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Riesz's Equality for the Hilbert Transform of the Finite Complex Measures

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Abstract. In the present paper using the notion of Q'-integration introduced by E.Titchmarsh we prove the analogue of Riesz's equality for the Hilbert transform of the finite complex measures.
Key Words and Phrases: Riesz's equality, Hilbert transform, finite complex measure, Q-integral, Q'-integral, analytic functions, nontangential boundary values.
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1. Introduction

Let ν be a complex Borel measure on the real axis R and the integral $\int_R \frac{d\nu(\tau)}{1+|\tau|}$ exist. The function

$$(H\nu)(t) = \frac{1}{\pi}\int_R \frac{d\nu(\tau)}{t-\tau}, t \in R,$$

is called the Hilbert transform of the measure ν. In particular, if a measure ν is absolutely continuous: $d\nu(t) = f(t)\,dt$, then $(H\nu)(t)$ is called the Hilbert transform of the function f and is denoted by $(Hf)(t)$. It is known (see [9, 15]) that $(H\nu)(t)$ exists for almost all $t \in R$, and for any $\lambda > 0$ the inequality

$$m\{t \in R: \ |(H\nu)(t)| > \lambda\} \le c_0 \frac{\|\nu\|}{\lambda}, \tag{1}$$

holds, where m stands for the Lebesgue measure, $\|\nu\|$ is the total variation of the measure ν, and c_0 is an absolute constant. M.Riesz (see, for example, [9, 11, 14]) proved that if a measure ν is absolutely continuous: $d\nu(t) = f(t)\,dt$ and $f \in L_p(R)$, $p > 1$, then $Hf \in L_p(R)$ and for any $g \in L_q(R)$ the following equation holds:

$$\int_R g(t)(Hf)(t)\,dt = -\int_R (Hg)(t) f(t)\,dt,$$

where $q = \frac{p}{p-1}$. If $f \in L_1(R)$ and $f \notin L_p(R)$ for any $p > 1$, then the function Hf doesn't even belong to the class of functions $L_1^{(loc)}(R)$. In this case, using the notion of A-integration, Anter Ali Alsayad (see [8]) proved the following theorem.

Theorem A [8]. *If $g \in L_p(R)$, $p \geq 1$ is a bounded function, its Hilbert's transform is also a bounded function, and $f \in L_1(R)$, then the function $g(t)(Hf)(t)$ is A-integrable on R and the following equation holds:*

$$(A) \int_R g(t)(Hf)(t)\,dt = -\int_R (Hg)(t) f(t)\,dt. \tag{2}$$

In the case where the measure ν is not absolutely continuous, the function $(H\nu)(t)$ does not satisfy the condition $\lambda m\{t \in R : |(H\nu)(t)| > \lambda\} = o(1)$ as $\lambda \to +\infty$, and therefore the formula (2) fails to hold. In [5], using the notion of Q'-integration introduced by E.Titchmarsh [20], the author proved that the Hilbert transform of the finite complex measure ν is Q'-integrable on the real axis R, and the Q'-integral of the function $H\nu$ is equal to zero.

In the present paper we prove that, if ν is a finite complex Borel measure on the real axis R, the function $g \in L_p(R)$, $p \geq 1$ is Hőlder continuous and $g(t)\ln(e + |t|)$ is bounded on R, then the function $g(t)(Hf)(t)$ is Q'-integrable on R and the following equation holds:

$$(Q') \int_R g(t)(Hf)(t)\,dt = -\int_R (Hg)(t) f(t)\,dt.$$

2. On the properties of Q- and Q'-integrals of the function measurable on the real axis

For a measurable complex function f on an interval $[a, b] \subset R$ we set
$[f(x)]_n = [f(x)]^n = f(x)$ for $|f(x)| \leq n$,
$[f(x)]_n = n \cdot sgnf(x)$, $[f(x)]^n = 0$ for $|f(x)| > n$, $n \in N$,
where $sgnz = \frac{z}{|z|}$ for $z \neq 0$ and $sgn0 = 0$.

