Azerbaijan Journal of Mathematics
V. 5, No 2, 2015, July
ISSN 2218-6816

$\mathcal{D}$-Hypercyclic and $\mathcal{D}$-Topologically Mixing Properties of Degenerate Multi-Term Fractional Differential Equations

M. Kostić

Abstract. In this paper, we introduce the notions of $\mathcal{D}$-hypercyclicity and $\mathcal{D}$-topologically mixing property of degenerate abstract multi-term fractional differential equations with Caputo fractional derivatives. The obtained results are illustrated with some examples.

Key Words and Phrases: abstract multi-term fractional differential equations, degenerate equations, Caputo fractional derivatives, hypercyclicity, topologically mixing property, well-posedness, separable locally convex spaces.
2010 Mathematics Subject Classifications: 47A16, 47D06, 47G20, 47D99

1. Introduction and preliminaries

We shall work in the setting of separable, infinite-dimensional, Hausdorff, sequentially complete locally convex spaces over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let E be such a space. Then a linear operator T on E is said to be hypercyclic if there exists an element $x \in D_\infty(T) \equiv \bigcap_{n\in\mathbb{N}} D(T^n)$ whose orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in E; T is said to be topologically transitive, resp. topologically mixing, if for every pair of open non-empty subsets U, V of E, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$, resp. if for every pair of open non-empty subsets U, V of E, there exists $n_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq n_0$, one has $T^n(U) \cap V \neq \emptyset$.

In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in engineering, physics, chemistry, biology and other sciences. Fairly complete information on fractional calculus and fractional differential equations can be found in [7], [16], [26]-[32] and [41]-[43]. On the other hand, in the past decades a great number of researchers from different areas have contributed to the field of linear dynamics. Concerning the linear dynamics of single operators, we can recommend for the reader the monographs [6] by F. Bayart, E. Matheron and [19] by K.-G. Grosse-Erdmann, A. Peris. The third chapter of monograph [25] contains the basic information about hypercyclic and topologically mixing properties of various classes of abstract Volterra integro-differential equations.

In this paper, we shall reconsider the notions of hypercyclicity and topologically mixing property of degenerate abstract multi-term fractional differential equations ([32]). Works

 © *2010* *AZJM* All rights reserved.

by K.-G. Grosse-Erdmann-S. G. Kim [18] and J. Bès-J. A. Conejero [8] were the motivation for writing this paper. The author would also like to acknowledge several fruitful discussions with Professor J. A. Conejero, which influenced him to write this paper. In [18], the authors have proposed the way of computing the orbit of a pair (x, y) under the action of a bilinear mapping $B : E \times E \to E$, with E being a separable Banach space. After that, the notion of bihypercyclicity of mapping B has been introduced. In the remaining part of [18], several interesting examples of bihypercyclic bilinear mappings have been presented; it has also been shown that every separable Banach space supports a bihypercyclic bilinear mapping and every separable Banach space E supports a bihypercyclic symmetric bilinear mapping whenever E supports a non-injective hypercyclic operator. In the setting of infinite-dimensional separable Fréchet spaces, a slightly different way of computing the orbit of a pair (x, y) under the action of bilinear mapping B has been proposed in [8, Definition 1]. The notion of orbit of a tuple $(x_1, x_2, \cdot \cdot \cdot, x_n) \in E^N$ under the action of an N-linear operator $M : E^n \to E$ as well as the notion of supercyclicity of the operator M has been introduced in the same definition, while the notion of N-linear Devaney chaos of M has been introduced in [8, Definition 18] $(N \geq 2)$. In [8, Theorem 5, Theorem 8], it has been proved that every separable infinite-dimensional Fréchet space E supports, for any integer $N \geq 2$, an N-linear operator having a residual set of supercyclic vectors and, for any integer $N \geq 2$ there exists an N-linear operator on the space $\omega = \mathbb{K}^{\mathbb{N}}$ (endowed with the product topology) that supports a dense N-linear orbit. The existence of hypercyclic N-linear operators on the Fréchet space $H(\mathbb{C})$ has been investigated in [8, Section 4]. Following the approaches used in [18] and [8], we define the orbits $\mathrm{Orb}(S; (B_i)_{1 \leq i \leq l})$ and $\mathrm{Orb}(S; (M_i)_{1 \leq i \leq l})$ for any non-empty subset S of E^N and any mappings $B_i : (E^N)^{b_i} \to E^N$, $M_i : E^N \to E$ $(1 \leq i \leq l)$; here it is worth noting that in our analysis these mappings need not be (separately) linear or continuous. Having this done, we have an open door (after a necessary patching up with some technicalities concerning the well-posedness of problem [(1)-(2)] below) to introduce the notions of $\mathfrak{D}$-hypercyclicity and $\mathfrak{D}$-topologically mixing property of degenerate abstract multi-term fractional problems (for more details, cf. Definition 2). In Theorem 1 and Theorem 2, we reformulate [18, Theorem 2] for our context, and prove the conjugacy lemma for abstract degenerate multi-term fractional differential equations. The main objective in Theorem 3 is to clarify the kind of Desch-Schappacher-Webb and Banasiak-Moszyński criteria ([15], [5], [14]) for $\mathfrak{D}$-topologically mixing of certain classes of abstract degenerate higher-order differential equations with integer order derivatives (the chaotic properties of abstract degenerate differential equations will not be considered within the framework of this paper; cf. [32] for more details). As explained in Remark 2(iii), Theorem 3 cannot be so easily transmitted to abstract degenerate differential equations with Caputo fractional derivatives. Finally, it should be noticed that we provide a large amount of relevant references on fractional calculus and fractional differential equations, degenerate differential equations, linear dynamics, and a large number of useful comments and remarks enriches our analysis.

Let $n \in \mathbb{N} \setminus \{1\}$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n$, $f \in C([0, \infty) : E)$, and let $A_0, A_1, \cdot \cdot \cdot, A_{n-1}, B$ be closed linear operators on E. The well-posedness of following multi-term fractional differential equation has been analyzed in a series of recent papers (cf. [26, Section 2.10] for an extensive survey of results):

$$\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=0}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) = f(t), \quad t \geq 0; \quad u^{(j)}(0) = u_j, \ j = 0, \cdot \cdot \cdot, \lceil \alpha_n \rceil - 1.$$

Set $m_i := \lceil \alpha_i \rceil$, $i \in \mathbb{N}_n^0$, $A_0 := A$, $A_n := B$, $T_{i,L}u(t) := A_i \mathbf{D}_t^{\alpha_i} u(t)$, if $t \geq 0$, $i \in \mathbb{N}_n^0$ and $\alpha_i > 0$, and $T_{i,R}u(t) := \mathbf{D}_t^{\alpha_i} A_i u(t)$, if $t \geq 0$ and $i \in \mathbb{N}_n^0$, where $\mathbb{N}_n := \{1, \cdot \cdot \cdot, n\}$ and $\mathbb{N}_n^0 := \mathbb{N}_n \cup \{0\}$. Henceforth we shall always assume that, for every $t \geq 0$ and $i \in \mathbb{N}_n^0$, $T_i u(t)$ denotes either $T_{i,L}u(t)$ or $T_{i,R}u(t)$. In this paper, we introduce and further analyze the notions of $\mathcal{D}$-hypercyclicity and $\mathcal{D}$-topologically mixing property of the following homogeneous degenerate abstract multi-term problem:

$$\sum_{i=0}^{n} T_i u(t) = 0, \quad t \geq 0, \tag{1}$$

thus continuing our previous research studies [29]-[32]. Although the introduced notions seem to be new even for non-degenerate abstract differential equations of first order ([40]), we shall focus our attention almost completely on degenerate multi-term problems.

Set

$$P_\lambda := \lambda^{\alpha_n} B + \sum_{i=0}^{n-1} \lambda^{\alpha_i} A_i, \quad \lambda \in \mathbb{C} \setminus \{0\},$$

$\mathcal{I} := \{i \in \mathbb{N}_n^0 : \alpha_i > 0$ and $T_{i,L}u(t)$ appears on the left hand side of (1)$\}$, $Q := \max \mathcal{I}$, if $\mathcal{I} \neq \emptyset$ and $Q := m_Q := 0$, if $\mathcal{I} = \emptyset$. We shall consider the equation (1) equipped with the following initial conditions (cf. [32] for more details):

$$u^{(j)}(0) = u_j, \ 0 \leq j \leq m_Q - 1 \text{ and } \big(A_i u\big)^{(j)}(0) = u_{i,j} \text{ if } m_i - 1 \geq j \geq m_Q. \tag{2}$$

If $T_n u(t) = T_{n,L}u(t)$, then (2) becomes:

$$u^{(j)}(0) = u_j, \ 0 \leq j \leq m_n - 1.$$

If this is not the case, then the choice (2) may be non-optimal and we cannot expect the existence of solutions of problem [(1)-(2)] in general ([32]).

