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The Problem of Optimization With Control of Mobile Sources For a Linear Parabolic Equation

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Abstract. In this paper, we consider a problem of optimal control of mobile sources for a linear parabolic equation. The variation method is applied to this problem. The necessary conditions for optimality are established in the form of the pointwise and integral maximum principles.

Key Words and Phrases: moving sources, integral identity, maximum principle, Hamilton – Pontryagin function, necessary optimality conditions.

2010 Mathematics Subject Classifications: 35K20, 49K20, 49J20

1. Introduction

Theoretical formulation of the problem of optimal control of mobile sources for systems with distributed parameters was first given in [1, 2] where numerous examples of systems with mobile sources of different physical nature are given and the main features of systems with movable control are revealed, which make it difficult or impossible to study them by known methods. One of the main features of the systems of optimal control of mobile sources is that they are non-linear with respect to the equation which governs the law of the motion of the source. This is particularly clear when formulating the control problem in terms of the moment problem where the latter becomes nonlinear. Thus, the moment method, widely used for finding the optimal control in linear systems with distributed and lumped parameters, is unsuitable for systems with control of mobile sources.

Optimal control problems of mobile sources have many important applications. Such problems are encountered in the process of optimization of thermal physics, diffusion, filtration, etc. In the study of these problems there arise a number of difficulties associated with their ill-posedness and nonlinearity. In [3, 5-7], the problem of optimal control of pointwise sources for parabolic equation is considered provided that the controls are only the intensity of motionless sources. In [4], questions of controllability of linear systems with the generalized influence are investigated. In [8-10], a variation method for solving the problem of optimal control of mobile sources for systems described by the heat equation and systems of ordinary differential equations is considered.

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In this paper we consider the variation method for solving the optimal control problem for systems described by a parabolic equation. We prove existence and uniqueness theorem for the solution of this problem, give sufficient conditions for Frechet differentiability of the performance criterion and expression for its gradient, and necessary conditions for optimality in the form of the pointwise and integral maximum principles.

2. Problem Statement

Let $l > 0,\ T > 0$ be the given numbers, $0 \le x \le l,\ 0 \le t \le T,\ \Omega_t = (0,\ l) \times (0,\ t)$, $\Omega = \Omega_T,\ L_2(\Omega)$ be the Banach space of all Lebesgue-measurable functions on Ω, with the finite norm $\|u\|_{L_2(\Omega)} = \left(\int_\Omega u^2 dx\right)^{1/2}$. If $\Omega = (a, b)$ is a segment of the real straight line, then we write $L_2(a, b)$ instead of $L_2(\Omega)$. In what follows, we will also need the functional spaces $V_2^{1,0}(\Omega),\ W_2^{1,0}(\Omega),\ W_2^{1,1}(\Omega)$ (see, e.g., [5]).

Consider the controlled process defined by the function $u(x, t)$ which satisfies the parabolic equation

$$\frac{\partial u(x,t)}{\partial t} - a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \sum_{k=1}^{n} p_k(t)\delta(x - s_k(t)),\ (x,t) \in \Omega, \tag{1}$$

with the boundary and initial conditions

$$\frac{\partial u(0,t)}{\partial x} = 0,\ \ \frac{\partial u(l,t)}{\partial x} = 0,\ \ 0 < t \le T, \tag{2}$$

$$u(x, o) = \varphi(x), 0 \le x \le l, \tag{3}$$

where $a > 0$ is a given number, $\varphi(x) \in L_2(0, l)$ is the a function, $\delta(\cdot)$ is the Dirac function, and $p(t) = (p_1(t), p_2(t), ..., p_n(t)) \in L_2(0, T; R^n),\ s(t) = (s_1(t), s_2(t), ..., s_n(t)) \in L_2(0, T; R^n)$ are the control functions.

The pair of functions $\vartheta = (p(t), s(t))$ is called the control. The Hilbert space of the pairs $\vartheta = (p(t), s(t))$ with the scalar product

$$< \vartheta^1, \vartheta^2 >_H = \int_0^T [p^1(t)p^2(t) + s^1(t)s^2(t)]dt$$

and norm $\|\vartheta\|_H = \sqrt{< \vartheta, \vartheta >_H} = \sqrt{\|p\|_{L_2}^2 + \|s\|_{L_2}^2}$ is denoted for brevity by $H = L_2(0, T; R^n) \times L_2(0, T; R^n)$. In H, we introduce a set of permissible controls

$$V = \left\{\vartheta = (p, s) \in H : 0 \le p_i \le A_i,\ 0 \le s_i \le B_i \le l,\ i = \overline{1, n}\right\}, \tag{4}$$

