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General Inverse Sturm-Liouville Problem with Symmetric Potential

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Abstract. The uniqueness theorems for an inverse nonselfadjoint Sturm-Liouville problem with symmetric potential and general boundary conditions are proved. The spectral data used for unique reconstruction of Sturm-Liouville problems are a spectrum and six eigenvalues. The uniqueness theorems for an inverse selfadjoint Sturm-Liouville problem with symmetric potential and nonseparated boundary conditions are also proved. These theorems use a spectrum and two (or three) eigenvalues for unique reconstruction of Sturm-Liouville problems. The theorems generalise G. Borg and N. Levinson's classical results to the case of Sturm-Liouville problem with general boundary conditions. Schemes for unique reconstruction of Sturm-Liouville problems with symmetric potential and general boundary conditions are given.

Key Words and Phrases: The inverse eigenvalue problem, the inverse Sturm-Liouville problem with symmetric potential, the nonseparated boundary conditions

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1. Introduction

Let L denote the Sturm-Liouville problem

$$ly = -y'' + q(x)\,y = \lambda\,y = s^2\,y, \tag{1}$$

$$U_i(y) = a_{i1}\,y(0) + a_{i2}\,y'(0) + a_{i3}\,y(\pi) + a_{i4}\,y'(\pi) = 0, \qquad i = 1, 2, \tag{2}$$

where $q(x) \in L_1(0, \pi)$ is a real function such that $q(x) = q(\pi - x)$ almost everywhere (a.e.) and the a_{ij} with $i = 1$, 2 and $j = 1$, 2, 3, 4 are complex constants.

The boundary value problem for differential equations of second order and the inverse Sturm-Liouville problem for L in the case of separated boundary conditions ($a_{13} = a_{14} = a_{21} = a_{22} = 0$) have been well studied (see [1, 2, 3, 6, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 24, 29, 30, 32, 34]). The inverse Sturm-Liouville problem with unknown coefficients in nonseparated boundary conditions was studied by V.A. Sadovnichii, V.A. Yurko, V.A. Marchenko, O.A. Plaksina, M.G. Gasymov, I.M. Guseinov, I.M. Nabiev, and other authors (see [5, 7, 8, 9, 18, 22, 25, 26, 27, 28, 31, 33]).

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For the inverse problem of reconstructing L in which all coefficients a_{ij} with $i = 1, 2$ and $j = 1, 2, 3, 4$ are unknown, no uniqueness theorems have been proved. Special cases of problem L with boundary conditions

$$V_1(y) = a_{11}\, y(0) + y'(0) + a_{13}\, y(\pi) = 0, \tag{3}$$

$$V_2(y) = a_{21}\, y(0) + a_{23}\, y(\pi) + y'(\pi) = 0, \tag{4}$$

and

$$P_1(y) = y(0) + \omega\, y(\pi) = 0 \tag{5}$$
$$P_2(y) = \overline{\omega}\, y'(0) + y'(\pi) + \alpha\, y(\pi) = 0, \tag{6}$$

have been earlier studied. Note that general selfadjoint nonseparated boundary conditions (2) can be reduced to one of the following types:

(i) the boundary conditions (3),(4), where a_{11} and a_{23} are any real numbers, $a_{13} \neq 0$ is any complex number, and $a_{21} = -\overline{a_{13}}$;

(ii) the boundary conditions (5), (6), where $\omega \neq 0$ is any complex number and α is any real number.

To uniquely reconstruct these boundary value problems with asymmetric potential, in addition to the spectrum of the problem itself, the spectra of two boundary value problems, a certain sequence of signs, and a certain real number were used (see, e.g., [22, 23]).

In this paper, we prove a theorem on the unique reconstruction of problem L with symmetric potential and general boundary conditions (2), which may be nonselfadjoint. As spectral data only the eigenvalues of three spectral problems are used.

2. Generalizations of Borg's Uniqueness Theorems

In 1946, Borg proved several uniqueness theorems for the solution of the inverse Sturm-Liouville problem [6, p. 69]. Two of them referred to the following spectral problems B_1 and B_2 with $q(x) \in L_1(0, \pi)$.