Let α be a fixed positive real number. The most important subcases of problem [(1)-(2)] are the following fractional Sobolev degenerate equations:

$$(\mathrm{DFP})_R : \begin{cases} \mathbf{D}_t^{\alpha} B u(t) = A u(t), \quad t \geq 0, \\ (Bu)^{(j)}(0) = B u_j, \ 0 \leq j \leq m-1, \end{cases}$$

and

$$(\mathrm{DFP})_L : \begin{cases} B \mathbf{D}_t^{\alpha} u(t) = A u(t), \quad t \geq 0, \\ u^{(j)}(0) = u_j, \ 0 \leq j \leq m-1, \end{cases}$$

where $m := \lceil \alpha \rceil$. In [32, Subsection 2.1-Subsection 2.2], we have recently considered the hypercyclicity and topologically mixing property of the equations $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$ with $x_0 = x$ and $x_1 = \cdots = x_{m-1} = 0$, as well as the problem

$$B\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=0}^{n-1} T_i u(t) = 0, \quad t \geq 0; \quad u^{(j)}(0) = u_j, \ j = 0, \cdot \cdot \cdot, m_n - 1, \tag{3}$$

provided that there exists an index $i \in \mathbb{N}^0_{m_n-1}$ such that $u_j = 0$, $j \in \mathbb{N}^0_{m_n-1} \setminus \{i\}$. In this paper, we continue studying hypercyclicity and topologically mixing property of problems (3) and $(\mathrm{DFP})_R$ by assuming that there exist two or more non-zero components of the tuple $(u_0, \cdot \cdot \cdot, u_{m_n-1})$ (i.e., the tuple $(Bu_0, \cdot \cdot \cdot, Bu_{m-1})$ in the case of problem $(\mathrm{DFP})_R$). The analysis of $\mathcal{D}$-hypercyclicity and $\mathcal{D}$-topologically mixing property of problem [(1)-(2)] is very complicated in general case and, with the exception of some minor facts and results concerning the existence and uniqueness of solutions, the most general abstract form of problem [(1)-(2)] will not be further considered here. For more details concerning the wellposedness of Sobolev first order degenerate equations, the reader may consult the monographs by A. Favini, A. Yagi [20], S. G. Krein [34], R. W. Carroll, R. W. Showalter [10], I. V. Melnikova, A. I. Filinkov [36] and G. A. Sviridyuk, V. E. Fedorov [46], as well as the papers [21], [37], [39], [44] and [52]. The well-posedness of various types of degenerate Sobolev equations of second order have been analyzed in [1], [10], [20], [22], [28], [38], [45] and [53]. The corresponding results on degenerate Sobolev equations with integer higher-order derivatives can be found in [2]-[3], [20, Section 5.7], [33], [46]-[50] and [53].

We use the standard terminology throughout the paper. For any $p \in \mathbb{N}$ and $r \in \mathbb{N}_p$, we define $\mathrm{Proj}_{r,p} : E^p \to E$ by $\mathrm{Proj}_{r,p}(x_1, \cdot \cdot \cdot, x_p) := x_r$, $\vec{x} = (x_1, \cdot \cdot \cdot, x_p) \in E^p$. If A is a linear operator acting on E, then the domain and point spectrum of A will be denoted by $D(A)$ and $\sigma_p(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. By I and E^* we denote the identity operator on E and the dual space of E, respectively. Given $s \in \mathbb{R}$ in advance, set $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leq l\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers; the convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)\,ds$. Set $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$, $0^\zeta := 0$ $(\zeta > 0,\ t > 0)$, $g_0(t) :=$ the Dirac δ-distribution, $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re z > 0\}$ and $L(z_0, \epsilon) := \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$ $(z_0 \in \mathbb{C},\ \epsilon > 0)$. For a number $\zeta > 0$ given in advance, the Caputo fractional derivative $\mathbf{D}_t^\zeta u$ ([7], [26]) is defined for those functions $u \in C^{\lceil \zeta \rceil - 1}([0,\infty) : E)$ for which $g_{\lceil \zeta \rceil - \zeta} * (u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1}) \in C^{\lceil \zeta \rceil}([0,\infty) : E)$:

$$\mathbf{D}_t^\zeta u(t) = \frac{d^{\lceil \zeta \rceil}}{dt^{\lceil \zeta \rceil}} \left[g_{\lceil \zeta \rceil - \zeta} * \left(u - \sum_{j=0}^{\lceil \zeta \rceil - 1} u^{(j)}(0) g_{j+1} \right) \right].$$

For any continuous E-valued function $t \mapsto u(t)$, $t \geq 0$, we define $\mathbf{D}_t^0 u(t) := u(t)$. If the Caputo fractional derivative $\mathbf{D}_t^\zeta u(t)$ exists, then for each number $\nu \in (0, \zeta)$ the Caputo fractional derivative $\mathbf{D}_t^{\nu}u(t)$ exists as well, and the following equality holds:

$$\mathbf{D}_t^{\nu}u(t)=\bigl(g_{\zeta-\nu}*\mathbf{D}_t^{\zeta}u(\cdot)\bigr)(t)+\sum_{j=\lceil\nu\rceil}^{\lceil\zeta\rceil-1}u^{(j)}(0)g_{j+1-\nu}(t),\quad t\geq 0. \tag{4}$$

It should be noted here that the term $\mathbf{D}_t^{\nu_1+\nu_2}u(t)$ need not be defined for some functions $t\mapsto u(t)$, $t\geq 0$, for which the term $\mathbf{D}_t^{\nu_1}\mathbf{D}_t^{\nu_2}u(t)$ is defined. Consider, for example, the case $\nu_1=\nu_2=1/2$, $\lambda>0$ and $u(t)=E_{1/2}(\lambda^{1/2}t^{1/2})$, $t\geq 0$ (cf. the next paragraph for the notion of Mittag-Leffler functions). Then [7, (1.25)] implies that $\mathbf{D}_t^{\nu_1}\mathbf{D}_t^{\nu_2}u(t)=\lambda E_{1/2}(\lambda^{1/2}t^{1/2})$, $t\geq 0$. On the other hand, $\mathbf{D}_t^{1}u(t)$ is not defined for $t\geq 0$ because the function $t\mapsto u(t)$, $t\geq 0$ is not continuously differentiable for $t\geq 0$. Even if we accept a slightly weaker definition of Caputo fractional derivatives from [7], when $\mathbf{D}_t^{1}E_{1/2}(\lambda^{1/2}t^{1/2})$ exists and equals to $\sum_{k=1}^{\infty}\frac{\lambda^{k/2}t^{(k/2)-1}}{\Gamma(k/2)}$ for $t>0$, the equality $\mathbf{D}_t^{1/2}\mathbf{D}_t^{1/2}E_{1/2}(\lambda^{1/2}t^{1/2})=\mathbf{D}_t^{1}E_{1/2}(\lambda^{1/2}t^{1/2})$, $t>0$ does not hold for any $\lambda>0$ because $\lambda E_{1/2}(\lambda^{1/2}t^{1/2})\sim\lambda$ as $t\to 0+$ while $\sum_{k=1}^{\infty}\frac{\lambda^{k/2}t^{(k/2)-1}}{\Gamma(k/2)}\sim(\frac{\lambda}{\pi t})^{1/2}$ as $t\to 0+$ (cf. [31], Remark 2(iv) and the equation (7) below).

The Mittag-Leffler function $E_{\beta,\gamma}(z)$ ($\beta>0$, $\gamma\in\mathbb{R}$) is defined by

$$E_{\beta,\gamma}(z):=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+\gamma)},\quad z\in\mathbb{C}.$$

In this place, we assume that $1/\Gamma(\beta k+\gamma)=0$ if $\beta k+\gamma\in-\mathbb{N}_0$. Set, for short, $E_{\beta}(z):=E_{\beta,1}(z)$, $z\in\mathbb{C}$. The asymptotic behaviour of the entire function $E_{\beta,\gamma}(z)$ is given in the following auxiliary lemma (see e.g. [26, Section 1.3]):

Lemma 1. *Let $0<\sigma<\frac{1}{2}\pi$. Then, for every $z\in\mathbb{C}\setminus\{0\}$ and $l\in\mathbb{N}\setminus\{1\}$,*

$$E_{\beta,\gamma}(z)=\frac{1}{\beta}\sum_{s}Z_s^{1-\gamma}e^{Z_s}-\sum_{j=1}^{l-1}\frac{z^{-j}}{\Gamma(\gamma-\beta j)}+O\bigl(|z|^{-l}\bigr),\quad |z|\to\infty,$$

where Z_s is defined by $Z_s:=z^{1/\beta}e^{2\pi is/\beta}$ and the first summation is taken over all those integers s satisfying $|\arg(z)+2\pi s|<\beta(\frac{\pi}{2}+\sigma)$.

The reader may consult [51] and [26] for further information concerning the Laplace transform and analytical properties of functions with values in sequentially complete locally convex spaces (cf. [4] for the Banach space case). By $\mathcal{L}$ and $\mathcal{L}^{-1}$ we denote the Laplace transform and its inverse transform, respectively.