where $A_i > 0, B_i > 0, i = \overline{1, n}$, are the given numbers, and consider the functional

$$J(\vartheta) = \int_0^l \int_0^T [u(x,t) - \tilde{u}(x,t)]^2\, dxdt + \sum_{k=1}^{n} \left\{\alpha_1 \int_0^T [p_k(t) - \tilde{p}_k(t)]^2\, dt + \right.$$

$$+\alpha_2 \int_0^T [s_k(t) - \tilde{s}_k(t)]^2 \, dt \Bigg\} \tag{5}$$

where $\vartheta = (p(t), s(t)) \in H$; $\alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 > 0$ are the given parameters, and $\tilde{u}(x,t)) \in L_2(\Omega)$, $\omega = (\tilde{p}(t), \tilde{s}(t)) \in H$, $\tilde{p}(t) = (\tilde{p}_1(t), \tilde{p}_2(t), ..., \tilde{p}_n(t)) \in L_2(0,T;R^n)$, $\tilde{s}(t) = (\tilde{s}_1(t), \tilde{s}_2(t), ..., \tilde{s}_n(t)) \in L_2(0,T;R^n)$ are the given functions.

We pose the following problem: given the constraints (1)-(3), determine a control $\vartheta = (p(t), s(t))$ from the set V and the function $u(x,t)$ such that the functional (5) assumes the least possible value.

3. Existence and uniqueness of the solution

To solve our problem, we will use the following known result.

Theorem 1 ([16])**.** *Let H be a uniformly convex Banach space, V be a closed bounded set on H, functional $I(\vartheta)$ be bounded below and lower semicontinuous on V, and $\alpha > 0, \beta \geq 1$ be the given numbers. Then, there exists a dense subset K of the space H such that for any $\omega \in K$ functional $J_\alpha(\vartheta) = I(\vartheta) + \alpha \|\vartheta - \omega\|_H^\beta$ reaches its minimum value on the set V. If $\beta > 1$, then the minimum value of the functional $J_\alpha(\vartheta)$ on the set V achieved in a single element.*

Definition 1. *Determination of the function $u(x,t) = u(x,t;\vartheta)$ from the conditions* (1)-(3) *under the given control $\vartheta \in V$ is called a reduced problem. A function $u(x,t) \in V_2^{1,0}(\Omega)$ which satisfies the integral identity*

$$\int_0^l \int_0^T \left[-u \frac{\partial \eta}{\partial t} + a^2 \frac{\partial u}{\partial x} \frac{\partial \eta}{\partial x} \right] dx dt = \int_0^l \varphi(x) \eta(x,0) dx + \sum_{k=1}^n \int_0^T p_k(t) \eta(s_k(t), t) dt, \tag{6}$$

$\forall \eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ *with* $\eta(x,T) = 0$, *is called the generalized solution of the reduced problem* (1)-(3) *corresponding to the control $\vartheta = (p(t), s(t)) \in V$.*

It should be noted that for each control $\vartheta \in V$ the existence of a unique generalized solution of the reduced problem (1)-(3) from $V_2^{1,0}(\Omega)$ follows from the results of [11, p.265-270]. In the sequel, we will use this fact throughout this paper. The purpose of this work is to study the optimal control problem (1)-(5). Therefore we will assume in what follows that the solution of the reduced problem exists and is unique.

Let the conditions (1)-(5) be satisfied. Then, the problem (1)-(5) has at least one solution. Note that for $\alpha_j = 0, j = \overline{1,2}$, the problem (1)-(5) is incorrect in the classical sense [13]. However, the following theorem is valid.

Theorem 2. *There exists a dense subset K of the space H such that the problem* (1)-(5) *has a unique solution for any $\omega \in K$ and $\alpha_i > 0$, $i = \overline{1,2}$.*

Proof. Let's prove the continuity of the functional

$$J_0(\vartheta) = \|u(x,t) - \tilde{u}(x,t)\|^2_{L_2(\Omega)}.$$

Let $\Delta\vartheta = (\Delta p, \Delta s) \in V$ be the increment of control on the element $\vartheta = (p,s) \in V$ such that $\vartheta + \Delta\vartheta \in V$. Denote

$$\Delta u \equiv \Delta u(x,t) = u(x,t;\vartheta + \Delta\vartheta) - u(x,t,\vartheta), u \equiv u(x,t;\vartheta), \Delta s_k \equiv \Delta s_k(t).$$