Problem B_1:

$$ly = -y'' + q(x)\, y = \lambda\, y, \quad y(0) = 0, \quad y(\pi) = 0, \qquad q(x) = q(\pi - x) \text{ a.e.}$$

Problem B_2:

$$ly = -y'' + q(x)\, y = \lambda\, y, \quad y'(0) = 0, \quad y'(\pi) = 0, \qquad q(x) = q(\pi - x) \text{ a.e.}$$

For these problems, Borg proved the following theorems P_1 and P_2 (in Borg's notation) [6, p. 69].

Theorem P_1. *The function $q(x)$ in (1) is uniquely determined by the spectrum of Problem B_1 if $q(x) = q(x - \pi)$ a.e.*

Theorem P_2. *The function* $q(x)$ *in (1) is uniquely determined by the spectrum of Problem* B_2 *if* $q(x) = q(x - \pi)$ *a.e.*

In this paper, we generalize these theorems to the case of general boundary conditions (2).

In what follows, we denote a problem of type L, but with different coefficients in the equation and different parameters in the boundary forms, by $\widetilde{L}$. Throughout the paper, we assume that if some symbol denotes an object from Problem L then the same symbol with the tilde $\tilde{}$ denotes the corresponding object from Problem $\widetilde{L}$.

Let A be the matrix composed of coefficients a_{lk} of the boundary conditions (2), i.e.,

$$A = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{matrix} \right\|, \tag{7}$$

and let M_{ij} be its minors composed of ith and jth columns:

$$M_{ij} = \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}, \qquad i, j = 1, 2, 3, 4.$$

Vectors are denoted by boldface letters. The symbol ^{T} denotes transposition. Column vectors are represented by rows with this superscript. For the rank of the matrix A we use the notation $\operatorname{rank} A$.

Together with Problems L, B_1, and B_2, we consider the following Problems L_1 and L_2.

Problem L_1.

$$\begin{aligned} ly &= -y'' + q(x)\, y = \lambda\, y, \\ U_{1,1}(y) &= y(0) - p(\lambda)\, y'(0) = 0, \\ U_{2,1}(y) &= y(\pi) = 0, \end{aligned}$$

Problem L_2.

$$\begin{aligned} ly &= -y'' + q(x)\, y = \lambda\, y, \\ U_{1,1}(y) &= y'(0) - p(\lambda)\, y(0) = 0, \\ U_{2,1}(y) &= y'(\pi) = 0. \end{aligned}$$

In Problems L_1 and L_2, $p(\lambda)$ is a polynomial of the form

$$p(\lambda) = M_{12} + (1 - M_{13})\, \lambda + (M_{14} - M_{32})\, \lambda^2 + M_{42}\, \lambda^3 + M_{34}\, \lambda^4.$$

Theorem 1. *If Problems* L *and* $\widetilde{L}$ *have a nonempty discrete spectrum; the spectra of Problems* L *and* $\widetilde{L}$, B_1 *and* $\widetilde{B}_1$, L_1 *and* $\widetilde{L}_1$ *coincide with algebraic multiplicities taken into account; and* $\operatorname{rank} A = 2$, *then these boundary value problems themselves coincide, i.e.,* $q(x) = \widetilde{q}(x)$ *a.e. and the matrices* $A = (a_{ij})_{2\times 4}$ *and* $\widetilde{A} = (\widetilde{a}_{ij})_{2\times 4}$ *of coefficients in the boundary conditions coincide up to a linear transformation of their rows.*

