left(\tilde{f}; \gamma\right)$. Let $x, y \in M$ be such that $(x^* \oplus y)^* \subseteq i_M\left(\tilde{f}; \gamma\right)$ and $y \in i_M\left(\tilde{f}; \gamma\right)$. Then $\tilde{f}(y) \supseteq \gamma$ and $\tilde{f}(a) \supseteq \gamma$ for all $a \in (x^* \oplus y)^*$. Thus $\bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a) \supseteq \gamma$, which implies from (5) that

$$\tilde{f}(x) \supseteq \left(\bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a) \right) \cap \tilde{f}(y) \supseteq \gamma.$$

Hence $x \in i_M\left(\tilde{f}; \gamma\right)$ and therefore $i_M\left(\tilde{f}; \gamma\right)$ is a weak hyper-MV-deductive system of M whenever $i_M\left(\tilde{f}; \gamma\right) \neq \emptyset$ for $\gamma \in P(U)$.

Conversely, suppose that $i_M\left(\tilde{f}; \gamma\right)$ is a weak hyper-MV-deductive system of M for all $\gamma \in P(U)$ with $i_M\left(\tilde{f}; \gamma\right) \neq \emptyset$. Clearly, $\tilde{f}(0) \supseteq \tilde{f}(x)$ for all $x \in M$. Let $\gamma = \left(\bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a) \right) \cap \tilde{f}(y)$. Then $\tilde{f}(y) \supseteq \gamma$ and for each $a \in (x^* \oplus y)^*$ we have

$$\tilde{f}(a) \supseteq \bigcap_{b \in (x^* \oplus y)^*} \tilde{f}(b) \supseteq \left(\bigcap_{b \in (x^* \oplus y)^*} \tilde{f}(b) \right) \cap \tilde{f}(y) = \gamma,$$

and so $a \in i_M\left(\tilde{f};\gamma\right)$. Thus $(x^* \oplus y)^* \subseteq i_M\left(\tilde{f};\gamma\right)$. It follows from (d2) that $x \in i_M\left(\tilde{f};\gamma\right)$, that is,

$$\tilde{f}(x) \supseteq \gamma = \left(\bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a)\right) \cap \tilde{f}(y).$$

Therefore $\left(\tilde{f}, M\right)$ is a weak int-soft hyper-MV-deductive system of M. ◀

Definition 6. *A soft set $\left(\tilde{f}, M\right)$ of M is called an int-soft hyper-MV-deductive system of M if it satisfies the condition* (5) *and*

$$(\forall x, y \in M)\left(x \ll y \;\Rightarrow\; \tilde{f}(x) \supseteq \tilde{f}(y)\right). \tag{7}$$

Example 5. *Consider a MV-algebra M which is given in Example 2. Let $\left(\tilde{f}, M\right)$ be a soft set in M defined by*

$$\tilde{f} = \begin{pmatrix} 0 & a & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix},$$

where $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3$ in $P(U)$. Then $\left(\tilde{f}, M\right)$ is an int-soft hyper-MV-deductive system of M.

Let $\tilde{f}$ be an int-soft hyper-MV-deductive system of M. Since $0 \ll x$ for all $x \in M$, it follows from (7) that $\tilde{f}(0) \supseteq \tilde{f}(x)$ for all $x \in M$. Hence every int-soft hyper-MV-deductive system of M is a weak int-soft hyper-MV-deductive system of M. But the converse is not valid as seen in the following example.

Example 6. *Let M be a hyper-MV-algebra in Example 2. Let $\left(\tilde{f}, M\right)$ be a soft set of M defined by*

$$\tilde{f} = \begin{pmatrix} 0 & a & 1 \\ \gamma_1 & \gamma_3 & \gamma_2 \end{pmatrix},$$

where $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3$ in $P(U)$. Then $\left(\tilde{f}, M\right)$ is a weak int-soft hyper-MV-deductive system of M, but not an int-soft hyper-MV-deductive system of M since $a \ll 1$ but $\tilde{f}(a) = \gamma_3 \subsetneq \gamma_2 = \tilde{f}(1)$.