It follows from (1)-(3) that the function Δu is a generalized solution of the boundary value problem

$$\frac{\partial \Delta u}{\partial t} = a^2 \frac{\partial^2 \Delta u}{\partial x^2} + \sum_{k=1}^{n} [(p_k + \Delta p_k)\delta(x - (s_k + \Delta s_k)) - p_k \delta(x - s_k)],$$

$$(x,t) \in \Omega, \frac{\partial \Delta u(0,t)}{\partial x} = 0, \quad \frac{\partial \Delta u(l,t)}{\partial x} = 0, \quad 0 < t \leq T, \tag{7}$$

$$\Delta u(x,0) = 0, x \in [0,l]. \tag{8}$$

Let's prove that the estimate

$$\|\Delta u\|_{V_2^{1,0}(\Omega)} \leq c_1 \|\Delta\vartheta\|_H \tag{9}$$

is true for the function Δu, where $c_1 \geq 0$ is some constant.

By multiplying both sides of (6) by $\eta = \eta(x,t)$ and integrating by parts, we obtain the relation

$$\int_0^l \int_0^T \left[\frac{\partial \Delta u}{\partial t}\eta + a^2 \frac{\partial \Delta u}{\partial x}\frac{\partial \eta}{\partial x} \right] dxdt = \sum_{k=1}^{n} \int_0^T [p_k + \Delta p_k)\eta(s_k + \Delta s_k, t) - p_k \eta(s_k,t)]\, dt, \tag{10}$$

$\forall \eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ with $\eta(x,T) = 0$.

Let $t_1, t_2 \in [0,T]$ be such that $t_1 \leq t_2$. We assume in equality (10) that

$$\eta(x,t) = \begin{cases} \Delta u(x,t) \;, & t \in (t_1,t_2], \\ 0\,, & t \in [0,t_1] \bigcup (t_2,T]. \end{cases}$$

Then, using the method of [12, p.166-168], we obtain the following integral identity:

$$\frac{1}{2} \int_0^l |\Delta u(x,t)|^2 \Big|_{t=t_1}^{t=t_2} dx + a^2 \int_0^l \int_0^t \left| \frac{\partial \Delta u}{\partial x} \right|^2 dxdt \Big|_{t=t_1}^{t=t_2} =$$

$$= \sum_{k=1}^{n} \int_{t_1}^{t_2} [(p_k + \Delta p_k)\Delta u(s_k + \Delta s_k, t) - p_k \Delta u(s_k,t)]\, dt. \tag{11}$$

Using formulas of finite increments

$$\Delta u(s_k+\Delta s_k,t)=\Delta u(s_k,t)+\frac{\partial \Delta u(\bar{s}_k,t)}{\partial x}\Delta s_k,\ \ \bar{s}_k=s_k+\theta\Delta s_k,\ \theta\in[0,1],$$

for the function $\Delta u(s_k+\Delta s_k,t)$, we can rewrite the right-hand side of (11) as follows:

$$\sum_{k=1}^{n}\int_{t_1}^{t_2}\left[(p_k+\Delta p_k)\Delta u(s_k+\Delta s_k,t)-p_k\Delta u(s_k,t)\right]dt=$$

$$=\sum_{k=1}^{n}\int_{t_1}^{t_2}\left[(p_k+\Delta p_k)(\Delta u(s_k,t)+\frac{\partial \Delta u(\bar{s}_k,t)}{\partial x}\Delta s_k)-p_k\Delta u(s_k,t)\right]dt=$$

$$=\sum_{k=1}^{n}\int_{t_1}^{t_2}\left[\frac{\partial \Delta u(\bar{s}_k,t)}{\partial x}(p_k+\Delta p_k)\Delta s_k+\Delta u(s_k,t)\Delta p_k\right]dt.$$

Considering this relation in (11), we obtain the energy balance equation for the problem (6)-(8):

$$\frac{1}{2}\left\|\Delta u(x,t)\right\|^2_{L_2(0,l)}\Big|_{t=t_1}^{t=t_2}+a^2\left\|\frac{\partial \Delta u(x,t)}{\partial x}\right\|^2_{L_2(\Omega_t)}\Big|_{t=t_1}^{t=t_2}=\sum_{k=1}^{n}\int_{t_1}^{t_2}[(p_k+\Delta p_k)\Delta s_k\times$$

$$\times\frac{\partial \Delta u(\bar{s}_k,t)}{\partial x}+\Delta p_k\Delta u(s_k,t)]dt, \tag{12}$$

where $\bar{s}_k=s_k+\theta\Delta s_k,\ \theta\in[0,1]$.