In 1929, E.Titchmarsh [20] introduced the notions of Q- and Q'-integrals.

Definition 1. *If a finite limit $\lim\limits_{n\to\infty} \int_a^b [f(x)]_n\,dx$ ($\lim\limits_{n\to\infty} \int_a^b [f(x)]^n\,dx$, respectively) exists, then f is said to be Q-integrable (Q'-integrable, respectively) on $[a, b]$, that is $f \in Q[a, b]$ ($f \in Q'[a, b]$), and the value of this limit is referred to as the Q-integral (Q'-integral) of this function and is denoted by*

$$(Q) \int_a^b f(x)\,dx \left((Q') \int_a^b f(x)\,dx \right).$$

In the same paper, E.Titchmarsh established that, when studying the properties of trigonometric series conjugate to Fourier series of Lebesgue integrable functions, Q-integration leads to a series of natural results. A very uncomfortable fact impeding the application of Q-integrals and Q'-integrals when studying diverse problems of function theory is the absence of the additivity property, that is, the Q-integrability (Q'-integrability) of two functions does not imply the Q-integrability (Q'-integrability) of their sum. If one adds the condition

$$\lambda\, m\left\{ x\in [a,b]:\ |f(x)|>\lambda \right\} = o(1),\, \lambda\to +\infty, \tag{3}$$

where m stands for the Lebesgue measure, to the definition of Q-integrability (Q'-integrability) of a function f on the interval $[a,b]$, then the Q-integral and Q'-integral coincide ($Q[a,b]=Q'[a,b]$), and these integrals become additive.

Definition 2. *If $f\in Q'[a,b]$ (or $f\in Q[a,b]$) and condition (3) holds, then f is said to be A-integrable on $[a,b]$, $f\in A[a,b]$, and the limit $\lim\limits_{n\to\infty}\int_a^b [f(x)]^n\,dx$ (or the limit $\lim\limits_{n\to\infty}\int_a^b [f(x)]_n\,dx$) is denoted in this case by $(A)\int_a^b f(x)\,dx$.*

As we noted above, the Q-integral and the Q'-integral do not have the additivity property. E.Titchmarsh in [20] for real functions and the author in [2] for complex functions established that, if $f\in Q[a,b]$ and $g\in L[a,b]$ (that is, g is Lebesgue integrable on the interval $[a,b]$), then $f+g\in Q[a,b]$ and the Q-integral of this sum is equal to the sum of the Q-integral of f and the Lebesgue integral of g. In [2], the author found a class of functions $M([a,b],C)$ such that, on this class, the Q'-integral coincides with the Q-integral, and proved that the Q'-integrability (Q-integrability) of a function $f\in M([a,b],C)$ and the A-integrability of a function g imply the Q'-integrability (Q-integrability) of their sum $f+g$, and the Q'-integral (Q-integral) of this sum is equal to the sum of the Q'-integral (Q-integral) of f and the A-integral of g. He also found a class of functions $SM([0,2\pi],C)\subset M([0,2\pi],C)$ such that the Q'-integral and the Q-integral have the additivity property on this class. The properties of Q- and Q'-integrals were investigated in [2, 10, 20], and for the applications of A-, Q- and Q'-integrals in the theory of functions of real and complex variables we refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 17, 18, 19, 21, 22, 23].

For a complex function f measurable on the real axis R we assume
$[f(x)]_{\delta,\lambda}=[f(x)]^{\delta,\lambda}=f(x)$ for $\delta\le|f(x)|\le\lambda$, $[f(x)]_{\delta,\lambda}=[f(x)]^{\delta,\lambda}=0$ for $|f(x)|<\delta$, $[f(x)]_{\delta,\lambda}=\lambda\, sgnf(x)$, $[f(x)]^{\delta,\lambda}=0$ for $|f(x)|>\lambda$, $0<\delta<\lambda$.