Lemma 1. *Every hyper-MV-deductive system D of M have the following condition:*

$$(\forall x, y \in M)\left(x \ll y,\; y \in D \;\Rightarrow\; x \in D\right). \tag{8}$$

Theorem 6. *Let $\left(\tilde{f}, M\right)$ be a soft set in M such that $i_M\left(\tilde{f};\gamma\right) \neq \emptyset$ for all $\gamma \in P(U)$. If $i_M\left(\tilde{f};\gamma\right)$ is a hyper-MV-deductive system of M for all $\gamma \in P(U)$, then $\left(\tilde{f}, M\right)$ is an int-soft hyper-MV-deductive system of M.*

Proof. Let $x, y \in M$ be such that $x \ll y$. Since $y \in i_M\left(\tilde{f}; \tilde{f}(y)\right)$ and $i_M\left(\tilde{f}; \tilde{f}(y)\right)$ is a hyper-MV-deductive system of M, it follows from (8) that $x \in i_M\left(\tilde{f}; \tilde{f}(y)\right)$. Hence $\tilde{f}(x) \supseteq \tilde{f}(y)$. For every $x, y \in M$, let

$$\gamma := \left(\bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a)\right) \cap \tilde{f}(y).$$

Then $y \in i_M\left(\tilde{f}; \gamma\right)$, and for each $b \in (x^* \oplus y)^*$ we have

$$\tilde{f}(b) \supseteq \bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a) \supseteq \left(\bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a)\right) \cap \tilde{f}(y) = \gamma.$$

Thus $b \in i_M\left(\tilde{f}; \gamma\right)$, i.e., $(x^* \oplus y)^* \subseteq i_M\left(\tilde{f}; \gamma\right)$ and so $(x^* \oplus y)^* \ll i_M\left(\tilde{f}; \gamma\right)$ by (b7). Since $i_M\left(\tilde{f}; \gamma\right)$ is a hyper-MV-deductive system of M, we obtain $x \in i_M\left(\tilde{f}; \gamma\right)$ by (d3). Therefore

$$\tilde{f}(x) \supseteq \gamma = \left(\bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a)\right) \cap \tilde{f}(y).$$

This completes the proof. ◄

Definition 7. *A soft set $\tilde{f}$ of M is called a previously weak int-soft hyper-MV-deductive system of M if it satisfies the condition* (4) *and*

$$(\forall x, y \in M)\,(\exists\, a \in (x^* \oplus y)^*)\left(\tilde{f}(x) \supseteq \tilde{f}(a) \cap \tilde{f}(y)\right). \tag{9}$$

Example 7. *The weak int-soft hyper-MV-deductive system $\tilde{f}$ of M in Example 2 is a previously weak int-soft hyper-MV-deductive system of M.*

Theorem 7. *Every previously weak int-soft hyper-MV-deductive system is a weak int-soft hyper-MV-deductive system.*

Proof. Let $\left(\tilde{f}, M\right)$ be a previously weak int-soft hyper-MV-deductive system of M and let $x, y \in M$. Then there exists $a \in (x^* \oplus y)^*$ such that $\tilde{f}(x) \supseteq \tilde{f}(a) \cap \tilde{f}(y)$. Note that $\tilde{f}(a) \supseteq \bigcap_{b \in (x^* \oplus y)^*} \tilde{f}(b)$, and so

$$\tilde{f}(x) \supseteq \left(\bigcap_{b \in (x^* \oplus y)^*} \tilde{f}(b)\right) \cap \tilde{f}(y).$$

Hence $\left(\tilde{f}, M\right)$ is a weak int-soft hyper-MV-deductive system of M. ◄

Theorem 8. *Let* $\left(\tilde{f}, M\right)$ *be a weak int-soft hyper-*MV*-deductive system of* M. *If* $\left(\tilde{f}, M\right)$ *satisfies the intersection property, then* $\left(\tilde{f}, M\right)$ *is a previously weak int-soft hyper-*MV*-deductive system of* M.