By applying the Cauchy-Bunyakovsky inequality to the right-hand side of (12), we obtain

$$\frac{1}{2}\left\|\Delta u(x,t)\right\|^2_{L_2(0,l)}\Big|_{t=t_1}^{t=t_2}+a^2\left\|\frac{\partial \Delta u(x,t)}{\partial x}\right\|^2_{L_2(\Omega_t)}\Big|_{t=t_1}^{t=t_2}\le$$

$$\le\sum_{k=1}^{n}\Big[\Big(\|p_k\|_{L_2(t_1,t_2)}+\|\Delta p_k\|_{L_2(t_1,t_2)}\Big)\max_{t_1\le t\le t_2}|\Delta s_k(t)|\left\|\frac{\partial \Delta u(\bar{s}_k,t)}{\partial x}\right\|_{L_2(t_1,t_2)}+$$

$$+\|\Delta p_k\|_{L_2(t_1,t_2)}\|\Delta u(s_k,t)\|_{L_2(t_1,t_2)}\Big]=$$

$$=\sum_{k=1}^{n}\Big[\Big(\|p_k\|_{L_2(t_1,t_2)}+\|\Delta p_k\|_{L_2(t_1,t_2)}\Big)\|\Delta s_k(t)\|_{C[t_1,t_2]}\left\|\frac{\partial \Delta u(\bar{s}_k,t)}{\partial x}\right\|_{L_2(t_1,t_2)}+$$

$$+\|\Delta p_k\|_{L_2(t_1,t_2)}\|\Delta u(s_k,t)\|_{L_2(t_1,t_2)}\Big]\ . \tag{13}$$

Also, it is not difficult to show that the following inequalities are true:

$$\|\Delta u(s_k,t)\|_{L_2(t_1,t_2)}\le c_2\|\Delta u\|_{V_2^{1,0}(\Omega)},$$

$$\left\| \frac{\partial \Delta u(\bar{s}_k,t)}{\partial x} \right\|_{L_2(t_1,t_2)} \le c_3 \left\| \Delta u \right\|_{V_2^{1,0}(\Omega)},$$

where $c_2 \ge 0, c_3 \ge 0$ are some constants. Then, the right-hand side of inequality (13) may be bounded from above:

$$\frac{1}{2} \left\| \Delta u(x,t) \right\|_{L_2(0,l)}^2 \Big|_{t=t_1}^{t=t_2} + a^2 \left\| \frac{\partial \Delta u(x,t)}{\partial x} \right\|_{L_2(\Omega_t)}^2 \Big|_{t=t_1}^{t=t_2} \le c_4 \left\| \Delta \vartheta \right\|_{L_2(t_1,t_2)} \left\| \Delta u \right\|_{V_2^{1,0}(\Omega)}, \quad (14)$$

for $\|\Delta \vartheta\|_{L_2(t_1,t_2)} \to 0$, where $c_4 > 0$ is some constant. Proceeding as in [12, p.166-168], for an arbitrary $t \in [0,T]$, we decompose the segment $[0,t]$ into a finite number of subsegments on each of which inequality (14) is satisfied. By summing up the resulting inequalities for each subsegment, we obtain

$$\frac{1}{2} \left\| \Delta u(x,t) \right\|_{L_2(0,l)}^2 + a^2 \left\| \frac{\partial \Delta u(x,t)}{\partial x} \right\|_{L_2(\Omega_T)}^2 \le c_4 \left\| \Delta \vartheta \right\|_H \left\| \Delta u \right\|_{V_2^{1,0}(\Omega)},$$

whence the inequality (9) follows. Then, $\|\Delta u\|_{V_2^{1,0}(\Omega)} \to 0$ for $\|\Delta \vartheta\|_H \to 0$. It follows by theorem of traces [14] that $\|\Delta u(x,t)\|_{L_2(\Omega)} \to 0$ for $\|\Delta \vartheta\|_H \to 0$.

The increment of the functional $J_0(\vartheta)$ can be represented as

$$J_0(\vartheta + \Delta \vartheta) - J_0(\vartheta) = 2 \int_0^l \int_0^T \left[u(x,t) - \tilde{u}(x,t) \right] \Delta u(x,t) dx dt + \left\| \Delta u(x,t) \right\|_{L_2(\Omega)}^2.$$

The continuity of the functional $J_0(\vartheta)$ follows from this and the fact that $\|\Delta u(x,t)\|_{L_2(\Omega)} \to 0$ for $\|\Delta \vartheta\|_H \to 0$.