Definition 3. *If a finite limit $\lim\limits_{\substack{\delta\to0+\\ \lambda\to+\infty}}\int_R [f(x)]_{\delta,\lambda}\,dx$ ($\lim\limits_{\substack{\delta\to0+\\ \lambda\to+\infty}}\int_R [f(x)]^{\delta,\lambda}\,dx$ respectively) exists, then f is said to be Q-integrable (Q'-integrable) on R, that is $f\in Q(R)$ ($f\in Q'(R)$), and the value of this limit is referred to as the Q-integral (Q'-integral) of this function and is denoted by*

$$(Q)\int_R f(x)\,dx\left((Q')\int_R f(x)\,dx\right).$$

Remark 1. *Let $h>0$ be any positive number. From the equalities*

$$\lim_{\substack{\delta\to0+\\ \lambda\to+\infty}}\int_R [f(x)]_{\delta,\lambda}\,dx=\lim_{\delta\to0+}\int_{\{x\in R:\ \delta\le|f(x)|\le h\}} f(x)\,dx+$$

$$+\lim_{\lambda\to+\infty}\int_{\{x\in R:\ |f(x)|>h\}}[f(x)]_\lambda\,dx, \tag{4}$$

$$\lim_{\substack{\delta\to0+\\ \lambda\to+\infty}}\int_R [f(x)]^{\delta,\lambda}\,dx=\lim_{\delta\to0+}\int_{\{x\in R:\ \delta\le|f(x)|\le h\}} f(x)\,dx+$$

$$+\lim_{\lambda\to+\infty}\int_{\{x\in R:\ |f(x)|>h\}}[f(x)]^{\lambda}\,dx, \tag{5}$$

it follows that if for some $h>0$ there exists the integral $\int_{\{x\in R:\ |f(x)|\le h\}} f(x)\,dx$, then Q- and Q'-integrals of the function f can be determined as follows

$$(Q)\int_R f(x)\,dx=\lim_{\lambda\to+\infty}\int_R [f(x)]_{\lambda}\,dx,\ (Q')\int_R f(x)\,dx=\lim_{\lambda\to+\infty}\int_R [f(x)]^{\lambda}\,dx,$$

where $[f(x)]_{\lambda}$ and $[f(x)]^{\lambda}$ are determined as in Definition 1, and if there exists the integral $\int_{\{x\in R:\ |f(x)|>h\}} f(x)\,dx$, then Q- and Q'-integrals of the function f can be determined as follows

$$(Q)\int_R f(x)\,dx=(Q')\int_R f(x)\,dx=\lim_{\delta\to0+}\int_{\{x\in R:\,|f(x)|\ge\delta\}} f(x)\,dx.$$

Note that, as in case of an interval, Q- and Q'-integrals of the functions measurable on the real axis do not satisfy the additivity property, that is the Q-integrability (Q'-integrability) of two functions does not imply the Q-integrability (Q'-integrability) of their sum. If one adds the conditions

$$\delta\, m\{\,x\in R:\ |f(x)|>\delta\,\}=o(1),\delta\to0+, \tag{6}$$

$$\lambda\, m\{\,x\in R:\ |f(x)|>\lambda\,\}=o(1),\lambda\to+\infty, \tag{7}$$

to the definition of Q-integrability (Q'-integrability) of a function f on R, then Q-integral and Q'-integral coincide ($Q(R)=Q'(R)$) and these integrals become additive (see [1]).