Proof. Applying Borg's uniqueness theorem P_1 [6, p. 69] for the inverse Sturm-Liouville problem with symmetric potential to Problem B_1, we see that

$$q(x) = \widetilde{q}(x) \quad a.e. \tag{8}$$

Let us show that, for the vectors $\mathbf{N} = (M_{12}, M_{13}, M_{14}, M_{32}, M_{42}, M_{34})^T$ and $\widetilde{\mathbf{N}} = \left(\widetilde{M}_{12}, \widetilde{M}_{13}, \widetilde{M}_{14}, \widetilde{M}_{32}, \widetilde{M}_{42}, \widetilde{M}_{34}\right)^T$ composed of the minors of the matrices $(a_{ij})_{2\times 4}$ and $(\widetilde{a}_{ij})_{2\times 4}$ respectively, we have

$$\mathbf{N} = \widetilde{\mathbf{N}}. \tag{9}$$

Let $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ be linearly independent solutions of Eq. (1) satisfying the conditions

$$y_1(0,\, \lambda) = 1, \quad y_1'(0,\, \lambda) = 0, \quad y_2(0,\, \lambda) = 0, \quad y_2'(0,\, \lambda) = 1. \tag{10}$$

The eigenvalues of Problem L are the roots of the entire function ([17, pp. 33–36], [19, p. 29])

$$\begin{gathered}\Delta(\lambda) = M_{12} + M_{34} + M_{32}\, y_1(\pi,\, \lambda) + M_{42}\, y_1'(\pi,\, \lambda) + \\ + M_{13}\, y_2(\pi,\, \lambda) + M_{14}\, y_2'(\pi,\, \lambda),\end{gathered} \tag{11}$$

and the eigenvalues of Problem L_1 are the roots of the entire function

$$\Delta_1(\lambda) = y_2(\pi,\, \lambda) - p(\lambda)\, y_1(\pi,\, \lambda).$$

If $\Delta(\lambda) \not\equiv 0$ (i.e., the spectrum of the boundary value problem is discrete), then, according to Hadamard's theorem, the function $\Delta(\lambda)$ (which is entire of order 1/2) can be reconstructed from its zeros up to a factor $C \neq 0$. Therefore, the functions $\Delta(\lambda)$ and $\widetilde{\Delta}(\lambda)$ are related by the identity

$$\Delta(\lambda) \equiv C\, \widetilde{\Delta}(\lambda), \tag{12}$$

where C is a nonzero constant.

If $\Delta(\lambda) \equiv 0$ (i.e., each λ is an eigenvalue of Problem L), then the condition that the eigenvalues of Problems L and $\widetilde{L}$ coincide also implies (12) (whence $\widetilde{\Delta}(\lambda) \equiv 0$).

Similarly, we have

$$\Delta_1(\lambda) \equiv C_1\, \widetilde{\Delta}_1(\lambda),$$

where C_1 is a nonzero constant.

The following asymptotic relations hold:

$$\begin{aligned}
y_1(x,\, \lambda) &= \cos sx + \frac{1}{s}\, u(x)\, \sin sx + \mathcal{O}\left(\frac{1}{s^2}\right), \\
y_2(x,\, \lambda) &= \frac{1}{s}\, \sin sx - \frac{1}{s^2}\, u(x)\, \cos sx + \mathcal{O}\left(\frac{1}{s^3}\right), \\
y_1'(x,\, \lambda) &= -s\, \sin sx + u(x)\, \cos sx + \mathcal{O}\left(\frac{1}{s}\right), \\
y_2'(x,\, \lambda) &= \cos sx + \frac{1}{s}\, u(x)\, \sin sx + \mathcal{O}\left(\frac{1}{s^2}\right),
\end{aligned}$$

$$\text{where } u(x) = \frac{1}{2}\int_0^x q(t)\,dt,$$

for sufficiently large $\lambda \in \mathbb{R}$ ([19, pp. 62–65]).

It follows from these relations that the functions $y_1(\pi, \lambda)$ and $y_2(\pi, \lambda)$ in the decomposition of the function $\Delta_1(\lambda)$ are linearly independent. Therefore,

$$M_{12} = \widetilde{M}_{12}, \quad M_{13} = \widetilde{M}_{13}, \quad M_{14} - M_{32} = \widetilde{M}_{14} - \widetilde{M}_{32}, \quad M_{42} = \widetilde{M}_{42}, \quad M_{34} = \widetilde{M}_{34}. \tag{13}$$