2. $\mathfrak{D}$-Hypercyclic and $\mathfrak{D}$-topologically mixing properties of degenerate Cauchy problems

We start this section by recalling the following definition of a strong solution of problem [(1)-(2)] (cf. [32, Definition 2]).

Definition 1. *A function $u \in C([0,\infty) : E)$ is said to be a strong solution of problem [(1)-(2)] iff the term $T_i u(t)$ is well defined and continuous for any $t \geq 0$, $i \in \mathbb{N}_n^0$, and [(1)-(2)] holds identically on* $[0,\infty)$.

Denote by $\mathfrak{T}$ the exact number of initial values subjected to the equation [(1)-(2)]; in other words, $\mathfrak{T}$ is the sum of number m_Q and the cardinality of set consisting of those pairs $(i,j) \in \mathbb{N}_n \times \mathbb{N}^0_{m_n-1}$ for which $m_i - 1 \geq j \geq m_Q$. More precisely, suppose that $\{i_1, \cdot\cdot\cdot, i_s\} = \{i \in \mathbb{N}_n : m_i - 1 \geq m_Q\}$ and $i_1 < \cdot\cdot\cdot < i_s$. Then the set of all initial values appearing in (2) is given by $\{u_0, \cdot\cdot\cdot, u_{m_Q-1}; u_{i_1,m_Q}, \cdot\cdot\cdot, u_{i_1,m_{i_1}-1}; \cdot\cdot\cdot; u_{i_s,m_Q}, \cdot\cdot\cdot, u_{i_s,m_{i_s}-1}\} = \{(u_j)_{0\leq j\leq m_Q-1}; (u_{i_{s'},j})_{1\leq s'\leq s, m_Q\leq j\leq m_{i_{s'}}-1}\}$ so that $\mathfrak{T} = m_{i_1} + \cdot\cdot\cdot + m_{i_s} + (1-s)m_Q$. Denote by $\mathfrak{Z}$ ($\mathfrak{Z}_{uniq}$) the set of all tuples of initial values $\vec{x} = ((u_j)_{0\leq j\leq m_Q-1}; (u_{i_{s'},j})_{1\leq s'\leq s, m_Q\leq j\leq m_{i_{s'}}-1}) \in E^{\mathfrak{T}}$ for which there exists a (unique) strong solution of problem [(1)-(2)]. Then $\mathfrak{Z}$ is a linear subspace of $E^{\mathfrak{T}}$ and $\mathfrak{Z}_{uniq} \subseteq \mathfrak{Z}$. The equality $\mathfrak{Z} = \mathfrak{Z}_{uniq}$ holds iff the zero function is a unique strong solution of the problem [(1)-(2)] with the initial value $\vec{x} = \vec{0}$. For any $\vec{x} \in \mathfrak{Z}$, we denote by $\mathfrak{S}(\vec{x})$ the set consisting of all strong solutions of problem [(1)-(2)] with the initial value $\vec{x}$.

In the remaining part of this paper, we shall only consider the problems (3) and $(\mathrm{DFP})_R$. Note that the problem $(\mathrm{DFP})_L$ is a very special case of problem (3) and that $\mathfrak{T} = m_n$ for problem (3), and $\mathfrak{T} = m$ for problem $(\mathrm{DFP})_R$. By (P) we denote either (3) or $(\mathrm{DFP})_R$. We shall always assume henceforth that $\emptyset \neq W \subseteq \mathbb{N}_{\mathfrak{T}}$, $\hat{E}_i$ is a linear subspace of E $(i \in W)$, $\tilde{E}$, $\check{E}$ are linear subspaces of $E^{\mathfrak{T}}$, as well as that $\vec{\beta} := (\beta_0, \beta_1, \cdot\cdot\cdot, \beta_{\mathfrak{T}-1}) \in [0,\alpha_n]^{\mathfrak{T}}$, $l \in \mathbb{N}$, $\emptyset \neq S \subseteq E^{\mathfrak{T}}$, $B_i : (E^{\mathfrak{T}})^{b_i} \to E^{\mathfrak{T}}$ and $M_i : E^{\mathfrak{T}} \to E$ are given mappings ($b_i \in \mathbb{N}$ for $1 \leq i \leq l$). Set $\mathfrak{B} := (\tilde{E}, \check{E}, S, (B_i)_{1\leq i\leq l}, \{\hat{E}_i : i \in W\}, \vec{\beta})$ and $\mathfrak{M} := (\tilde{E}, \check{E}, S, (M_i)_{1\leq i\leq l}, \{\hat{E}_i : i \in W\}, \vec{\beta})$. Let $\mathfrak{P} : \mathfrak{Z} \to P(\cup_{\vec{x}\in\mathfrak{Z}} \mathfrak{S}(\vec{x}))$ be a fixed mapping satisfying $\emptyset \neq \mathfrak{P}(\vec{x}) \subseteq \mathfrak{S}(\vec{x})$, $\vec{x} \in \mathfrak{Z}$.

Following K.-G. Grosse-Erdmann-S. G. Kim [18, pp. 701-702], we introduce the set $\mathbb{U}_p(S)$ $(p \in \mathbb{N}_0)$ recursively by $\mathbb{U}_0(S) := S$, $\mathbb{U}_{p+1}(S) := \mathbb{U}_p(S) \cup \{B_i(\vec{x_1}, \cdot\cdot\cdot, \vec{x_{b_i}}) : 1 \leq i \leq l,\ \vec{x_1}, \cdot\cdot\cdot, \vec{x_{b_i}} \in \mathbb{U}_p(S)\}$. If $\mathfrak{T} \geq 2$, then we introduce the set $\mathbf{U}_p(S)$ $(p \in \mathbb{N}_0)$ following the approach of J. Bès-J. A. Conejero [8, pp. 2-3]: $\mathbf{U}_0(S) =: S$, $\mathbf{U}_{p+1}(S) := \mathbf{U}_p(S) \cup \{(x_2, x_3, \cdot\cdot\cdot, x_{\mathfrak{T}}, M_i(x_1, x_2, \cdot\cdot\cdot, x_{\mathfrak{T}})) : 1 \leq i \leq l,\ (x_1, x_2, \cdot\cdot\cdot, x_{\mathfrak{T}}) \in \mathbf{U}_p(S)\}$. If $\mathfrak{T} = 1$, then $\mathbf{U}_p(S) := S$, $p \in \mathbb{N}_0$. Define

$$\mathrm{Orb}\big(S; (B_i)_{1\leq i\leq l}\big) := \bigcup_{p\in\mathbb{N}_0} \mathbb{U}_p(S), \quad \mathrm{Orb}\big(S; (M_i)_{1\leq i\leq l}\big) := \bigcup_{p\in\mathbb{N}_0} \mathbf{U}_p(S),$$

and denote by $\mathcal{M}_{\mathfrak{B}}$ ($\mathcal{M}_{\mathfrak{M}}$) the set consisting of those tuples $\vec{x} \in \mathrm{Orb}(S; (B_i)_{1\leq i\leq l}) \cap \mathfrak{Z}$ ($\vec{x} \in \mathrm{Orb}(S; (M_i)_{1\leq i\leq l}) \cap \mathfrak{Z}$) for which $\mathrm{Proj}_{i,\mathfrak{T}}(\vec{x}) \in \hat{E}_i$, $i \in W$. In the sequel, we shall denote by D_i ($\mathfrak{D}$) either B_i or M_i ($\mathfrak{B}$ or $\mathfrak{M}$) and, in the case that $l = 1$, we shall also write $\mathrm{Orb}(S; B_1)$, $\mathrm{Orb}(S; M_1)$ and $\mathrm{Orb}(S; D_1)$ in place of $\mathrm{Orb}(S; (B_i)_{1\leq i\leq l})$, $\mathrm{Orb}(S; (M_i)_{1\leq i\leq l})$ and $\mathrm{Orb}(S; (D_i)_{1\leq i\leq l})$, respectively. A similar terminological agreement will be used in the case where the set W is a singleton.

Motivated by some results from the theory of abstract higher-order differential equations with integer order derivatives, obtained in the usual way, i.e. converting higher-order equations into first order matrix differential equations by introducing the first derivative, the second derivative, ... , the $(n-1)$th derivative of the unknown E-valued function as a part of a new enlarged unknown E^n-valued function (cf. [51, pp. 79-83], [20, Section 5.7], [46, Theorem 5.6.3] and Theorem 3 below for further information), we would like to propose the following definition (concerning the abstract multi-term differential equations with Caputo fractional derivatives, we do not yet know what the ideal option for work is).