Definition 4. *If $f\in Q'(R)$ (or $f\in Q(R)$) and the conditions* (6) *and* (7) *hold, then f is said to be A-integrable on R, $f\in A(R)$ and the limit $\lim\limits_{\substack{\delta\to0+\\ \lambda\to+\infty}}\int_R [f(x)]^{\delta,\lambda}\,dx$ (or the limit $\lim\limits_{\substack{\delta\to0+\\ \lambda\to+\infty}}\int_R [f(x)]_{\delta,\lambda}\,dx$) is denoted in this case by $(A)\int_R f(x)\,dx$.*

For the real function f measurable on R we assume

$$(f>\lambda)=\{\,t\in R:\ f(t)>\lambda\,\},(f<\lambda)=\{\,t\in R:\ f(t)<\lambda\,\},$$
$$(f\ge\lambda)=\{\,t\in R:\ f(t)\ge\lambda\,\},(f\le\lambda)=\{\,t\in R:\ f(t)\le\lambda\,\},$$
$$(\delta\le f\le\lambda)=\{\,t\in R:\ \delta\le f(t)\le\lambda\,\}.$$

Definition 5. *We denote by $M(R;C)$ the class of measurable complex-valued functions f on R for which the finite limits $\lim\limits_{\lambda\to+\infty}\lambda\, m(|f|>\lambda)$ and $\lim\limits_{\delta\to0+}\delta\, m(|f|>\delta)$ exist.*

It is known that the distribution function $m\{t\in R:\ |(H\nu)(t)|>\lambda\}$ of Hilbert transform of the complex measure ν satisfies the following equality (see [12, 16]):

$$\lim_{\lambda\to+\infty}\lambda m\{t\in R:\ |(H\nu)(t)|>\lambda\}=\frac{2}{\pi}\|\nu_s\|, \tag{8}$$

where ν_s stands for the singular part of the measure ν. In the paper [5] it is proved that the equality

$$\lim_{\delta\to 0+} \delta m\left\{t \in R:\ \left|(H\nu)(t)\right| > \delta\right\} = \frac{2}{\pi}\left|\nu(R)\right|, \tag{9}$$

holds.

The equalities (8) and (9) show that the Hilbert transform of the finite complex measure belongs to the class $M(R;C)$.

Theorem 1. *The Q-integral and the Q'-integral coincide in the function class $M(R;C)$, that is, if $f \in M(R;C)$, then for the existence of the integral $(Q)\int_R f(x)\,dx$ it is necessary and sufficient that the integral $(Q')\int_R f(x)\,dx$ exist, and in that case the following equation holds:*

$$(Q)\int_R f(x)\,dx = \left(Q'\right)\int_R f(x)\,dx. \tag{10}$$

Proof of Theorem 1. Let $h > 0$ be any positive number. If $f \in Q'(R)$, then from (5) it follows that there exist finite limits $\lim\limits_{\delta\to 0+}\int_{(\delta\le|f|\le h)} f(x)\,dx$ and $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}[f(x)]^{\lambda}\,dx$. Similar to the proof of Theorem 1 in [2], one can prove that the existence of the limit $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}[f(x)]^{\lambda}\,dx$ implies the existence of the limit $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}[f(x)]_{\lambda}\,dx$ and their equality. Hence, from (4) it follows that the function f is Q-integrable and equation (10) holds.

It remains to prove that, in the function class $M(R;C)$, it follows from $f \in Q(R)$ that $f \in Q'(R)$. From (4) we obtain that if $f \in Q(R)$ and $f \in M(R;C)$, then there exist finite limits $\lim\limits_{\delta\to 0+}\int_{(\delta\le|f|\le h)} f(x)\,dx$, $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}[f(x)]_{\lambda}\,dx$ and $\lim\limits_{\lambda\to+\infty}\lambda\, m(|f|>\lambda)$. Similar to the proof of Theorem 2 in [2], one can prove that the existence of the limit $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}[f(x)]_{\lambda}\,dx$ implies the existence of the limit $\lim\limits_{\lambda\to+\infty}\int_{(|f|>h)}[f(x)]^{\lambda}\,dx$. Hence, from (5) it follows that the function f is Q'-integrable and the equation (10) holds. This completes the proof of Theorem 1. ◀

We need the following theorems proved by the author in [4] and [5].