The functions $y_1(\pi, \lambda) = y_2'(\pi, \lambda)$, $y_1'(\pi, \lambda)$, $y_2(\pi, \lambda)$, and 1 in the decomposition of $\Delta(\lambda)$ are linearly independent as well (the relation $y_1(\pi, \lambda) = y_2'(\pi, \lambda)$ holds if and only if $q(x) = q(x - \pi)$ [31, Lemma 4]). This observation, together with (11) and (12), implies

$$\begin{gathered} M_{12} + M_{34} = C\,(\widetilde{M}_{12} + \widetilde{M}_{34}), \quad M_{14} + M_{32} = C\,(\widetilde{M}_{14} + \widetilde{M}_{32}), \\ M_{42} = C\,\widetilde{M}_{42}, \quad M_{13} = C\,\widetilde{M}_{13}. \end{gathered} \tag{14}$$

At least one of the numbers $M_{12} + M_{34}$, $M_{32} + M_{14}$, M_{42}, and M_{13} is different from zero. Otherwise, we would have $\Delta(\lambda) \equiv 0$ in contradiction to the assumption of the theorem that Problems L and $\widetilde{L}$ have discrete spectrum. This observation, together with (13) and (14), implies

$$\begin{gathered} C = 1, \quad M_{12} = \widetilde{M}_{12}, \quad M_{13} = \widetilde{M}_{13}, \quad M_{14} = \widetilde{M}_{14}, \\ M_{32} = \widetilde{M}_{32}, \quad M_{42} = \widetilde{M}_{42}, \quad M_{34} = \widetilde{M}_{34}, \end{gathered}$$

whence we obtain (9).

It follows from (9) (see [4, p.32]) that the matrices $(a_{ij})_{2\times4}$ and $(\widetilde{a}_{ij})_{2\times4}$ coincide up to a linear transformation of the rows. Combining this with (8), we see that the boundary value problems L and $\widetilde{L}$ coincide.◀

Under certain conditions, the following theorem (stronger than Theorem 1) holds true.

Theorem 2. *If λ_0 is an eigenvalue of Problem L, λ_i ($i = 1, 2, 3, 4, 5$) are any pairwise distinct eigenvalues of Problem L_1, $y_1(\pi, \lambda_i) \neq 0$, $i = 0, 1, 2, 3, 4, 5$, and $q(x) = q(\pi - x)$ a.e., then Problems L, B_1, and L_1 are uniquely determined by the spectrum of Problem B_1 and λ_i, $i = 0, 1, 2, 3, 4, 5$, i.e. the function $q(x)$ is uniquely determined and the matrix $(a_{ij})_{2\times4}$ is determined up to a linear transformation of the rows.*

Proof. Applying Borg's uniqueness theorem P_1 [6] for the inverse Sturm-Liouville problem with symmetric potential to Problem B_1, we see that the function $q(x)$ in (1) is uniquely determined by the spectrum of Problem B_1.

The numbers λ_i ($i = 1, 2, 3, 4, 5$) are eigenvalues of Problem L_1, so $\Delta_1(\lambda_i) = 0$, $i = 1, 2, 3, 4, 5$. It now follows that

$$M_{12} + (1 - M_{13})\,\lambda_i + (M_{14} - M_{32})\,\lambda_i^2 + M_{42}\,\lambda_i^3 + M_{34}\,\lambda_i^4 = \frac{y_2(\pi, \lambda_i)}{y_1(\pi, \lambda_i)}. \tag{15}$$

The determinant of system (15) with respect to the unknowns M_{12}, $(1 - M_{13})$, $(M_{14} - M_{32})$, M_{42}, M_{34} is the fifth-order Vandermonde determinant equal to $\prod_{k_1 > k_2}(\lambda_{k_1} - \lambda_{k_2})$. Therefore, system (15) has a unique solution, which can be found by the Cramer formulas.

Since λ_0 is an eigenvalue of Problem L, it follows from (11) that

$$(M_{32}+M_{14})\, y_1(\pi,\,\lambda_0) = -(M_{12}+M_{34}) - M_{42}\, y_1'(\pi,\,\lambda_0) - M_{13}\, y_2(\pi,\,\lambda_0) \qquad (16)$$

$(y_1(\pi,\,\lambda_0) = y_2'(\pi,\,\lambda_0)$ iff $q(x) = q(\pi - x)$ a.e.).