Definition 2. *The abstract Cauchy problem (3) is said to be:*

(i) $(\mathfrak{D},\mathfrak{P})$*-hypercyclic iff there exist a tuple* $\vec{x}\in\mathcal{M}_{\mathfrak{D}}\cap\tilde{E}$ *and a function* $u(\cdot;\vec{x})\in\mathfrak{P}(\vec{x})$ *such that* $\{((\mathbf{D}_s^{\beta_0}u(s;\vec{x}))_{s=t},(\mathbf{D}_s^{\beta_1}u(s;\vec{x}))_{s=t},\cdots,(\mathbf{D}_s^{\beta_{\mathfrak{T}-1}}u(s;\vec{x}))_{s=t}):t\geq 0\}$ *is a dense subset of* $\check{E}$*; such a vector is called a* $(\mathfrak{D},\mathfrak{P})$*-hypercyclic vector of problem (3).*

(ii) $\mathfrak{D}$*-hypercyclic iff it is* $(\mathfrak{D},\mathfrak{S})$*-hypercyclic; any* $(\mathfrak{D},\mathfrak{S})$*-hypercyclic vector of problem (3) will be also called a* $\mathfrak{D}$*-hypercyclic vector of problem (3).*

(iii) $\mathfrak{D}_{\mathfrak{P}}$*-topologically transitive iff for every pair of open non-empty subsets* U *and* V *of* $E^{\mathfrak{T}}$ *satisfying* $U\cap\tilde{E}\neq\emptyset$ *and* $V\cap\check{E}\neq\emptyset$*, there exist a tuple* $\vec{x}\in\mathcal{M}_{\mathfrak{D}}$*, a function* $u(\cdot;\vec{x})\in\mathfrak{P}(\vec{x})$ *and a number* $t\geq 0$ *such that* $\vec{x}\in U\cap\tilde{E}$ *and* $((\mathbf{D}_s^{\beta_0}u(s;\vec{x}))_{s=t},(\mathbf{D}_s^{\beta_1}u(s;\vec{x}))_{s=t},\cdots,(\mathbf{D}_s^{\beta_{\mathfrak{T}-1}}u(s;\vec{x}))_{s=t})\in V\cap\check{E}$.

(iv) $\mathfrak{D}$*-topologically transitive iff it is* $\mathfrak{D}_{\mathfrak{S}}$*-topologically transitive.*

(v) $\mathfrak{D}_{\mathfrak{P}}$*-topologically mixing iff for every pair of open non-empty subsets* U *and* V *of* $E^{\mathfrak{T}}$ *satisfying* $U\cap\tilde{E}\neq\emptyset$ *and* $V\cap\check{E}\neq\emptyset$*, there exists a number* $t_0\geq 0$ *such that, for every number* $t\geq t_0$*, there exist a tuple* $\vec{x_t}\in\mathcal{M}_{\mathfrak{D}}$ *and a function* $u(\cdot;\vec{x_t})\in\mathfrak{P}(\vec{x_t})$ *such that* $\vec{x_t}\in U\cap\tilde{E}$ *and* $((\mathbf{D}_s^{\beta_0}u(s;\vec{x_t}))_{s=t},(\mathbf{D}_s^{\beta_1}u(s;\vec{x_t}))_{s=t},\cdots,(\mathbf{D}_s^{\beta_{\mathfrak{T}-1}}u(s;\vec{x_t}))_{s=t})\in V\cap\check{E}$.

(vi) $\mathfrak{D}$*-topologically mixing iff it is* $\mathfrak{D}_{\mathfrak{S}}$*-topologically mixing.*

In our previous study [32], we have seen that there is no substantial difference in the analysis of hypercyclic and topologically mixing properties of problems $(\mathrm{DFP})_R$ and $(\mathrm{DFP})_L$; the analysis carried out in this paper basically shows the same thing. If $\mathfrak{Q}(\vec{x})$ is any non-empty subset consisting of solutions of problem $(\mathrm{DFP})_R$ with the initial value $\vec{x}$, then we denote by $\mathfrak{Q}_s(\vec{x})$ the set $\mathfrak{Q}(\vec{x})\cap C^{m-1}([0,\infty):E)$ $(\vec{x}\in\mathfrak{Z})$. We introduce the notions of $(\mathfrak{D},\mathfrak{P}_s)$-hypercyclicity, $\mathfrak{D}_{\mathfrak{P}_s}$-topological transitivity and $\mathfrak{D}_{\mathfrak{P}_s}$-topologically mixing property of problem $(\mathrm{DFP})_R$ in the same way as in Definition 2, with the sets $\mathfrak{P}(\vec{x})$ and $\mathfrak{P}(\vec{x_t})$ replaced respectively by $\mathfrak{P}_s(\vec{x})$ and $\mathfrak{P}_s(\vec{x_t})$. Finally, we say that the problem $(\mathrm{DFP})_R$ is $\mathfrak{D}$-hypercyclic ($\mathfrak{D}$-topologically transitive, $\mathfrak{D}$-topologically mixing) iff it is $\mathfrak{D}_{\mathfrak{S}_s}$-hypercyclic ($\mathfrak{D}_{\mathfrak{S}_s}$-topologically transitive, $\mathfrak{D}_{\mathfrak{S}_s}$-topologically mixing).

Remark 1.

(i) We have presented only one way for the computing the orbit $Orb(S;(D_i)_{1\leq i\leq l})$*, some other possibilities will take more space than we actually have here. In the case where* $\mathfrak{T}\geq 2$*, the orbit* $Orb(S;(D_i)_{1\leq i\leq l})$ *can have a very unpleasant form and it is very difficult to say, in general, whether there exists an element of* $Orb(S;(D_i)_{1\leq i\leq l})$ *that is a* $(\mathfrak{D},\mathfrak{P})$*-hypercyclic*

vector of problem (3). On the other hand, in the definition of $\mathcal{M}_{\mathfrak{D}}$ we can take any non-empty subset S' of $E^{\mathfrak{T}}$ instead of $Orb(S;(D_i)_{1\leq i\leq l})$. But this is a very special case of our definition with $\mathfrak{D}=\mathfrak{B}$, $l=1$, $b_1=1$ and $B_1: E^{\mathfrak{T}}\rightarrow E^{\mathfrak{T}}$ being the identity mapping. It is also worth noting that the continuous version of Herrero-Bourdon theorem [19, Theorem 7.17, pp. 190-191] suggests us to define the set $\mathcal{M}_{\mathfrak{D}}$ as the union of those vectors $\vec{x}\in span\{Orb(S;(D_i)_{1\leq i\leq l})\}\cap\mathfrak{Z}$ for which $Proj_{i,\mathfrak{T}}(\vec{x})\in\hat{E}_i$, $i\in W$. If we define $\mathcal{M}_{\mathfrak{D}}$ in such a way, then the assertion of Theorem 2 below continues to hold, the assertion of Theorem 1 continues to hold with the mapping $\mathfrak{P}'(\cdot)=c\mathfrak{P}(\cdot/c)$ replaced by $\mathfrak{P}(\cdot)$, while the assertion of Theorem 3 continues to hold if we assume that $\{\vec{x_\lambda}:\lambda\in\Omega\}\subseteq Orb(S;(D_i)_{1\leq i\leq l})$. Also note that the notion introduced in [32, Definition 3, Definition 10] is a special case of the notion introduced in Definition 2 and that of [30, Theorem 2.4] can be formulated in our context.

(ii) Let $0\leq\beta\leq\alpha<2$, and let the requirements of [32, Theorem 5] hold (in (iii) and in the sequel of (ii) of this remark, we will use almost same terminology as in [32]; the only exception will be the notation used to denote the space $\tilde{E}$). Applying Lemma 1 (cf. also the asymptotic expansion formulae [7, (1.26)-(1.28)]), we get $\lim_{t\rightarrow+\infty}\frac{t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(\lambda^\alpha t^\alpha)}{E_\alpha(\lambda^\alpha t^\alpha)}=\lambda^{\beta-\alpha}$, $\lambda\in\mathbb{C}_+$ and $\lim_{t\rightarrow+\infty}t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(\lambda^\alpha t^\alpha)=0$, $\lambda\in\Omega_{0,-}$. Using the identity $\mathbf{D}_t^\beta E_\alpha(\lambda^\alpha t^\alpha)=\lambda^\alpha t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(\lambda^\alpha t^\alpha)$, $t\geq 0$, $\lambda\in\mathbb{C}\setminus(-\infty,0]$ (which can be proved directly, or by applying (4) and [7, (1.25)]) and [32, Lemma 4], we may conclude by a careful inspection of the proof of [32, Theorem 5] that the problem (DFP)$_L$ is $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing, provided that $\vec{\beta}=(\beta,\beta)$, $W=\{1\}$, $\hat{E}_1=\overline{span\{f(\lambda^\alpha):\lambda\in\Omega\}}$, $\tilde{E}=\hat{E}_1\times\{0\}$, $\check{E}=\{(z,z):z\in\hat{E}_1\}$, $\hat{E}_1\times\{0\}\subseteq Orb(S;(D_i)_{1\leq i\leq l})$ and $\mathfrak{P}((\sum_{i=1}^m\alpha_i f(\lambda_i^\alpha),0))=\{\sum_{i=1}^m\alpha_i E_\alpha(\cdot^\alpha\lambda_i^\alpha)f(\lambda_i^\alpha)\}$ ($m\in\mathbb{N}$, $\alpha_i\in\mathbb{C}$, $\lambda_i\in\Omega$ for $1\leq i\leq m$). This, in turn, implies that the problem (DFP)$_R$ is $\mathfrak{D}_{\mathfrak{P}_s}$-topologicially mixing (the only thing worth noticing here is that, given y and z as in the proof of [32, Theorem 5], the vector $\vec{x_t}$ can be chosen in such a way that $\vec{x_t}=(y+\sum_{j=1}^m\frac{\gamma_j}{\tilde{\lambda_j}^\beta E_\alpha(\tilde{\lambda_j}^\alpha t^\alpha)}f(\tilde{\lambda_j}^\alpha),0)$, $t>0$ is sufficiently large). This is a slight improvement of above-mentioned result, which can be applied in the analysis of fractional analogons of the linearized Boussinesq equation $(\sigma^2\Delta-1)u_{tt}+\gamma^2\Delta u=0$ on symmetric spaces of non-compact type (cf. [35] for the notion, as well as [29, Example 2.5(i)-(ii)] and [32, Example 7-Example 8] for some other applications). Finally, note that Definition 2 and [32, Definition 3, Definition 10] have some advantages over [29, Definition 2.2] and [31, Definition 2.2]. For example, an application of [32, Theorem 5] shows that the abstract Cauchy problems (DFP)$_R$ and (DFP)$_L$, with $1<\alpha<2$, $E=L^2(\mathbb{R})$, $B=I$ and $A=\mathcal{A}_c$ being the bounded perturbation of the one-dimensional Ornstein-Uhlenbeck operator from [29, Example 2.5(iii)], are both topologically mixing in the sense of [32, Definition 3]. The topologically mixing property of corresponding problems in the sense of [29, Definition 2.2] can be proved only in the case where $0<\alpha\leq 1$, cf. [13] and [29].