Proof. Since $\left(\tilde{f}, M\right)$ satisfies the intersection property, there exists $b \in (x^* \oplus y)^*$ such that $\tilde{f}(b) = \bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a)$. It follows from (5) that

$$\tilde{f}(x) \supseteq \left(\bigcap_{a \in (x^* \oplus y)^*} \tilde{f}(a) \right) \cap \tilde{f}(y) = \tilde{f}(b) \cap \tilde{f}(y).$$

Hence $\left(\tilde{f}, M\right)$ is a previously weak int-soft hyper-MV-deductive system of M. ◀

Corollary 1. *Every int-soft hyper-*MV*-deductive system satisfying the intersection property is a previously weak int-soft hyper-*MV*-deductive system.*

Theorem 9. *If* $\left(\tilde{f}, M\right)$ *is an int-soft hyper-*MV*-deductive system of* M *satisfying the intersection property, then the* γ*-inclusive set* $i_M\left(\tilde{f}; \gamma\right)$ *is a hyper-*MV*-deductive system of* M *for all* $\gamma \in P(U)$ *with* $i_M\left(\tilde{f}; \gamma\right) \neq \emptyset$.

Proof. Assume that $i_M\left(\tilde{f}; \gamma\right) \neq \emptyset$ for $\gamma \in P(U)$. Then there exists $a \in i_M\left(\tilde{f}; \gamma\right)$ and so $\tilde{f}(a) \supseteq \gamma$. Hence $\tilde{f}(0) \supseteq \tilde{f}(a) \supseteq \gamma$, i.e., $0 \in i_M\left(\tilde{f}; \gamma\right)$. Let $x, y \in M$ be such that $(x^* \oplus y)^* \ll i_M\left(\tilde{f}; \gamma\right)$ and $y \in i_M\left(\tilde{f}; \gamma\right)$. Then there exist $w \in (x^* \oplus y)^*$ and $z \in i_M\left(\tilde{f}; \gamma\right)$ such that $w \ll z$. Note that $\left(\tilde{f}, M\right)$ is a weak int-soft hyper-MV-deductive system of M satisfying the intersection property. Thus $\left(\tilde{f}, M\right)$ is a previously weak int-soft hyper-MV-deductive system of M by Theorem 8. Using (7) and (9), we have

$$\tilde{f}(x) \supseteq \tilde{f}(w) \cap \tilde{f}(y) \supseteq \tilde{f}(z) \cap \tilde{f}(y) \supseteq \gamma$$

and thus $x \in i_M\left(\tilde{f}; \gamma\right)$. Therefore $i_M\left(\tilde{f}; \gamma\right)$ is a hyper-MV-deductive system of M. ◀

Acknowledgements

The author wishes to thank the anonymous reviewers for their valuable suggestions. Also, the author would like to acknowledge financial support for this work, from the Deanship of Scientific Research (DRS), University of Tabuk, Tabuk, Saudi Arabia, under grant no. S/0123/1436.

References

[1] C.C. Chang, *Algebraic analysis of many valued logic*, Trans. Amer. Math. Soc., **88**, 1958, 467–490.

[2] Sh. Ghorbani, *On hyper MV-algebras, Ph. D. thesis*, Shahbid Bahonar University of Kerman, 2007.

[3] Sh. Ghorbani, A. Hasankhani, E. Eslami, *Hyper MV-algebras*, Set-Valued Math. Appl., **1**, 2008, 205–222.

[4] Sh. Ghorbani, E. Eslami, A. Hasankhani, *Quotient hyper MV-algebras*, Sci. Math. Jpn., **66(3)**, 2007, 371–386.

[5] Y.B. Jun, M.S. Kang, H.S. Kim, *Hyper MV-deductive systems of hyper MV-algebras*, Commun. Korean Math. Soc., **25**, 2010, 537–545.

[6] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm, 1934, 45–49.

[7] D. Molodtsov, *Soft set theory - First results*, Comput. Math. Appl., **37**, 1999, 19–31.

[8] L. Torkzadeh, A. Ahadpanah, *Hyper MV-ideals in hyper MV-algebras*, Math. Log. Quart., **56**, 2010, 51–62.

G. Muhiuddin
Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia
E-mail: chishtygm@gmail.com

Received 15 February 2015
Accepted 23 June 2015