The functional $J_0(\vartheta)$ is bounded from below and, by virtue of the above proven fact, is continuous in V. Additionally, H is a uniformly convex reflexive Banach space [15]. Then, it follows from Theorem 1 that there exists a dense subset K of the space H such that for any $\omega = (\tilde{p}(t), \tilde{s}(t)) \in H$ the problem (1)-(5) has a unique solution for $\alpha_i > 0, i = \overline{1,2}$. ◀

4. Necessary optimality conditions

Let $\psi = \psi(x,t)$ be the generalized solution from $V_2^{1,0}(\Omega)$ of the problem conjugate to (1)-(3):

$$\frac{\partial \psi(x,t)}{\partial t} + a^2 \frac{\partial^2 \psi(x,t)}{\partial x^2} = -2[u(x,t) - \tilde{u}(x,t)], \quad (x,t) \in \Omega, \quad (15)$$

$$\frac{\partial \psi(0,t)}{\partial x} = 0, \quad \frac{\partial \psi(l,t)}{\partial x} = 0, \quad t \in [0,T), \quad (16)$$

$$\psi(x,T) = 0, \quad x \in [0,\ell]. \quad (17)$$

Definition 2. *A function $\psi(x,t) \in V_2^{1,0}(\Omega)$, which satisfies the integral identity*

$$\int_0^l \int_0^T \left(\psi \frac{\partial \eta_1}{\partial t} + a^2 \frac{\partial \psi}{\partial x} \frac{\partial \eta_1}{\partial x} \right) dxdt = 2 \int_0^l \int_0^T [u(x,t) - \tilde{u}(x,t)] \eta_1(x,t) dxdt, \tag{18}$$

$\forall \eta_1 = \eta_1(x,t) \in W_2^{1,1}(\Omega)$ *with* $\eta_1(x,0) = 0$, *is called the generalized solution of the problem* (15)-(17) *corresponding to the control* $\vartheta = (p(t), s(t)) \in V$.

The conjugate problem (15)-(17) is a mixed problem for the linear parabolic equation. Therefore it follows from the facts established for the problem (1)-(3) that for every $\vartheta = (p(t), s(t)) \in V$ the problem (15)-(17) has a unique solution from $V_2^{1,0}(\Omega)$ [12].

We call the function

$$H(t, \psi, \vartheta) = -\sum_{k=1}^{n} \left[\psi(s_k(t), t) p_k(t) + \alpha_1 \left(p_k(t) - \tilde{p}_k(t) \right)^2 + \alpha_2 \left(s_k(t) - \tilde{s}_k(t) \right)^2 \right] \tag{19}$$

the Hamilton-Pontryagin function of the problem (1)-(5). Now let's find sufficient conditions for Frechet differentiability of the functional (5) and the expression for its gradient.

Theorem 3. *If $\psi(x,t)$ is the solution from $V_2^{1,0}(\Omega)$ of the conjugate problem* (15)-(17), *then the functional* (5) *is Frechet-differentiable on the set V, and the following relation is valid for its gradient:*

$$J'(\vartheta) = \left(\frac{\partial J(\vartheta)}{\partial p}, \frac{\partial J(\vartheta)}{\partial s} \right) = \left(-\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial s} \right), \tag{20}$$

where

$$\frac{\partial H}{\partial p} = \left(\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial p_2}, ..., \frac{\partial H}{\partial p_n} \right), \quad \frac{\partial H}{\partial s} = \left(\frac{\partial H}{\partial s_1}, \frac{\partial H}{\partial s_2}, ..., \frac{\partial H}{\partial s_n} \right),$$

$$\frac{\partial H}{\partial p_k} = -\psi(s_k(t), t) - 2\alpha_1 \left(p_k(t) - \tilde{p}_k(t) \right),$$

$$\frac{\partial H}{\partial s_k} = -\frac{\partial \psi(s_k(t), t)}{\partial x} p_k(t) - 2\alpha_2 \left(s_k(t) - \tilde{s}_k(t) \right), k = \overline{1, n}.$$

Proof. Let as consider an increment of the functional

$$\Delta J(\vartheta) \equiv J(\vartheta + \Delta\vartheta) - J(\vartheta) = 2 \int_0^l \int_0^T \left[u(x,t) - \tilde{u}(x,t) \right] \Delta u(x,t) dxdt +$$

$$+ \int_0^l \int_0^T \left| \Delta u(x,t) \right|^2 dxdt + \sum_{k=1}^n \left\{ 2\alpha_1 \int_0^T \left[p_k(t) - \tilde{p}_k(t) \right] \Delta p_k(t) dt + \alpha_1 \int_0^T \left| \Delta p_k(t) \right|^2 dt + \right.$$

$$\left. + 2\alpha_2 \int_0^T \left[s_k(t) - \tilde{s}_k(t) \right] \Delta s_k(t) dt + \alpha_2 \int_0^T \left| \Delta s_k(t) \right|^2 dt \right\}, \tag{21}$$

where $\vartheta = (p, s) \in V$, $\vartheta + \Delta\vartheta \in V$, $\Delta u(x,t) \equiv u(x,t; \vartheta + \Delta\vartheta) - u(x,t;\vartheta)$, $u \equiv u(x,t;\vartheta)$.

