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On a Uniform Approximation of Entire Function Associated with the Riemann Zeta Function

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Abstract. In this paper, some uniform approximations of entire functions associated with the Riemann zeta function are presented.

Key Words and Phrases: Riemann zeta function, entire function, approximation.

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For $Rez > 1$, the Riemann zeta function $\varsigma(z)$ is defined by the equality

$$\varsigma(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

The definition implies that $\varsigma(z)$ is a regular function in the half-plane $Rez > 1$. As is known (see, e.g., [1-4]), the function $\xi(z) = \frac{1}{2} z(z-1)\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\varsigma(z)$ is an entire function and $\xi(z) = \xi(1-z)$, i.e. the function $\varsigma(z)$ is analytically continued to the whole complex plane except $z = 1$, where it has a simple pole. Consequently, the entire function $\Xi(z) = \xi\left(\frac{1}{2} + iz\right)$ is even. In 1859, Riemann proposed a hypothesis that all the zeros of the function $\Xi(z)$ are real, which is still not proven.

It is well known (see, e.g., [5, §3.4.4, Problem 203]) that to prove the reality of all the zeros of some entire function, it is sufficient to show that it can be uniformly approximated on any compact set of the complex plane by entire functions having only real zeros. This is not difficult to deduce from the Rouche's or the Hurwitz's theorem (see [1, §3.45]).

In the present paper, a uniform approximation of the function $\Xi(z)$ on any compact set K of the complex plane $\mathbb{C}$ is considered.

We'll use the following representation of the function $\Xi(z)$ (see [2, §10.1]):

$$\Xi(z) = 2\int_0^{+\infty} \Phi(u)\cos uz du, \tag{1}$$

where

$$\Phi(u) = 2\sum_{n=1}^{\infty}\left(2n^4\pi^2 e^{\frac{9}{2}u} - 3n^2\pi e^{\frac{5}{2}u}\right) e^{-\pi n^2 e^{2u}}.$$

Denote

$$P_{2m}(u)=\sum_{s=0}^{2m}\frac{u^s}{s!},\,P^+_{2m}(u)=\sum_{s=0}^{m}\frac{u^{2s}}{(2s)\,!},\,P^-_{2m}(u)=\sum_{s=1}^{m}\frac{u^{2s-1}}{(2s-1)\,!},$$

and transform an even function $\Phi(u)$:

$$2\Phi(u)=\Phi(u)+\Phi(-u)=2\sum_{n=1}^{\infty}\left(2n^4\pi^2e^{\frac{9}{2}u}-3n^2\pi e^{\frac{5}{2}u}\right)\,e^{-\pi n^2e^{2u}}+$$

$$+2\sum_{n=1}^{\infty}\left(2n^4\pi^2e^{-\frac{9}{2}u}-3n^2\pi e^{-\frac{5}{2}u}\right)\,e^{-\pi n^2e^{-2u}}=$$

$$=2\sum_{n=1}^{\infty}\left(2n^4\pi^2e^{\frac{9}{2}u}-3n^2\pi e^{\frac{5}{2}u}\right)\,e^{-\pi n^2\left[e^{2u}-P_{2m}(2u)\right]-n^2\pi P_{2m}(2u)}+$$

$$+2\sum_{n=1}^{\infty}\left(2n^4\pi^2e^{-\frac{9}{2}u}-3n^2\pi e^{-\frac{5}{2}u}\right)\,e^{-\pi n^2\left[e^{-2u}-P_{2m}(-2u)\right]-n^2\pi P_{2m}(-2u)}=$$

$$=\Phi^+_m(u)+2R_m(u)+2R_m(-u)\,, \tag{2}$$

where

$$\Phi^+_m(u)=$$

$$=4\sum_{n=1}^{\infty}e^{-\pi n^2P^+_{2m}(2u)}\left\{2n^4\pi^2cosh\left(\pi n^2P^-_{2m}(2u)-\frac{9}{2}u\right)-3\pi n^2cosh\left(\pi n^2P^-_{2m}(2u)-\frac{5}{2}u\right)\right\}, \tag{3}$$

$$R_m(u)=$$

$$=\sum_{n=1}^{\infty}\left(2n^4\pi^2e^{\frac{9}{2}u}-3n^2\pi e^{\frac{5}{2}u}\right)\,\left(e^{-\pi n^2\left[e^{2u}-P_{2m}(2u)\right]}-1\right)\,e^{-n^2\pi P_{2m}(2u)}. \tag{4}$$

Denote

$$\Xi_m(z)=\int_0^{+\infty}\Phi^+_m(u)\cos uzdu. \tag{5}$$

Theorem 1. *For every compact $K\subset\mathbb{C}$,*

$$\lim_{m\to\infty}\sup_{z\in K}|\,\Xi(z)-\Xi_m(z)|=0.$$

Proof. From (1)-(5) we have

$$\Xi(z) = \Xi_m(z) + 2\int_0^{+\infty} R_m(u)\cos uz du + 2\int_0^{+\infty} R_m(-u)\cos uz du.$$