(iii) Consider the situation of [32, Theorem 11] with the second equality in [32, (10)] replaced by

$$\lim_{t\rightarrow+\infty}\mathbf{D}_t^{\beta_j}H_i(\lambda,t)=0,\ \lambda\in\Omega_-,\ 0\leq j\leq\mathfrak{T}-1,$$

and with the first equality in [32, (10)] replaced by

$$\lim_{t\to+\infty}\bigl|F(\lambda,t)\bigr| = +\infty,\ \lambda\in\Omega_+,\ 0\le j\le \mathfrak{T}-1,$$

where $a>0$ *and* $F:\Omega_+\times(a,+\infty)\to\mathbb{C}$ *is some function. Set* $E_0 := \overline{span\{f(\lambda):\lambda\in\Omega\}}$. *If we suppose additionally that there exist complex numbers* $G_{\beta_0},\cdot\cdot\cdot,G_{\beta_{\mathfrak{T}-1}}$ *such that*

$$\lim_{t\to+\infty}\frac{\mathbf{D}_t^{\beta_j}H_i(\lambda,t)}{F(\lambda,t)} = G_{\beta_j},\ \lambda\in\Omega_+,\ 0\le j\le\mathfrak{T}-1,$$

then the proof of [32, Theorem 11] shows that the abstract Cauchy problem (3) is $\mathfrak{D}_{\mathfrak{P}}$*-topologically mixing, provided that* $W=\{i\}$, $\hat{E}_i=E_0$, $\tilde{E}=\{\vec{x}\in E^{\mathfrak{T}}:Proj_{i,\mathfrak{T}}(\vec{x})\in E_0$ *and* $Proj_{j,\mathfrak{T}}(\vec{x})=0$ *for* $j\in\mathbb{N}_{\mathfrak{T}}\setminus\{i\}\}$, $\check{E}=\{(G_{\beta_0}z,\cdot\cdot\cdot,G_{\beta_{\mathfrak{T}-1}}z):z\in\hat{E}_1\}$, $\tilde{E}\subseteq Orb(S;(D_j)_{1\le j\le l})$ *and* $\mathfrak{P}((\sum_{j=1}^{m}\alpha_j f(0,\cdot\cdot\cdot,\lambda_j,\cdot\cdot\cdot,0))=\{\sum_{j=1}^{m}\alpha_j H_i(\lambda_j,\cdot)f(\lambda_j)\}$ *(*$m\in\mathbb{N}$, $\alpha_j\in\mathbb{C}$, $\lambda_j\in\Omega$ *for* $1\le j\le m$*), where* λ_j *appears on i-th place starting from zero (there exists a great number of concrete examples in which the above conditions hold with* $\vec{\beta}$ *being the constant multiple of* $(1,1,\cdot\cdot\cdot,1)$*, see e.g. our analysis of topologically mixing properties of strongly damped Klein-Gordon equation [32, Example 13]; we refer the reader to Theorem 3 and Example 1 for the case in which* $\vec{\beta}$ *is not of the form described above). Also, it should be noted that the comments from (ii) and (iii) can be formulated in the light of [29, Remark 1(iii)] and [31, Remark 1;3.], and the proof of [32, Theorem 11] implies that*

$$\mathbf{D}_t^{\beta_l}H_i(\lambda,t)=\mathcal{L}^{-1}\left(\frac{z^{\alpha_n+\beta_l-i-1}+\sum_{j\in D_i}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j+\beta_l-i-1}-\chi_{\mathcal{D}_i}(0)f_n(\lambda)z^{\alpha+\beta_l-i-1}}{z^{\alpha_n}+\sum_{j=1}^{n-1}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j}-f_n(\lambda)z^{\alpha}}\right)(t),$$

for $t\ge0$, $\lambda\in\Omega$, $l\in\mathbb{N}^0_{\mathfrak{T}-1}$, $\lceil\beta_l\rceil<i-1$, *and*

$$\mathbf{D}_t^{\beta_l}H_i(\lambda,t)=\mathcal{L}^{-1}\left(\frac{-\sum_{j\in\mathbb{N}_{n-1}\setminus D_i}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j+\beta_l-i-1}-f_n(\lambda)z^{\alpha+\beta_l-i-1}\bigl(\chi_{\mathcal{D}_i}(0)-1\bigr)}{z^{\alpha_n}+\sum_{j=1}^{n-1}\frac{f_n(\lambda)}{f_j(\lambda)}z^{\alpha_j}-f_n(\lambda)z^{\alpha}}\right),$$

for $t\ge0$, $\lambda\in\Omega$, $l\in\mathbb{N}^0_{\mathfrak{T}-1}$, $\lceil\beta_l\rceil\ge i-1$.

(iv) As indicated in [32], it is much better to introduce the notions of $\mathfrak{D}$*-hypercyclicity,* $\mathfrak{D}$*-topological transitivity and* $\mathfrak{D}$*-topologically mixing property of problem (P) with the set* $\mathfrak{Z}$ *than with* $\mathfrak{Z}_{uniq}$ *(the choice of strong solutions in Definition 2 is almost inevitable; cf. [32]). Consider now, for the sake of brevity, the abstract Cauchy problem (3). If* $\mathfrak{Z}=\mathfrak{Z}_{uniq}$*, then we define the operator* $T(t):\mathcal{M}_{\mathfrak{D}}\to E^{\mathfrak{T}}$ *by* $T(t)\vec{x}:=((\mathbf{D}_s^{\beta_0}u(s;\vec{x}))_{s=t},(\mathbf{D}_s^{\beta_1}u(s;\vec{x}))_{s=t},\cdot\cdot\cdot,(\mathbf{D}_s^{\beta_{\mathfrak{T}-1}}u(s;\vec{x}))_{s=t})$ $(t\ge0)$*, where* $u(\cdot;\vec{x})$ *denotes the unique strong solution of problem (3) with the initial value* $\vec{x}$*. Let* $(O_n)_{n\in\mathbb{N}}$ *be an open base of the topology of* $E^{\mathfrak{T}}$ *(*$O_n\neq\emptyset$, $n\in\mathbb{N}$*). If* $\tilde{E}=\check{E}=E^{\mathfrak{T}}$ *and if we denote by* $HC_{\mathfrak{D}}$ *the set which consists of all* $\mathfrak{D}$*-hypercyclic vectors of problem (3), then we have the obvious equality* $HC_{\mathfrak{D}}=\bigcap_{n\in\mathbb{N}}\bigcup_{t\ge0}T(t)^{-1}(O_n)$*; cf. also [18, Theorem 1], [8, Proposition 4] and [32]. Even in the framework of Fréchet*

spaces, we cannot conclude from the above that $\mathfrak{D}$-topological transitivity of problem (3) implies its $\mathfrak{D}$-hypercyclicity (in this place, it is worth noting that there exists a continuous linear operator on the space $\varphi = \bigoplus_{n\in\mathbb{N}}\mathbb{K}$ that is topologically transitive but not hypercyclic [9, Theorem 2.2], so that the connections between $\mathfrak{D}$-hypercyclicity and $\mathfrak{D}$-topological transitivity seem to be more complicated in non-metrizable locally convex spaces).