If we assume $\eta_1 = \Delta u(x,t)$ in (18) and $\eta = \psi(x,t)$ in (10), and subtract the resulting relations, then we have

$$\int_0^l \int_0^T \left(\psi \frac{\partial \Delta u}{\partial t} + a^2 \frac{\partial \psi}{\partial x} \frac{\partial \Delta u}{\partial x} \right) dxdt = 2 \int_0^l \int_0^T \left[u(x,t) - \tilde{u}(x,t) \right] \Delta u(x,t) dxdt,$$

$$\int_0^l \int_0^T \left[\frac{\partial \Delta u}{\partial t} \psi + a^2 \frac{\partial \Delta u}{\partial x} \frac{\partial \psi}{\partial x} \right] dxdt = \sum_{k=1}^{n} \int_0^T \left[p_k + \Delta p_k) \eta(s_k + \Delta s_k, t) - p_k \eta(s_k, t) \right] dt,$$

$$2 \int_0^l \int_0^T \left[u(x,t) - \tilde{u}(x,t) \right] \Delta u(x,t) dxdt =$$
$$= \sum_{k=1}^{n} \int_0^T \left[(p_k + \Delta p_k) \psi(s_k + \Delta s_k, t) - p_k \psi(s_k, t) \right] dt. \tag{22}$$

It is clear that under the above assumptions the following Taylor expansion is valid:

$$\psi(s_k + \Delta s_k, t) = \psi(s_k, t) + \frac{\partial \psi(s_k, t)}{\partial x} \Delta s_k + o\left(\|\Delta s_k\|_{L_2(0,T)} \right).$$

In view of this fact, from (22) we obtain that

$$2 \int_0^l \int_0^T \left[u(x,t) - \tilde{u}(x,t) \right] \Delta u(x,t) dxdt = \sum_{k=1}^{n} \int_0^T \left[\frac{\partial \psi(s_k(t), t)}{\partial x} p_k(t) \Delta s_k(t) + \right.$$
$$\left. + \psi(s_k(t), t) \Delta p_k(t) \right] dt + R_1, \tag{23}$$

where $R_1 = \sum_{k=1}^{n} \left(\int_0^T \frac{\partial \psi(s_k(t),t)}{\partial x} \Delta p_k(t) \Delta s_k(t) dt) + o\left(\|\Delta s_k\|_{L_2(0,T)} \right) \right)$.

It is clear that $R_1 = o\left(\|\Delta \vartheta\|_H \right)$ as $\|\Delta \vartheta\|_H \to 0$. On the other hand, from the estimate (9) follows that

$$\|\Delta u(x,t)\|_{L_2(\Omega)} \le c_5 \|\Delta \vartheta\|_H,$$

where $c_5 > 0$ is some constant. By substituting the resulting relations in (21), we get

$$\Delta J(\vartheta) = \sum_{k=1}^{n} \left(J_1(k) + J_2(k) \right) + o(\|\Delta \vartheta\|_H) \quad \text{as} \quad \|\Delta \vartheta\|_H \to 0,$$

where

$$J_1(k) = \int_0^T \left[\psi(s_k(t), t) + 2\alpha_1 \left(p_k(t) - \tilde{p}_k(t) \right) \right] \Delta p_k(t) dt,$$

$$J_2(k) = \int_0^T \left[\frac{\partial \psi(s_k(t), t)}{\partial x} p_k(t) + 2\alpha_2 \left(s_k(t) - \tilde{s}_k(t) \right) \right] \Delta s_k(t) dt.$$

Taking into account the expression for the Hamilton-Pontryagin function, we obtain

$$\Delta J = \left(-\frac{\partial H}{\partial \vartheta}, \Delta\vartheta\right)_H + o\left(\|\Delta\vartheta\|_H\right) \quad \text{for} \quad \|\Delta\vartheta\|_H \to 0,$$

which shows that the functional (5) is Frechet differentiable and the formula (20) is valid. ◀

Theorem 4. *Let the functions* $u^*(x,t)$, $\psi^*(x,t)$ *be solutions of problems* (1)-(3) *and* (15)-(17) , *respectively, for* $\vartheta = \vartheta^* \in V$. *Then for the control* ϑ^* *to be optimal, it is necessary that the condition*

$$H(t,\psi^*,\vartheta^*) = \max_{\vartheta\in V} H(t,\psi^*,\vartheta), \forall (x,t) \in \Omega, \tag{24}$$

hold.