Therefore, it suffices to prove that

$$\lim_{m\to\infty} \sup_{z\in K}\left|\int_0^{+\infty} R_m(\pm u)\cos uz du\right| = 0. \qquad (6)$$

Let $K \subset \mathbb{C}$ be any fixed compact. Then, there exists $r > 0$ such that $K \subset \{z : |z| \le r\}$. If A is a sufficiently large positive fixed number, then we have

$$\sup_{z\in K}\left|\int_0^{+\infty} R_m(u)\cos uz du\right| \le \int_0^A |R_m(u)|\, e^{ru} du + \int_A^{+\infty} |R_m(u)|\, e^{ru} du. \qquad (7)$$

If $u > A$, using the formula (4) and the inequalities $e^{2u} > 1 + 2u + 2u^2$, $P_{2m}(2u) > 1 + 2u + 2u^2$, we obtain:

$$|R_m(u)| =$$

$$= \left|\sum_{n=1}^{\infty}\left(2n^4\pi^2 e^{\frac{9}{2}u} - 3n^2\pi e^{\frac{5}{2}u}\right)\left(e^{-\pi n^2 e^{2u}} - e^{-n^2\pi P_{2m}(2u)}\right)\right| \le$$

$$\le \sum_{n=1}^{\infty}\left(2n^4\pi^2 e^{\frac{9}{2}u} + 3n^2\pi e^{\frac{5}{2}u}\right)\cdot 2e^{-\pi n^2\left(1+2u+2u^2\right)} \le$$

$$\le \left(\sum_{n=1}^{\infty} 8n^4\pi^2 e^{-\pi n^2}\right) e^{-\pi u^2} = c_1 e^{-\pi u^2}, \quad c_1 = \sum_{n=1}^{\infty} n^4 e^{-\pi n^2}\cdot 8\pi^2.$$

Further, let ε be an arbitrary positive number. Let us choose $A > 0$ so that the following inequality is fulfilled:

$$\int_A^{+\infty} |R_m(u)|\, e^{ru} du \le \int_A^{+\infty} c_1 e^{-\pi u^2 + ru} du < \frac{\varepsilon}{2}. \qquad (8)$$

Let us estimate the first integral in the right-hand side of (7) (for chosen A). As ($u \in (0, A)$)

$$1 - e^{-\pi n^2\left(e^{2u} - P_{2m}(2u)\right)} = \int_0^{\pi n^2\left(e^{2u} - P_{2m}(2u)\right)} e^{-\xi} d\xi \le \pi n^2\left(e^{2u} - P_{2m}(2u)\right),$$

from (4) we have:

$$|R_m(u)| \le \sum_{n=1}^{\infty}\left(2n^4\pi^2 e^{\frac{9}{2}u} - 3n^2\pi e^{\frac{5}{2}u}\right)\pi n^2\left(e^{2u} - P_{2m}(2u)\right) e^{-\pi n^2 P_{2m}(2u)} \le$$

$$\leq \sum_{s=2m+1}^{\infty} \frac{(2A)^s}{s!} \sum_{n=1}^{\infty} 4n^4\pi^2 \cdot \pi n^2 e^{\frac{9}{2}u}\, e^{-\pi n^2(1+2u)} \leq$$

$$\leq \sum_{s=2m+1}^{\infty} \frac{(2A)^s}{s!} \sum_{n=1}^{\infty} 4\pi^3 n^6\, e^{-\pi n^2} = c_2 \sum_{s=2m+1}^{\infty} \frac{(2A)^s}{s!},$$

where $c_2 = 4\pi^3 \sum_{n=1}^{\infty} n^6 e^{-\pi n^2}$.

Therefore, the following estimate is valid:

$$\int_0^A |R_m(u)|\, e^{ru} du \leq c_2 e^{rA} A \sum_{s=2m+1}^{\infty} \frac{(2A)^s}{s!}. \tag{9}$$

Thus, from (7) - (9) we obtain

$$\sup_{z\in K} \left| \int_0^{\infty} R_m(u) \cos uz\, du \right| \leq c_2 e^{rA} A \sum_{s=2m+1}^{\infty} \frac{(2A)^s}{s!} + \frac{\varepsilon}{2}.$$

Passing to the limit as $m \to \infty$, we have

$$\overline{\lim_{m\to\infty}} \sup_{z\in K} \left| \int_0^{+\infty} R_m(u) \cos uz\, du \right| \leq \frac{\varepsilon}{2},$$

and since ε is an arbitrary positive number, we get

$$\lim_{m\to\infty} \sup_{z\in K} \left| \int_0^{+\infty} R_m(u) \cos uz\, du \right| = 0.$$