Combining (15) and (16), we see that the unknowns M_{12}, M_{13}, M_{14} M_{32}, M_{42}, M_{34} are uniquely determined. It follows (see [4, p.32]) that the matrix $(a_{ij})_{2\times 4}$ is determined up to a linear transformation of the rows. Hence Problems L, B_1, and L_1 are uniquely determined by the spectrum of Problem B_1 and λ_i, $i = 0, 1, 2, 3, 4, 5$.◄

Remark 1. *If $L = B_1$, then $L = B_1 = L_1$. Therefore, Borg's uniqueness theorems P_1 is a special case of Theorems 1 and 2 proved above.*

Scheme for identification of Problems L, L_1, and B_1.

On the basis of the theorems proved above, it is easy to obtain a procedure for reconstructing Problem L. The potential $q(x)$ can be reconstructed by any known method for reconstructing the potential of a Sturm-Liouville problem (see [6, 7, 34]), and the boundary conditions can be reconstructed by the methods for identifying boundary conditions (see, e.g., [4]).

By Theorem 2, we can give the Scheme for identification of Problems L, L_1, and B_1:

Step 1. By Borg's theorem P_1 or M.G.Gasymov, I.M.Guseinov, and I.M.Nabiev's method [7], we find $q(x)$.

Step 2. By the function $q(x)$, we find the linearly independent solutions $y_1(x,\,\lambda)$ and $y_2(x,\,\lambda)$ of equation (1), satisfying conditions (10).

Step 3. By the numbers λ_i, $i = 0, 1, 2, 3, 4, 5$ satisfying the conditions of Theorem 2, we find the solution of systems (15) and (16) (M_{12}, M_{13}, M_{14} M_{32}, M_{42}, M_{34}).

Step 4. By the determinants M_{12}, M_{13}, M_{14} M_{32}, M_{42}, M_{34}, we find the matrix $(a_{ij})_{2\times 4}$ determined up to a linear transformation of the rows. The matrix $(a_{ij})_{2\times 4}$ is determined by the matrix identification methods (see [4, pp.33-34])).

Theorem 3. *If Problems L and $\widetilde{L}$ have a nonempty discrete spectrum; the spectra of Problems L and $\widetilde{L}$, B_2 and $\widetilde{B}_2$, L_2 and $\widetilde{L}_2$ coincide with algebraic multiplicities taken into account; and* $\operatorname{rank} A = 2$, *then these boundary value problems themselves coincide, i.e., $q(x) = \widetilde{q}(x)$ a.e. and the matrices $A = (a_{ij})_{2\times 4}$ and $\widetilde{A} = (\widetilde{a}_{ij})_{2\times 4}$ of coefficients in the boundary conditions coincide up to a linear transformation of their rows.*

Proof. Applying Borg's uniqueness theorem P_2 [6] for the inverse Sturm-Liouville problem with symmetric potential to Problem B_2, we see that (8) holds almost everywhere. Relation (9) is proved in the same way as in the proof of Theorem 1 with the only difference that the characteristic function $\Delta_1(\lambda)$ should be replaced by

$$\Delta_2(\lambda) = -y_1'(\pi,\,\lambda) - p(\lambda)\, y_2'(\pi,\,\lambda)$$

and instead of the linear independence of the functions $y_1(\pi, \lambda)$ and $y_2(\pi, \lambda)$ in the decomposition of $\Delta_1(\lambda)$ the linear independence of the functions $y_1'(\pi, \lambda)$ and $y_2'(\pi, \lambda)$ in the decomposition of $\Delta_2(\lambda)$ should be used.◄

Under certain conditions, the following theorem (stronger than Theorem 3) holds true.