It is worth noting that the assertion of [18, Theorem 2] admits an adequate reformulation in our context. Before we state the corresponding theorem, it would be very helpful to introduce the sets $[(B_i)_{1\leq i\leq l}]_p(S)$ and $[(M_i)_{1\leq i\leq l}]_p(S)$ ($p \in \mathbb{N}_0$) recursively by $[(B_i)_{1\leq i\leq l}]_0(S) := [(M_i)_{1\leq i\leq l}]_0(S) := S$, $[(B_i)_{1\leq i\leq l}]_{p+1}(S) := \{B_i(\vec{x_1},\cdots,\vec{x_{b_i}}) : 1 \leq i \leq l, \vec{x_1} \in [(B_i)_{1\leq i\leq l}]_{j_1}(S),\cdots,\vec{x_{b_i}} \in [(B_i)_{1\leq i\leq l}]_{j_{b_i}}(S)$ for some numbers $j_1,\cdots,j_{b_i} \in \mathbb{N}_p$ with $j_1+\cdots+j_{b_i} = p\}$, $[(M_i)_{1\leq i\leq l}]_{p+1}(S) := \mathbf{U}_{p+1}(S) \setminus \mathbf{U}_p(S)$, $p \in \mathbb{N}_0$. Then the set $[(B_i)_{1\leq i\leq l}]_p(S)$ ($[(M_i)_{1\leq i\leq l}]_p(S)$) contains all the elements from $\mathrm{Orb}(S;(B_i)_{1\leq i\leq l})$ ($\mathrm{Orb}(S;(M_i)_{1\leq i\leq l})$) obtained by n applications of operators $B_1,\cdots,B_l$ ($M_1,\cdots,M_l$), totally counted, and the following holds:

$$\mathrm{Orb}\big(S;(B_i)_{1\leq i\leq l}\big) = \bigcup_{p\in\mathbb{N}_0}\big[(B_i)_{1\leq i\leq l}\big]_p(S), \quad \mathrm{Orb}\big(S;(M_i)_{1\leq i\leq l}\big) = \bigcup_{p\in\mathbb{N}_0}\big[(M_i)_{1\leq i\leq l}\big]_p(S). \tag{5}$$

Suppose now that $c \in \mathbb{K}\setminus\{0\}$, and $B_i' : (E^{\mathfrak{T}})^{b_i} \to E^{\mathfrak{T}}$ and $M_i' : E^{\mathfrak{T}} \to E$ satisfy

$$B_i'\big(c\vec{x_1},\cdots,c\vec{x_{b_i}}\big) = cB_i\big(\vec{x_1},\cdots,\vec{x_{b_i}}\big),$$

provided that $\vec{x_1},\cdots,\vec{x_{b_i}} \in E^{\mathfrak{T}}$, $1 \leq i \leq l$, and

$$\big(cx_2,\cdots,cx_{\mathfrak{T}},M_i'(cx_1,cx_2,\cdots,cx_{\mathfrak{T}})\big) = c\big(x_2,\cdots,x_{\mathfrak{T}},M_i(x_1,x_2,\cdots,x_{\mathfrak{T}})\big),$$

provided that $(x_1,x_2,\cdots,x_{\mathfrak{T}}) \in E^{\mathfrak{T}}$, $1 \leq i \leq l$. Define $S_c := \{c\vec{x} : \vec{x} \in S\}$. Then we can inductively prove that $[(B_i')_{1\leq i\leq l}]_p(S_c) = \{c\vec{x} : \vec{x} \in [(B_i)_{1\leq i\leq l}]_p(S)\}$ and $[(M_i')_{1\leq i\leq l}]_p(S_c) = \{c\vec{x} : \vec{x} \in [(M_i)_{1\leq i\leq l}]_p(S)\}$ for all $p \in \mathbb{N}_0$, so that (5) implies

$$\Big\{c\vec{x} : \vec{x} \in \mathrm{Orb}\big(S;(B_i)_{1\leq i\leq l}\big)\Big\} = \mathrm{Orb}\big(S_c;(B_i')_{1\leq i\leq l}\big)$$

and

$$\Big\{c\vec{x} : \vec{x} \in \mathrm{Orb}\big(S;(M_i)_{1\leq i\leq l}\big)\Big\} = \mathrm{Orb}\big(S_c;(M_i')_{1\leq i\leq l}\big).$$

Now it is very simple to prove the following

Theorem 1. *Set $\mathfrak{D}' := (\tilde{E},\check{E},S_c,(D_i')_{1\leq i\leq l},\{\hat{E}_i : i \in W\},\vec{\beta})$ and $\mathfrak{P}' : \mathfrak{Z} \to P(\cup_{\vec{x}\in\mathfrak{Z}}\mathfrak{S}(\vec{x}))$, by $\mathfrak{P}'(\vec{x}) := c\mathfrak{P}(\vec{x}/c)$, $\vec{x} \in \mathfrak{Z}$. Then the abstract Cauchy problem (3), resp. (DFP)$_R$, is $\mathfrak{D}$-hypercyclic ($(\mathfrak{D},\mathfrak{P})$-hypercyclic, $\mathfrak{D}_{\mathfrak{P}}$-topologically transitive, resp. $\mathfrak{D}_{\mathfrak{P}_s}$-topologically transitive, $\mathfrak{D}$-topologically transitive, $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing, resp. $\mathfrak{D}_{\mathfrak{P}_s}$-topologically mixing, $\mathfrak{D}$-topologically mixing) iff the abstract Cauchy problem (3), resp. (DFP)$_R$, is $\mathfrak{D}'$-hypercyclic ($(\mathfrak{D}',\mathfrak{P}')$-hypercyclic, $\mathfrak{D}'_{\mathfrak{P}'}$-topologically transitive, resp. $\mathfrak{D}_{\mathfrak{P}'_s}$-topologically transitive, $\mathfrak{D}'$-topologically transitive, $\mathfrak{D}'_{\mathfrak{P}'}$-topologically mixing, resp. $\mathfrak{D}'_{\mathfrak{P}'_s}$-topologically mixing, $\mathfrak{D}'$-topologically mixing).*

Suppose now that X is another locally convex space over the field of $\mathbb{K}$ and $\phi : X \to E$ is a linear topological homeomorphism. Then the mapping $\phi^{\mathfrak{T}} : X^{\mathfrak{T}} \to E^{\mathfrak{T}}$, defined in the very obvious way, is a linear topological homeomorphism between the spaces $X^{\mathfrak{T}}$ and $E^{\mathfrak{T}}$. Define $S_\phi := (\phi^{\mathfrak{T}})^{-1}(S)$ and the closed linear operators A_i^X on X by $D(A_i^X) := \phi^{-1}(D(A_i))$ and $A_i^X x = y$ iff $A_i(\phi x) = \phi y$ $(0 \leq i \leq n)$. For any E-valued function $t \mapsto u(t)$, $t \geq 0$ we define the X-valued function $t \mapsto u_\phi(t)$, $t \geq 0$ by $u_\phi(t) := \phi^{-1}(u(t))$, $t \geq 0$. Then it is easily verified that the Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$ is defined for $t \geq 0$ iff the Caputo fractional derivative $\mathbf{D}_t^\alpha u_\phi(t)$ is defined for $t \geq 0$. If this is the case, then we have $\mathbf{D}_t^\alpha u_\phi(t) = \phi^{-1}(\mathbf{D}_t^\alpha u(t))$, $t \geq 0$. Using this fact, we can easily prove that the function $t \mapsto u(t)$, $t \geq 0$ is a strong solution of problem (P) with the initial value $\vec{x} = (x_1, \cdots, x_{\mathfrak{T}}) \in E^{\mathfrak{T}}$ iff the function $t \mapsto u_\phi(t)$, $t \geq 0$ is a strong solution of problem $(\mathrm{P})_\phi$ with the initial value $\vec{x}^\phi := (\phi^{-1}(x_1), \cdots, \phi^{-1}(x_{\mathfrak{T}})) \in X^{\mathfrak{T}}$, where the abstract Cauchy problem $(\mathrm{P})_\phi$ is defined by replacing all the operators A_i in the problem (P) with the operators A_i^X $(0 \leq i \leq n)$. If we denote by $\mathfrak{Z}^\phi$ $(\mathfrak{Z}^\phi_{uniq})$ the set consisting of those tuples $\vec{x}^\phi \in X^{\mathfrak{T}}$ for which there exists a (unique) strong solution of the problem $(\mathrm{P})_\phi$, then the above implies $\mathfrak{Z}^\phi = (\phi^{\mathfrak{T}})^{-1}\mathfrak{Z}$ $(\mathfrak{Z}^\phi_{uniq} = (\phi^{\mathfrak{T}})^{-1}\mathfrak{Z}_{uniq})$.