It remains to prove the validity of the second equality of (6), i.e. the following equality:

$$\lim_{m\to\infty} \sup_{z\in K} \left| \int_0^{+\infty} R_m(-u) \cos uz\, du \right| = 0. \tag{10}$$

According to (4), we have:

$$R_m(-u) =$$

$$= \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{-\frac{9}{2}u} - 3n^2\pi e^{-\frac{5}{2}u}\right) \left(e^{-\pi n^2\left[e^{-2u} - P_{2m}(-2u)\right]} - 1\right) e^{-n^2\pi P_{2m}(-2u)} =$$

$$= \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{-\frac{9}{2}u} - 3n^2\pi e^{-\frac{5}{2}u}\right) e^{-\pi n^2 e^{-2u}} \left(1 - e^{-\pi n^2\left[P_{2m}(-2u) - e^{-2u}\right]}\right). \tag{11}$$

We first estimate $R_m(-u)$ as $m \to \infty$. Taking into account the equality $\Phi(u) = \Phi(-u)$, the formula (11) can be written as

$$R_m(-u) = \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{\frac{9}{2}u} - 3n^2\pi e^{\frac{5}{2}u}\right) \ e^{-n^2\pi\, e^{2u}} -$$

$$-\sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{-\frac{9}{2}u} - 3n^2\pi e^{-\frac{5}{2}u}\right) \ e^{-n^2\pi\, P_{2m}(-2u)}.$$

It is obvious that there exists $B_1 > 0$ such that for all $u > B_1$ the inequality $P_{2m}(-2u) \geq 1 + u^2 \ (m \geq 1)$ holds. Therefore for $u > B_1$, in view of the inequality $e^{2u} > 1 + 2u + 2u^2$, we have:

$$|R_m(-u)| \leq \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{\frac{9}{2}u}\right) \ e^{-\pi\, n^2\left(1+2u+2u^2\right)} +$$

$$+\sum_{n=1}^{\infty} 4n^4\pi^2 \ e^{-\pi\, n^2\left(1+u^2\right)} \leq \sum_{n=1}^{\infty} 6n^4\pi^2 \ e^{-\pi\, n^2} e^{-\pi\, u^2} = c_3 e^{-\pi\, u^2},$$

where $c_3 = \sum_{n=1}^{\infty} 6n^4\pi^2 \ e^{-\pi\, n^2}$. Consequently, the following integral converges:

$$\int_{B_1}^{+\infty} |R_m(-u)|\, e^{ru} du \leq c_3 \int_{B_1}^{+\infty} e^{-\pi u^2 + ru} du.$$

Let ε as valid,be an arbitrary positive number. Then there is a number $B > B_1$ such that

$$\int_{B}^{+\infty} |R_m(-u)|\, e^{ru} du < \frac{\varepsilon}{2}. \tag{12}$$

Further, let $u \in (0, B)$ and $m + 1 > B$. We have

$$P_{2m}(-2u) - e^{-2u} = -\sum_{s=2m+1}^{\infty} \frac{(-2u)^s}{s!} = \frac{(2u)^{2m+1}}{(2m+1)!} - \frac{(2u)^{2m+2}}{(2m+2)!} + \frac{(2u)^{2m+3}}{(2m+3)!} -$$

$$-\frac{(2u)^{2m+4}}{(2m+4)!} + ... = \frac{(2u)^{2m+1}}{(2m+1)!}\left(1 - \frac{2u}{2m+2}\right) + \frac{(2u)^{2m+3}}{(2m+3)!}\left(1 - \frac{2u}{2m+4}\right) + ... >$$

$$> \frac{(2u)^{2m+1}}{(2m+1)!}\left(1 - \frac{B}{m+1}\right) + \frac{(2u)^{2m+3}}{(2m+3)!}\left(1 - \frac{B}{m+2}\right) + ... > 0.$$

Therefore

$$1 - e^{-\pi n^2\left(P_{2m}(-2u) - e^{-2u}\right)} = \int_0^{\pi n^2\left(P_{2m}(-2u) - e^{-2u}\right)} e^{-\xi} d\xi \leq \pi n^2 \left(P_{2m}(-2u) - e^{-2u}\right).$$

Then from the formula (11) we have ($u \in (0, B)$):

$$|R_m(-u)| \le \sum_{n=1}^{\infty} 4n^6\pi^3\ e^{-\pi n^2 e^{-2B}} \sum_{s=2m+1}^{\infty} \frac{(2B)^s}{s\,!} = c_4 \sum_{s=2m+1}^{\infty} \frac{(2B)^s}{s\,!},$$

where

$$c_4 = \sum_{n=1}^{\infty} 4n^6\pi^3\ e^{-\pi n^2 e^{-2B}}.$$

Consequently,

$$\int_0^B |R_m(-u)|\, e^{ru} du \le e^{rB} c_4 B \sum_{s=2m+1}^{\infty} \frac{(2B)^s}{s\,!}. \tag{13}$$