Theorem 4. *If λ_0 is an eigenvalue of Problem L, λ_i ($i = 1, 2, 3, 4, 5$) are any pairwise distinct eigenvalues of Problem L_1, $y_1(\pi, \lambda_i) \neq 0$, $i = 0, 1, 2, 3, 4, 5$, and $q(x) = q(\pi - x)$ a.e., then Problems L, B_2, and L_2 are uniquely determined by the spectrum of Problem B_2 and λ_i, $i = 0, 1, 2, 3, 4, 5$, i.e. the function $q(x)$ is uniquely determined and the matrix $(a_{ij})_{2\times 4}$ is determined up to a linear transformation of the rows.*

Proof. This theorem is proved in the same way as Theorem 3 with the only difference that the system (15) should be replaced by

$$M_{12} + (1 - M_{13})\,\lambda_i + (M_{14} - M_{32})\,\lambda_i^2 + M_{42}\,\lambda_i^3 + M_{34}\,\lambda_i^4 = -\frac{y_1'(\pi, \lambda_i)}{y_1(\pi, \lambda_i)} \tag{17}$$

and instead of the linear independence of the functions $y_1(\pi, \lambda)$ and $y_2(\pi, \lambda)$ in the decomposition of $\Delta_1(\lambda)$ the linear independence of the functions $y_1'(\pi, \lambda)$ and $y_2'(\pi, \lambda)$ in the decomposition of $\Delta_2(\lambda)$ should be used.◄

Remark 2. *If $L = B_2$, then $L = B_2 = L_2$. Therefore, Borg's uniqueness theorem P_2 is a special case of Theorems 3 and 4.*

Scheme for identification of Problems L, L_2, and B_2 is similar to the scheme for identification of Problems L, L_1, and B_1.

3. Generalizations of Levinson's Uniqueness Theorem

In 1949, Levinson considered the following Sturm-Liouville problem L_0 with symmetric potential [12].

Problem L_0:

$$ly = -y'' + q(x)\,y = \lambda\,y, \quad y'(0) - h\,y(0) = 0, \quad y'(\pi) + h\,y(\pi) = 0, \qquad h \in \mathbb{R}.$$

For this problem, Levinson proved the following theorem.

Theorem. If $q(x) = q(x - \pi)$, then the function $q(x)$ and the number h are uniquely determined by the spectrum of Problem L_0.

This section contains generalizations of this theorem to the case of noseparated boundary conditions.

Consider the following spectral problem.

Problem Y_1 :

$$ly \;=\; -y'' + q(x)\,y = \lambda\,y,$$

$$\begin{aligned} U_{1,1}(y) &= a_{11}\, y(0) + y'(0) - a_{21}\, y(\pi) = 0, \\ U_{2,1}(y) &= a_{21}\, y(0) + a_{23}\, y(\pi) + y'(\pi) = 0, \qquad a_{11}, a_{21}, a_{23} \in \mathbb{R}. \end{aligned}$$

The boundary conditions of Problem Y_1 coincide with the boundary conditions (3),(4), where a_{11}, a_{23}, a_{13}, and a_{21} are any real numbers and $a_{21} = -a_{13}$. Yurko showed in [33] that Problem Y_1 can be uniquely reconstructed from two spectra and a sequence of signs, namely, from the spectrum of Problem Y_1, the spectrum $\{z_n\}$ of the problem for Eq. (2) and boundary conditions $y'(0) + a_{11}\, y(0) = y(\pi) = 0$, and the sequence of signs $\omega_n = \operatorname{sign}\left(|\theta'(\pi, z_n)| - |a_{21}|\right)$, where $\theta(x, \lambda)$ is the solution of Eq. (2) under the boundary conditions $\theta(0, \lambda) = 1$, $\theta'(0, \lambda) = -a_{11}$.

In what follows, we show that if the potential of Problem Y_1 is symmetric, then Problem Y_1 can be reconstructed from two spectra (a sequence of signs is not needed in this case).

Let Y_2 denote the following spectral problem.