Theorem B [4, Theorem 2.3]. *If a function $f \in M(R;C)$ is Q'-integrable on R and a function g is A-integrable on R, then their sum $f+g \in M(R;C)$ is Q'-integrable on R, and the following equation holds:*

$$\left(Q'\right)\int_R [f(x)+g(x)]\,dx = \left(Q'\right)\int_R f(x)\,dx + (A)\int_R g(x)\,dx.$$

Theorem C [5, Theorem 4]. *Let ν be a finite complex measure on the real axis R. Then the equation*

$$\left(Q'\right)\int_R (H\nu)(t)\,dt = 0,$$

holds.

3. Riesz's equality for the Hilbert transform of the finite complex measures

Theorem 2. *Let ν be a finite complex measure on the real axis R, the function $g \in L_p(R)$, $p \geq 1$ be Hőlder continuous and $g(t)\ln(e+|t|)$ be bounded on R. Then the function $g(t)(H\nu)(t)$ is Q'-integrable on R and the following equation holds:*

$$(Q') \int_R g(t)(H\nu)(t)\,dt = -\int_R (Hg)(t)\,d\nu(t). \tag{11}$$

Remark 2. *Note that from the conditions of the theorem it follows that the function $(Hg)(t)$ is bounded on R and therefore the integral on the right-hand side of* (11) *exists.*

Proof of Theorem 2. Let us consider the measure $d\mu(t) = g(t)\,d\nu(t)$. Then

$$(H\mu)(t) = \frac{1}{\pi}\int_R \frac{g(\tau)\,d\nu(\tau)}{t-\tau} = \frac{1}{\pi}\int_R \frac{g(\tau)-g(t)}{t-\tau}d\nu(\tau) + g(t)(H\nu)(t) =$$

$$= J(t) + g(t)(H\nu)(t), \tag{12}$$

where

$$J(t) = \frac{1}{\pi}\int_R \frac{g(\tau)-g(t)}{t-\tau}d\nu(\tau) = \frac{1}{\pi}\int_{(|t-\tau|\leq 1)} \frac{g(\tau)-g(t)}{t-\tau}d\nu(\tau) + \frac{1}{\pi}\int_{(|t-\tau|>1)} \frac{g(\tau)\,d\nu(\tau)}{t-\tau} -$$

$$-\frac{1}{\pi}\int_{(|t-\tau|>1)} \frac{g(t)\,d\nu(\tau)}{t-\tau} = J_1(t) + J_2(t) - J_3(t). \tag{13}$$

At first consider the case of $\mu(R) = 0$. In this case, for every $t \neq 0$ we have the equality

$$J_2(t) = \frac{1}{\pi}\int_{(|t-\tau|>1)} \frac{g(\tau)\,d\nu(\tau)}{t-\tau} - \frac{1}{\pi}\int_R \frac{g(\tau)\,d\nu(\tau)}{t+sgnt} =$$

$$= \frac{1}{\pi}\int_{(|t-\tau|>1)} \frac{\tau+sgnt}{(t-\tau)(t+sgnt)}g(\tau)\,d\nu(\tau) - \frac{1}{\pi}\int_{(|t-\tau|\leq 1)} \frac{g(\tau)\,d\nu(\tau)}{t+sgnt}. \tag{14}$$

Then, by the conditions of the theorem, it follows that the integrals

$$\int_R \left(\int_{(|t-\tau|\leq 1)} \left|\frac{g(\tau)-g(t)}{t-\tau}\right| dt\right) d\nu(\tau), \int_R \left(\int_{(|t-\tau|>1)} \left|\frac{\tau+sgnt}{(t-\tau)(t+sgnt)}\right| dt\right) g(\tau)\,d\nu(\tau),$$

$$\int_R \left(\int_{(|t-\tau|\leq 1)} \left|\frac{1}{t+sgnt}\right| dt\right) g(\tau)\,d\nu(\tau), \int_R \left(\int_{(|t-\tau|>1)} \left|\frac{g(t)}{t-\tau}\right| dt\right) d\nu(\tau),$$