Define the mappings $B_{i,\phi} : (X^{\mathfrak{T}})^{b_i} \to X^{\mathfrak{T}}$ and $M_{i,\phi} : X^{\mathfrak{T}} \to X$ by

$$B_{i,\phi}\big(\vec{x_1}, \cdots, \vec{x_{b_i}}\big) := \big(\phi^{\mathfrak{T}}\big)^{-1}\Big(B_i\big(\phi^{\mathfrak{T}}\vec{x_1}, \cdots, \phi^{\mathfrak{T}}\vec{x_{b_i}}\big)\Big) \text{ and } M_{i,\phi}(\vec{x}) := \phi^{-1}\big(M_i(\phi^{\mathfrak{T}}\vec{x})\big),$$

for any $\vec{x_1}, \cdots, \vec{x_{b_i}}, \vec{x} \in X^{\mathfrak{T}}$, $1 \leq i \leq l$, as well as the mappings $\mathfrak{P}_\phi : \mathfrak{Z}^\phi \to P(\{\mathfrak{S}(\vec{x}^\phi) : \vec{x}^\phi \in \mathfrak{Z}^\phi\})$ and $(\mathfrak{P}_\phi)_s : \mathfrak{Z}^\phi \to P(\{\mathfrak{S}(\vec{x}^\phi) : \vec{x}^\phi \in \mathfrak{Z}^\phi\})$ by $\mathfrak{P}_\phi((\phi^{\mathfrak{T}})^{-1}\vec{x}) := \{u_\phi(\cdot) : u(\cdot) \in \mathfrak{P}(\vec{x})\}$ and $(\mathfrak{P}_\phi)_s((\phi^{\mathfrak{T}})^{-1}\vec{x}) := \{u_\phi(\cdot) : u(\cdot) \in \mathfrak{P}_s(\vec{x})\}$ $(\vec{x} \in \mathfrak{Z})$, respectively. Set

$$\mathfrak{D}_\phi := \Big((\phi^{\mathfrak{T}})^{-1}(\tilde{E}), (\phi^{\mathfrak{T}})^{-1}(\check{E}), S_\phi, (D_{i,\phi})_{1\leq i\leq l}, \{\phi^{-1}(\hat{E}_i) : i \in W\}, \vec{\beta}\Big).$$

Making use of the argumentation similar to that used in the proof of [18, Theorem 3], we can show that

$$\phi^{\mathfrak{T}}\big(\mathrm{Orb}\big(S_\phi; (D_{i,\phi})_{1\leq i\leq l}\big)\big) = \mathrm{Orb}\big(S; (D_i)_{1\leq i\leq l}\big).$$

Now it is quite simple to prove the following conjugacy lemma for abstract degenerate multi-term fractional differential equations (cf. [24, Lemma 1.4] for a pioneering result in this direction):

Theorem 2. *The abstract Cauchy problem (3), resp. (DFP)$_R$, is $\mathfrak{D}$-hypercyclic ($(\mathfrak{D}, \mathfrak{P})$-hypercyclic, $\mathfrak{D}_{\mathfrak{P}}$-topologically transitive, resp. $\mathfrak{D}_{\mathfrak{P}_s}$-topologically transitive, $\mathfrak{D}$-topologically transitive, $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing, resp. $\mathfrak{D}_{\mathfrak{P}_s}$-topologically mixing, $\mathfrak{D}$-topologically mixing) iff the abstract Cauchy problem (3)$_\phi$, resp. (DFP)$_{R,\phi}$, is $\mathfrak{D}_\phi$-hypercyclic ($(\mathfrak{D}_\phi, \mathfrak{P}_\phi)$-hypercyclic, $\mathfrak{D}_{\phi_{\mathfrak{P}_\phi}}$-topologically transitive, resp. $\mathfrak{D}_{\phi_{(\mathfrak{P}_\phi)_s}}$-topologically transitive, $\mathfrak{D}_\phi$-topologically transitive, $\mathfrak{D}_{\phi_{\mathfrak{P}_\phi}}$-topologically mixing, resp. $\mathfrak{D}_{\phi_{(\mathfrak{P}_\phi)_s}}$-topologically mixing, $\mathfrak{D}_\phi$-topologically mixing).*

We continue by stating the following theorem.

Theorem 3. *Let $\alpha_i = i$ for all $i \in \mathbb{N}_n$, let Ω be an open non-empty subset of $\mathbb{K} = \mathbb{C}$ intersecting the imaginary axis, and let $f : \Omega \to E$ be an analytic mapping satisfying*

$$P_\lambda f(\lambda) = \left(\lambda^{\alpha_n} B + \sum_{i=0}^{n-1} \lambda^{\alpha_i} A_i\right) f(\lambda) = 0, \quad \lambda \in \Omega. \tag{6}$$

Set $\vec{x_\lambda} := [f(\lambda)\ \lambda f(\lambda)\ \cdots\ \lambda^{n-1} f(\lambda)]^T$ $(\lambda \in \Omega)$, $E_0 := span\{\vec{x_\lambda} : \lambda \in \Omega\}$, $\tilde{E} := \check{E} := \overline{E_0}$, $\vec{\beta} := (0, 1, \cdots, n-1)$, $W := \mathbb{N}_n$ and $\hat{E}_i := span\{f(\lambda) : \lambda \in \Omega\}$, $i \in W$. Let $\emptyset \neq S \subseteq E^n$ be such that $E_0 \subseteq Orb(S; (D_i)_{1 \leq i \leq l})$. Then $\vec{x_\lambda} \in \mathfrak{M}_{\mathfrak{D}}$, $\lambda \in \Omega$ and the abstract Cauchy problem (3) is $\mathfrak{D}_{\mathfrak{P}}$-topologically mixing provided that $\sum_{j=1}^{q} e^{\lambda_j \cdot} f(\lambda_j) \in \mathfrak{P}(\sum_{j=1}^{q} \vec{x_{\lambda_j}})$ for any $\sum_{j=1}^{q} \vec{x_{\lambda_j}} \in E_0$ $(q \in \mathbb{N};\ \lambda_j \in \Omega,\ 1 \leq j \leq q)$.

Proof. We shall content ourselves with sketching it. Consider the operator matrices

$$\mathcal{A} := \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{n-1} \end{bmatrix}$$

and

$$\mathcal{B} := \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdot\cdot I & 0 \\ 0 & 0 & 0 & \cdots & B \end{bmatrix},$$

acting on E^n with their maximal domains. Then the operator matrix $\mathcal{A}$ is closable, the operator matrix $\mathcal{B}$ is closed and, due to (6), $\overline{\mathcal{A}}\vec{x_\lambda} = \lambda \mathcal{B}\vec{x_\lambda}$, $\lambda \in \Omega$. Furthermore, if we suppose that Ω_0 is an arbitrary open connected subset of Ω which admits a cluster point in Ω, then the linear span of the set $\{\vec{x_\lambda} : \lambda \in \Omega_0\}$ is dense in $\tilde{E}$. Now the statement follows similarly as in the proof of [32, Theorem 5].◄

Remark 2. *(i) The assertions of [32, Theorem 5, Theorem 11] continue to hold, with appropriate modifications, in the setting of separable sequentially complete locally convex spaces. The conclusion of Theorem 3 remains true if we consider the equation (3) with the same initial conditions and with the term $B\frac{d^n}{dt^n}u(t)$ replaced by $\frac{d^n}{dt^n}Bu(t)$ (cf. [32, Remark 12(i)]).*

(ii) Suppose $\alpha_1 \in (0, 1)$ and $\alpha_i = i\alpha_1$, $i \in \mathbb{N}_n$. Then the argumentation used in the proofs of Theorem 3 and [32, Theorem 5] enables one to deduce some results about $\mathfrak{D}$-topologically mixing properties of the problem

$$\begin{aligned} &B\big(\mathbf{D}_t^{\alpha_1}\big)^n u(t) + \sum_{i=0}^{n-1} A_i \big(\mathbf{D}_t^{\alpha_1}\big)^i u(t) = 0, \quad t \geq 0, \\ &\Big(\big(\mathbf{D}_t^{\alpha_1}\big)^j u(t)\Big)_{t=0} = u_j,\ j = 0, \cdots, n-1, \end{aligned} \tag{7}$$

and its analogons obtained by replacing, optionally, some of the terms $B(\mathbf{D}_t^{\alpha_1})^n u(t)$ *and* $A_i(\mathbf{D}_t^{\alpha_1})^i u(t)$ *by* $(\mathbf{D}_t^{\alpha_1})^n Bu(t)$ *and* $(\mathbf{D}_t^{\alpha_1})^i A_i u(t)$, *respectively* $(0 \leq i \leq n-1)$. *The case* $\alpha_1 \in (1,2)$ *can be considered quite similarly.*