Problem Y_2 :

$$-y'' + q(x)\, y = \lambda\, y, \quad a_{11}\, y(0) + y'(0) = 0, \quad -a_{11}\, y(\pi) + y'(\pi) = 0.$$

Theorem 5. *If $q(x) = q(\pi - x)$, $\widetilde{q}(x) = \widetilde{q}(\pi - x)$, and the spectra of Problems Y_1 and $\widetilde{Y}_1$, Y_2 and $\widetilde{Y}_2$ coincide with algebraic multiplicities taken into account, then these boundary value problems themselves coincide, i.e., $q(x) = \widetilde{q}(x)$, $a_{11} = \widetilde{a}_{11}$, $a_{21} = \widetilde{a}_{21}$, and $a_{23} = \widetilde{a}_{23}$.*

Proof. Applying Levinson's uniqueness theorem [12] to problem Y_2, we see that, for the inverse Sturm-Liouville problem with symmetric potential,

$$q(x) = \widetilde{q}(x), \quad a_{11} = \widetilde{a_{11}}. \tag{18}$$

To prove the theorem, it remains to prove the relations $a_{21} = \widetilde{a}_{21}$ and $a_{23} = \widetilde{a}_{23}$. The eigenvalues of Problem Y_1 are the roots of the entire function

$$\begin{aligned} \Delta_3(\lambda) = -a_{21} - a_{23}\, y_1(\pi,\, \lambda) - y_1'(\pi,\, \lambda) + \\ +(a_{11}\, a_{23} + a_{21}^2)\, y_2(\pi,\, \lambda) + a_{11}\, y_2'(\pi,\, \lambda). \end{aligned} \tag{19}$$

According to Hadamard's theorem, the function $\Delta(\lambda)$ (which is entire of order 1/2) can be reconstructed from its zeros up to a multiplier $C \neq 0$. Therefore, the functions $\Delta_3(\lambda)$ and $\widetilde{\Delta}_3(\lambda)$ are related by the identity

$$\Delta_3(\lambda) \equiv C_3\, \widetilde{\Delta}_3(\lambda), \tag{20}$$

where C is a nonzero constant.

It follows from the asymptotic relations that the functions $y_1'(\pi,\, \lambda)$, $y_2'(\pi,\, \lambda) \equiv y_1(\pi,\, \lambda)$, $y_2(\pi,\, \lambda)$, and 1 are linearly independent. Therefore, $C_3 = 1$, $a_{21} = \widetilde{a}_{21}$, $a_{23} = \widetilde{a}_{23}$.◄

Under certain conditions, stronger results than Theorem 5 hold true.

Suppose value a_{11} and function $q(x)$ are reconstructed. Then linearly independent solutions $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ of equation (1) under conditions (10) are known. So we can state the following conditions.

Condition 1. Numbers λ_1 and λ_2 satisfy equations

$$y_2(\pi, \lambda_1) = y_2(\pi, \lambda_2) = 0.$$

and inequalities

$$y_1(\pi, \lambda_2) - y_1(\pi, \lambda_1) \neq 0.$$

Condition 2. Numbers λ_1, λ_2 and λ_3 satisfy equation

$$\begin{vmatrix} 1 & y_1(\pi, \lambda_1) - a_{11}\, y_2(\pi, \lambda_1) & y_2(\pi, \lambda_1) \\ 1 & y_1(\pi, \lambda_2) - a_{11}\, y_2(\pi, \lambda_2) & y_2(\pi, \lambda_2) \\ 1 & y_1(\pi, \lambda_3) - a_{11}\, y_2(\pi, \lambda_3) & y_2(\pi, \lambda_3) \end{vmatrix} \neq 0.$$

Theorem 6. *If λ_1 and λ_2 are eigenvalues of Problem Y_1 and satisfy Condition 1, then Problems Y_1 and Y_2 (the function $q(x)$ and coefficients a_{11}, a_{21} and a_{23}) are uniquely determined by the spectrum of Problem Y_2 and λ_i, $i = 1, 2$.*

Proof. Applying Levinson's uniqueness theorem [12] to problem Y_2, we see that the function $q(x)$ and the coefficient a_{11} are uniquely determined by the spectrum of Problem Y_2. To prove the theorem, it remains to find the coefficients a_{21} and a_{23}.