exist. Therefore it follows from Fubini's theorem (see, for example, [13], Ch.5, §6) and from (14) that the functions $J_1(t)$, $J_2(t)$ and $J_3(t)$ are Lebesgue integrable. Hence we obtain that the function $J(t)$ in (13) is also Lebesgue integrable on R. It follows from the equality (12) and Theorems B and C that the function $g(t)(H\nu)(t)$ is Q'-integrable on R and

$$(Q')\int_R g(t)(H\nu)(t)\,dt=(Q')\int_R (H\mu)(t)\,dt-\int_R J(t)\,dt=-\int_R J(t)\,dt=$$

$$=-\int_R J_1(t)\,dt-\int_R J_2(t)\,dt+\int_R J_3(t)\,dt.$$

Then from the equations

$$\int_R J_1(t)\,dt=\frac{1}{\pi}\int_R\left(\int_{(|t-\tau|\le 1)}\frac{g(\tau)-g(t)}{t-\tau}d\nu(\tau)\right)dt=$$

$$=\frac{1}{\pi}\int_R\left(\int_{(|t-\tau|\le 1)}\frac{g(\tau)-g(t)}{t-\tau}dt\right)d\nu(\tau),$$

$$\int_R J_2(t)\,dt=\frac{1}{\pi}\int_R\left(\int_{(|t-\tau|>1)}\frac{\tau+sgnt}{(t-\tau)(t+sgnt)}g(\tau)\,d\nu(\tau)-\int_{(|t-\tau|\le 1)}\frac{g(\tau)\,d\nu(\tau)}{t+sgnt}\right)dt=$$

$$=\frac{1}{\pi}\int_R\left(\int_{(|t-\tau|>1)}\frac{\tau+sgnt}{(t-\tau)(t+sgnt)}dt-\int_{(|t-\tau|\le 1)}\frac{dt}{t+sgnt}\right)g(\tau)\,d\nu(\tau),$$

$$\int_R J_3(t)\,dt=\frac{1}{\pi}\int_R\left(\int_{(|t-\tau|>1)}\frac{g(t)}{t-\tau}d\nu(\tau)\right)dt=\frac{1}{\pi}\int_R\left(\int_{(|t-\tau|>1)}\frac{g(t)}{t-\tau}dt\right)d\nu(\tau),$$

we have

$$(Q')\int_R g(t)(H\nu)(t)\,dt=-\frac{1}{\pi}\int_R\left(\int_{(|t-\tau|\le 1)}\frac{g(\tau)-g(t)}{t-\tau}dt\right)d\nu(\tau)-$$

$$-\frac{1}{\pi}\int_R\left(\int_{(|t-\tau|>1)}\frac{\tau+sgnt}{(t-\tau)(t+sgnt)}dt-\int_{(|t-\tau|\le 1)}\frac{dt}{t+sgnt}\right)g(\tau)\,d\nu(\tau)+$$

$$+\frac{1}{\pi}\int_R\left(\int_{(|t-\tau|>1)}\frac{g(t)}{t-\tau}dt\right)d\nu(\tau)=-\int_R (Hg)(t)\,d\nu(t).$$

That is, the equation (11) holds in case $\mu(R)=0$.

Now let's consider the case $\mu(R)=d_0\neq 0$. Denote by ν_1 the absolutely continuous measure satisfying the condition $\int_R g(t)\,d\nu_1(t)=d_0$, and by ν_2 the difference $\nu_2=\nu-\nu_1$. Let $d\mu_i(t)=g(t)\,d\nu_i(t)$, $i=\overline{1,2}$. Then $\mu_2(R)=0$ and, according to the above case, we have the equation