Proof. Inside the domain Ω, we fix the Lebesgue point (σ,θ) of all functions involved in the conditions of problems (1)-(3) and (15)-(17). Let

$$\Pi_\varepsilon \equiv \left\{(x,t) : \sigma - \frac{\varepsilon}{2} < x < \sigma + \frac{\varepsilon}{2}, \theta - \frac{\varepsilon}{2} < t < \theta + \frac{\varepsilon}{2}\right\} \subset \Omega,$$

where $\varepsilon > 0$ is a sufficiently small number.

We construct the pulse variation of control

$$\vartheta^\varepsilon \equiv (p^\varepsilon, s^\varepsilon) = \begin{cases} \vartheta, \quad if \quad (x,t) \in \Pi_\varepsilon, \\ \vartheta^*, \quad if \quad (x,t) \notin \Pi_\varepsilon. \end{cases}$$

where ϑ is some constant vector, and denote $\Delta u_\varepsilon \equiv u_\varepsilon(x,t) - u^*(x,t)$, where $u_\varepsilon(x,t) = u(x,t;\vartheta^\varepsilon)$. Then, the function Δu_ε satisfies the identity

$$\begin{aligned} &\textstyle\int_0^l \int_0^T \left[-\Delta u_\varepsilon \frac{\partial \eta}{\partial t} + a^2 \frac{\partial \Delta u_\varepsilon}{\partial x}\frac{\partial \eta}{\partial x}\right] dxdt = \\ &= \textstyle\sum_{k=1}^n \int_0^T \left[p_k^\varepsilon + \Delta p_k^\varepsilon)\eta(s_k^\varepsilon + \Delta s_k^\varepsilon, t) - p_k^\varepsilon \eta(s_k^\varepsilon, t)\right] dt, \end{aligned} \tag{25}$$

$\forall \eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ with $\eta(x,T) = 0$.

Proceeding as in the proof of (9), we establish that the estimate

$$\|\Delta u_\varepsilon\|_{V_2^{1,0}(\Omega_T)} \le c_6 \|\Delta\vartheta^\varepsilon\|_{L_2(\Pi_\varepsilon)},$$

where $c_6 > 0$ is some constant, is valid for the function $\Delta u_\varepsilon(x,t)$. Then, the fact that $(\sigma,\theta) \in \Omega$ is the Lebesgue point implies $\Delta u_\varepsilon \to 0$ in $V_2^{1,0}(\Omega)$ as $\varepsilon \to 0$.

Let $\psi_\varepsilon = \psi_\varepsilon(x,t) \in V_2^{1,0}(\Omega)$ be the solution of the integral identity

$$\int_0^l \int_0^T \left(\psi \frac{\partial \eta_1}{\partial t} + a^2 \frac{\partial \psi}{\partial x} \frac{\partial \eta_1}{\partial x} \right) dxdt = 2 \int_0^l \int_0^T \left[u_\varepsilon(x,t) - \tilde{u}(x,t) + \frac{1}{2} \Delta u_\varepsilon(x,t) \right] \eta_1(x,t) dxdt, \tag{26}$$

$\forall \eta_1 = \eta_1(x,t) \in W_2^{1,1}(\Omega)$ with $\eta_1(x,T) = 0$. The difference $\psi_\varepsilon - \psi^*$satisfies the integral identity similar to (26). Then, the fact that $\Delta u_\varepsilon \to 0$ in $V_2^{1,0}(\Omega)$ as $\varepsilon \to 0$ implies $\psi_\varepsilon \to \psi^*$ in $V_2^{1,0}(\Omega)$ as $\varepsilon \to 0$.

We calculate the increment of the functional (5):

$$\Delta J(\overline{\vartheta}^*) \equiv J(\overline{\vartheta}^\varepsilon) - J(\overline{\vartheta}^*) = 2 \int_0^l \int_0^T \left[u^*(x,t) - \tilde{u}(x,t) + \tfrac{1}{2} \Delta u_\varepsilon(x,t) \right] \Delta u_\varepsilon(x,t) dxdt +$$
$$+ \sum_{k=1}^n \Big\{ 2\alpha_1 \int_0^T \left[p_k^*(t) - \tilde{p}_k(t) \right] \left[p_k^\varepsilon(t) - p_k^*(t) \right] dt + \alpha_1 \int_0^T \left[p_k^\varepsilon(t) - p_k^*(t) \right]^2 dt +$$
$$+ 2\alpha_2 \int_0^T \left[s_k^*(t) - \tilde{s}_k^*(t) \right] \left[s_k^\varepsilon(t) - s_k^*(t) \right] dt + \alpha_2 \int_0^T \left[s_k^\varepsilon(t) - s_k^*(t) \right]^2 dt \Big\} . \tag{27}$$