The numbers λ_i $(i = 1, 2)$ are eigenvalues of Problem Y_1, so $\Delta_3(\lambda_i) = 0$, $i = 1, 2$. It now follows that

$$a_{21} + a_{23}\, y_1(\pi,\, \lambda_i) = -y_1'(\pi,\, \lambda_i) + a_{11}\, y_2'(\pi,\, \lambda_i), \qquad i = 1, 2. \tag{21}$$

The determinant of system (21) with respect to the unknowns a_{21} and a_{23} is equal to $\big(y_1(\pi, \lambda_2) - y_1(\pi, \lambda_1)\big) \neq 0$. Therefore, system (21) has a unique solution, which can be found by the Cramer formulas.

Hence Problems Y_1 and Y_2 are uniquely determined by the spectrum of Problem Y_2 and two eigenvalues of Problem Y_1.◄

Theorem 7. *If λ_1, λ_2 and λ_3 are eigenvalues of Problem Y_1 and satisfy Condition 2, then Problems Y_1 and Y_2 (the function $q(x)$ and coefficients a_{11}, a_{21} and a_{23}) are uniquely determined by the spectrum of Problem Y_2 and λ_i, $i = 1, 2, 3$.*

Proof. This theorem is proved in the same way as Theorem 6 with the only difference that the system (21) should be replaced by the system of three equations

$$a_{21} + a_{23}\, \big(y_1(\pi, \lambda_i) - a_{11}\, y_2(\pi, \lambda_i)\big) - a_{21}^2\, y_2(\pi,\, \lambda) = -y_1'(\pi,\, \lambda_i) + a_{11}\, y_2'(\pi,\, \lambda_i) \tag{22}$$

with respect to three unknowns a_{21}, a_{23}, a_{21}^2. ◄

Remark 3. *Note that Levinson's uniqueness theorem is a special case of Theorems 5, 6 and 7 proved above. Indeed, for $L = L_0$, i.e., in the case where $a_{11} = -h$, $a_{12} = 1$, $a_{13} = a_{14} = 0$, $a_{21} = a_{22} = 0$, $a_{23} = h$, and $a_{24} = 1$, we have $Y_1 = Y_2$; therefore, the problem of identification from the spectrum of Problem Y_2 and the eigenvalues of Problem Y_1 reduces to the problem of identification from one spectrum.*

By Theorems 6 and 7 we can give the Scheme for identification of Problems Y_1 and Y_2:

Step 1. By N.Levinson's theorem, we find $q(x)$ and coefficient a_{11}.

Step 2. By the function $q(x)$, we find the linearly independent solutions $y_1(x, \lambda)$ and $y_2(x, \lambda)$ of equation (1), satisfying conditions (10).

Step 3. By the eigenvalues λ_i, $i = 1, 2$ (or $i = 1, 2, 3$) of Problem Y_1 satisfying Condition 1 (or Condition 2), we find the solution of system (21) or (22) (the coefficients a_{21} and a_{23}).

4. Further Generalisations

In [18], Borg's theorems P_1 and P_2 and M.G.Gasymov, I.M.Guseinov, and I.M.Nabiev's results [7] were generalized to the case where Eq. (1) is replaced by

$$ly = -y'' + 2\,s\,q_1(x) + q(x)\,y = s^2\,y, \tag{23}$$

where $q_1(x) \in W_2^1(0, \pi)$, and $q(x) \in L_2(0, \pi)$ are real functions, $q(x) = q(\pi - x)$, and $q_1(x) = q_1(\pi - x)$. It can be shown by using these results and the methods applied in the proofs of Theorems 1 and 3 that, in this case, the three spectra uniquely determine not only the function $q(x)$ and the boundary conditions (2), but also the function $q_1(x)$.

It follows from [18, Remark 2, p. 40] that Theorem 5 can be generalized to the case where the equation (1) is replaced by the equation (23). In this case, not only the function $q(x)$ and the boundary condition, but also the function $q_1(x)$ can be uniquely reconstructed using two spectra

The corresponding analogues of Theorems 2, 4, 6 and 7 can be also proved.

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