$$(Q')\int_R g(t)(H\nu_2)(t)\,dt=-\int_R (Hg)(t)\,d\nu_2(t). \tag{15}$$

As the measure ν_1 is absolutely continuous, then by Theorem A the equation

$$(A)\int_R g(t)(H\nu_1)(t)\,dt = -\int_R (Hg)(t)\,d\nu_1(t), \tag{16}$$

holds. From the equations (15), (16) and by Theorem C it follows that

$$(Q')\int_R g(t)(H\nu)(t)\,dt = (A)\int_R g(t)(H\nu_1)(t)\,dt + (Q')\int_R g(t)(H\nu_2)(t)\,dt =$$

$$= -\int_R (Hg)(t)\,d\nu_1(t) - \int_R (Hg)(t)\,d\nu_2(t) = -\int_R (Hg)(t)\,d\nu(t).$$

This completes the proof of Theorem 2. ◄

Theorem 3. *Let ν be a finite complex measure on the real axis R, and the function $g \in L_p(R)$, $p \geq 1$ be Hőlder continuous on R. Then the integral $\int_{(|g\cdot H\nu|\leq 1)} g(t)(H\nu)(t)\,dt$ exists.*

Proof of Theorem 3. Denote

$$I_1 = \int_{(|g\cdot H\nu|\leq 1)\bigcap(|H\nu|>1)} g(t)(H\nu)(t)\,dt, I_2 = \int_{(|g\cdot H\nu|\leq 1)\bigcap(|H\nu|\leq 1)} g(t)(H\nu)(t)\,dt.$$

Then it follows from the inequality (1) that

$$|I_1| \leq m\,(|H\nu| > 1) \leq c_0 \cdot \|\nu\| < \infty.$$

If $g \in L_1(R)$, then

$$|I_2| \leq \int_{(|g\cdot H\nu|\leq 1)\bigcap(|H\nu|\leq 1)} |g(t)|\,dt \leq \|g\|_1 < \infty,$$

and if $g \in L_p(R)$, $p > 1$, then it follows from the Hőlder's inequality and the inequality (1) that

$$|I_2| \leq \int_{(|g\cdot H\nu|\leq 1)\bigcap(|H\nu|\leq 1)} |g(t)|\cdot|(H\nu)(t)|\,dt \leq \left(\int_{(|H\nu|\leq 1)} |(H\nu)(t)|^{p'}\,dt\right)^{\frac{1}{p'}} \|g\|_p$$

$$= \left(\sum_{k=0}^{\infty}\int_{\left(2^{-k-1}<|H\nu|\leq 2^{-k}\right)} |(H\nu)(t)|^{p'}\,dt\right)^{\frac{1}{p'}} \|g\|_p \leq \left(\sum_{k=0}^{\infty} 2^{-kp'} m\left(|H\nu| > 2^{-k-1}\right)\right)^{\frac{1}{p'}} \|g\|_p$$

$$\leq \left(2c_0\,\|\nu\| \sum_{k=0}^{\infty} 2^{-k(p'-1)}\right)^{\frac{1}{p'}} \|g\|_p < \infty.$$

It follows from these estimates that the integral $\int_{(|g\cdot H\nu|\leq 1)} g(t)(H\nu)(t)\,dt$ exists. This completes the proof of Theorem 3. ◄

Remark 3. *It follows from Theorem 3 and Remark 1 that the equation* (11) *in Theorem 2 can be rewritten in the following way:*

$$\lim_{\lambda\to+\infty}\int_{(|g\cdot H\nu|\le\lambda)} g\,(t)\,(H\nu)\,(t)\,dt = -\int_R (Hg)\,(t)\,d\nu\,(t)\,.$$

Remark 4. *In the class $M\,(R;\,C)$, the Q'-integral coincides with the Q-integral (see Theorem 1). Then, under conditions of Theorem 2, the function $g\,(t)\,(H\nu)\,(t)$ is Q-integrable on R and the following equation holds:*

$$(Q)\int_R g\,(t)\,(H\nu)\,(t)\,dt = -\int_R (Hg)\,(t)\,d\nu\,(t)\,.$$

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