Thus, from (12) and (13) it follows that

$$\sup_{z\in K}\left|\int_0^{\infty} R_m(-u)\cos uz\, du\right| \le \int_0^B |R_m(-u)|\, e^{ru}\, du +$$

$$+\int_B^{\infty} |R_m(-u)|\, e^{ru}\, du \le c_4 e^{rB} B \sum_{s=2m+1}^{\infty} \frac{(2B)^s}{s!} + \frac{\varepsilon}{2}.$$

Now it is not difficult to deduce the relation (10). Theorem 1 is proved. ◀

Let us introduce the notation

$$\Xi_m^{(l)}(z) =$$

$$= 4\int_0^{+\infty} \sum_{n=1}^{l} e^{-\pi n^2 P_{2m}^{+}(2u)} \left\{ 2n^4\pi^2 cosh\left(\pi n^2 P_{2m}^{-}(2u) - \frac{9}{2}u\right) - \right.$$

$$\left. -3n^2\pi cosh\left(\pi n^2 P_{2m}^{-}(2u) - \frac{5}{2}u\right)\right\} \cos uz du. \tag{14}$$

Theorem 2. *There exists m_0 such that for any compact $K \subset \mathbb{C}$*

$$\lim_{l\to\infty} \sup_{m\ge m_0} \sup_{z\in K} \left|\Xi_m^{(l)}(z) - \Xi_m(z)\right| = 0.$$

Proof. From the relations (3), (5) and (14) we have

$$\Xi_m(z) - \Xi_m^{(l)}(z) = 4\int_0^{\infty} \sum_{n=l+1}^{\infty} F_n^{+}(u)\cos uz du,$$

where

$$F_n^{+}(u) =$$

$$= e^{-\pi n^2 P_{2m}^{+}(2u)} \left\{ 2n^4\pi^2 cosh\left(\pi n^2 P_{2m}^{-}(2u) - \frac{9}{2}u\right) - 3n^2\pi cosh\left(\pi n^2 P_{2m}^{-}(2u) - \frac{5}{2}u\right)\right\}.$$

As, for $u > B_1$ the estimation $P_{2m}(-2u) > 1 + u^2$ is valid, where B_1 is a sufficiently large number, let us estimate $F_n^+(u)$ for $u > B_1$:

$$F_n^+(u) \leq e^{-\pi n^2 P_{2m}^{+}(2u)} 2n^4\pi^2 e^{\pi n^2 P_{2m}^{-}(2u) + \frac{9}{2}u} =$$

$$= 2n^4\pi^2 e^{-\pi n^2 P_{2m}(-2u) + \frac{9}{2}u} \leq 2n^4\pi^2 e^{-\pi n^2\left(1+u^2\right) + \frac{9}{2}u}.$$

For $0 < u < B_1$, we use the inequality

$$P_{2m}(-2u) - e^{-2u} > 0 \quad (1 + m > B_1):$$

$$F_n^+(u) \leq 2n^4\pi^2 e^{-\pi n^2 P_{2m}(-2u) + \frac{9}{2}u} = 2n^4\pi^2 e^{-\pi n^2\left(P_{2m}(-2u) - e^{-2u}\right)} e^{-\pi n^2 e^{-2u} + \frac{9}{2}u} \leq$$

$$\leq 2n^4\pi^2 e^{-\pi n^2 e^{-2u} + \frac{9}{2}u} \leq 2n^4\pi^2 e^{-\pi n^2 e^{-2B_1} + \frac{9}{2}B_1}.$$

Taking into account these estimates for $F_n^+(u)$, we can write

$$\sup_{z\in K}\left|\Xi_m(z) - \Xi_m^{(l)}(z)\right| \leq \int_0^{B_1} \sum_{n=l+1}^{\infty} F_n^+(u)\, e^{ru} du + \int_{B_1}^{+\infty} \sum_{n=l+1}^{\infty} F_n^+(u)\, e^{ru} du \leq$$

$$\leq 2\pi^2 e^{\left(\frac{9}{2}+r\right)B_1} B_1 \sum_{n=l+1}^{\infty} n^4 e^{-\pi n^2 e^{-B_1}} + 2\pi^2 \int_{B_1}^{\infty} e^{-\pi u^2 + \frac{9}{2}u + ru} du \sum_{n=l+1}^{\infty} n^4 e^{-\pi n^2}.$$

Hence, we get the proof of Theorem 2. Note that we can take $m_0 = B_1 - 1$. ◄

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