Proceeding as in the proof of (23) and using the identity (26), we obtain

$$2 \int_0^\ell \int_0^T \left[u^*(x,t) - \tilde{u}(x,t) + \frac{1}{2} \Delta u_\varepsilon(x,t) \right] \Delta u_\varepsilon(x,t) dxdt =$$
$$= \sum_{k=1}^n \int_{\Pi_\varepsilon} \left[\frac{\partial \psi_\varepsilon(s_k^\varepsilon(t), t)}{\partial x} p_k^\varepsilon(t) \Delta s_k^\varepsilon(t) + \psi_\varepsilon(s_k^\varepsilon, t) \Delta p_k^\varepsilon \right] dt.$$

Considering this relation in (27), we get

$$\Delta J(\vartheta^*) \equiv J(\vartheta^\varepsilon) - J(\vartheta^*) = \sum_{k=1}^n \int_{\Pi_\varepsilon} \left[\tfrac{\partial \psi_\varepsilon(s_k^\varepsilon(t), t)}{\partial x} p_k^\varepsilon(t) \Delta s_k^\varepsilon(t) + \psi_\varepsilon(s_k^\varepsilon, t) \Delta p_k^\varepsilon \right] dt +$$
$$+ \sum_{k=1}^n \Big\{ 2\alpha_1 \int_0^T \left[p_k^*(t) - \tilde{p}_k(t) \right] \left[p_k^\varepsilon(t) - p_k^*(t) \right] dt + \alpha_1 \int_0^T \left[p_k^\varepsilon(t) - p_k^*(t) \right]^2 dt +$$
$$+ 2\alpha_2 \int_0^T \left[s_k^*(t) - \tilde{s}_k^*(t) \right] \left[s_k^\varepsilon(t) - s_k^*(t) \right] dt + \alpha_2 \int_0^T \left[s_k^\varepsilon(t) - s_k^*(t) \right]^2 dt \Big\} .$$

Then, we get from the expression for the Hamilton-Pontryagin function (19) that

$$\Delta J(\vartheta^*) = - \int_{\Pi_\varepsilon} \left[H(t, \psi_\varepsilon, \vartheta^\varepsilon) - H(t, \psi_\varepsilon, \vartheta^* \right] dt.$$

By virtue of the fact that $\psi_\varepsilon \to \psi^*$ in $V_2^{1,0}(\Omega)$, we obtain the formula for the variation of functional (5):

$$\delta J(\vartheta^*) = \lim_{\varepsilon \to 0} \frac{\Delta J(\vartheta^*)}{\varepsilon} = - \left[H(\theta, \psi^*, \vartheta) - H(\theta, \psi^*, \vartheta^* \right] .$$

It follows from the optimality of the control $\vartheta \in V$ that $\delta J(\vartheta^*) \geq 0$. From this fact and the fact that the Lebesgue points are dense everywhere in Ω, we get the validity of (24). ◄

Theorem 5. *For the control* $\vartheta^* = (p^*(t), s^*(t)) \in V$ *to be optimal, it is necessary that the condition*

$$< J'(\vartheta^*), \vartheta - \vartheta^* >_H = \sum_{k=1}^{n} \int_0^T \{[\psi^*(s_k^*(t), t) + 2\alpha_1 (p_k^*(t) - \tilde{p}_k(t))] (p_k(t) - p_k^*(t)) +$$

$$+ [\psi_x^*(s_k^*(t), t) p_k^*(t) + 2\alpha_2 (s_k^*(t) - \tilde{s}_k(t)] (s_k(t) - s_k^*(t))\} \, dt \geq 0, \tag{28}$$

$\forall \vartheta \in V$ *hold. Here* $\psi^*(x,t)$ *is the solution of the problem* (15)-(17) *for* $\vartheta = \vartheta^* \in V$.

Proof. By virtue of the well-known theorem of [17, p.28], for the control $\vartheta^* = (p^*(t), s^*(t)) \in V$ to be optimal, it is necessary that the inequality

$$< J'(\vartheta^*), \vartheta - \vartheta^* >_H \geq 0, \forall \vartheta \in V \tag{29}$$

hold.

Using (20) and the Hamilton-Pontryagin function, we calculate the gradient of the functional (5) and substitute it in (29) to demonstrate the validity of inequality (28). ◄

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