(iii) *It should be emphasized that Theorem 3 cannot be so simply reformulated in the case where there exists an index* $i \in \mathbb{N}_n$ *such that* $\alpha_i \notin \mathbb{N}$. *Speaking matter-of-factly, probably the only way to exploit (6) is to find analytic functions* $F_i : \Omega \to \mathbb{C}$ $(0 \leq i \leq m_n - 1)$ *such that the equation (3), equipped with the initial conditions* $u^{(i)}(0) = F_i(\lambda)f(\lambda)$, $0 \leq i \leq m_n - 1$, *has a strong solution of the form* $u(t;\lambda) = G(\lambda,t)f(\lambda)$, $t \geq 0$ $(\lambda \in \Omega)$, *where*

$$\lambda^{-\alpha_n}\mathbf{D}_t^{\alpha_n}G(\lambda,t) = \cdots = \lambda^{-\alpha_1}\mathbf{D}_t^{\alpha_1}G(\lambda,t) = G(\lambda,t), \quad t \geq 0 \quad (\lambda \in \Omega). \qquad (8)$$

By [16, Theorem 7.2], the validity of (8) would imply that for each $t \geq 0$, $\lambda \in \Omega$ *and* $i \in \mathbb{N}_n$, *we have:*

$$G(\lambda,t) = F_0(\lambda)E_{\alpha_i}\left(\lambda^{\alpha_i}t^{\alpha_i}\right) + \sum_{k=1}^{m_i-1} F_k(\lambda)\int_0^t \frac{(t-s)^{k-1}}{(k-1)!}E_{\alpha_i}\left(\lambda^{\alpha_i}s^{\alpha_i}\right)ds,$$

i.e., that for each $t \geq 0$, $\lambda \in \Omega$ *and* $i \in \mathbb{N}_n$, *we have:*

$$G(\lambda,t) = \sum_{l=0}^{\infty}\sum_{k=0}^{m_i-1} \lambda^{\alpha_i l}F_k(\lambda)\frac{t^{\alpha_i l+k}}{\Gamma(\alpha_i l + k + 1)}.$$

The function $t \mapsto G(\lambda,t) - F_0(\lambda)$, $t \geq 0$ *behaves asymptotically like* $\lambda^{\alpha_1}F_0(\lambda)g_{\alpha_1+1}(t)$ *as* $t \to 0+$, *so the number* α_1 *cannot be an element of the interval* $(0,1)$ *(to see this, consider the asymptotic behaviour of function* $t \mapsto G(\lambda,t) - F_0(\lambda)$, $t \geq 0$ *as* $t \to 0+$, *with the number* α_1 *replaced by* α_2*). Considering the asymptotic behaviour of function* $t \mapsto G(\lambda,t) - F_0(\lambda) - tF_1(\lambda)$, $t \geq 0$ $(t \mapsto G(\lambda,t) - F_0(\lambda) - tF_1(\lambda) - (t^2/2)F_2(\lambda)$, $t \geq 0; \cdots)$ *as* $t \to 0+$, *we obtain similarly that* α_1 *cannot be an element of the interval* $(1,2)$ $((2,3); \cdots)$. *Consequently,* $\alpha_1 \in \mathbb{N}$. *Ignoring the first order* α_1, *and repeating the same procedure with the order* α_2, *we get that* $\alpha_2 \in \mathbb{N}$. *A similar line of reasoning shows that* $\alpha_3, \cdots, \alpha_n \in \mathbb{N}$.

(iv) *Hypercyclic and topologically mixing properties of higher-order non-degenerate differential equations with integer order derivatives have been considered in a series of recent papers by using the usual reduction into first order matrix differential equations (cf. [11]-[12], [26, Section 3.2] and the references therein). To the best knowledge of the author, Theorem 3 is new and not considered elsewhere within the framework of the theory of abstract degenerate differential equations. It should be noted that we can prove a slight extension of this theorem by using the analyses from [29, Remark 1(iii)] and [31, Remark 1;3.] (cf. also Remark 1(iii) and Example 1(i) below).*

(v) *In [32], we have recently reconsidered the well known assertion of S. El Mourchid [17, Theorem 2.1] concerning the connection between the imaginary point spectrum* *and hypercyclicity of strongly continuous semigroups. An analogon of [17, Theorem 2.1] for abstract degenerate differential equations of first order has been formulated in [32, Theorem 18]. On the basis of this result, we can state some new facts about $\mathcal{D}$-topologically mixing properties of problem (3) considered in Theorem 3, provided that the equation (6) holds for all values of complex parameter λ belonging to some subinterval of imaginary axis. Details are left to the interested reader.*

We close the paper by providing some illustrative examples.

Example 1. *(i) Consider the equation (3) with $\alpha_i = i$, $i \in \mathbb{N}_n$ and with the operator $A_0 = A$ replaced by $-A$. Although it may seem contrary, Theorem 3 is not so easily comparable to [32, Theorem 11] in this case. For example, in the situation of [32, Example 14] with $P(z) = -z$ and $\alpha = 1$, the assumptions of [32, Theorem 11] are satisfied with $E := L^2(\mathbb{R})$, $c_1 > c > \frac{b}{2} > 0$, $A := (c - c_1)I$, $B := \mathcal{A}_c - A$, $A_1 := -\mathcal{A}_c + cI$, where the operator $\mathcal{A}_c$ is defined by $D(\mathcal{A}_c) := \{u \in L^2(\mathbb{R}) \cap W^{2,2}_{loc}(\mathbb{R}) : \mathcal{A}_c u \in L^2(\mathbb{R})\}$ and $\mathcal{A}_c u := u'' + bxu' + cu$, $u \in D(\mathcal{A}_c)$, $\Omega := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ \lambda \neq c - c_1,\ \Re\lambda < c - \frac{b}{2}\}$, $f(\lambda) := g_1(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}}\xi|\xi|^{-(2+\frac{\lambda-c}{b})})(\cdot)$, $\lambda \in \Omega$ or $f(\lambda) := g_2(\lambda) := \mathcal{F}^{-1}(e^{-\frac{\xi^2}{2b}}|\xi|^{-(1+\frac{\lambda-c}{b})})(\cdot)$, $\lambda \in \Omega$ (here $\mathcal{F}^{-1}$ denotes the inverse Fourier transform on the real line), $f_1(\lambda) := (c-c_1)/(c-\lambda)$ and $f_2(\lambda) := (c-c_1)/(\lambda-(c-c_1))$ $(\lambda \in \Omega)$. In particular, there is no open connected subset Ω' of Ω satisfying $\Omega' \cap i\mathbb{R} \neq \emptyset$ and $(\lambda^2/(f_2(\lambda)) + \lambda/(f_1(\lambda)) - 1)Af(\lambda) = 0$, $\lambda \in \Omega'$, i.e., the equation (6) does not hold with this choice of $f(\lambda)$. This is quite predictable because the equation (6), with the set Ω and the function $f(\cdot)$ replaced respectively by ${}'\Omega$ and ${}'f(\cdot)$ therein (and in our further analysis, for the sake of consistency of notation), is equivalent to say that $(\lambda^2 - \lambda)\mathcal{A}_c\, {}'f(\lambda) = (\lambda^2(c-c_1) - \lambda c + (c-c_1))'f(\lambda)$, $\lambda \in {}'\Omega$. Denote by Λ the set of all complex numbers $z \in i\mathbb{R} \setminus \{0\}$ for which there exists $\delta(z) > 0$ such that $\{0,1\} \cap L(z,\delta(z)) = \emptyset$, and for each $\lambda \in L(z,\delta(z))$ we have $\Re(c - c_1 - \frac{c_1}{\lambda-1} + \frac{c-c_1}{\lambda^2-\lambda}) < c - \frac{b}{2}$. Recalling that $\{z \in \mathbb{C} : \Re z < c - \frac{b}{2}\} \subseteq \sigma_p(\mathcal{A}_c)$, it readily follows that Theorem 3 can be also applied here with ${}'\Omega := \bigcup_{z\in\Lambda} L(z,\delta(z))$ and ${}'f_i(\lambda) := g_i(c - c_1 - \frac{c_1}{\lambda-1} + \frac{c-c_1}{\lambda^2-\lambda})$, $\lambda \in {}'\Omega$ $(i = 1,2)$, producing slightly different results from those obtained by applying [32, Theorem 11] (with $\hat{E} = \check{E} = \overline{span\{[{}'f_i(\lambda)\ \lambda\, {}'f_i(\lambda)]^T : \lambda \in {}'\Omega,\ i = 1,2\}}$, the subspace of E^2 whose first and second projections equal to E; cf. [13] and [29]). On the other hand, there exists a great number of very simple (non-)degenerate equations where we can apply Theorem 3 but not [32, Theorem 11]. Consider, for example, the equation $u'''(t) + (c_2 - \mathcal{A}_c)u'(t) + c_1u(t) = 0$, $t \geq 0$, where $c_1 \in \mathbb{C} \setminus \{0\}$ and $c_2 \in \mathbb{C}$. Using the same arguments as in the analysis carried out in [31, Remark 1(vi)] (cf. also [26, Theorem 3.3.9, Remark 3.3.10(v)]), with $c_3 = 0$, we obtain that there do not exist an open connected subset Ω_- of $\mathbb{C}$ and an index $i \in \{0,1,2\}$ such that the second equality in [32, (10)] holds. Contrary to this, there exist $t > 0$ and $\epsilon > 0$ such that the equation (6) holds with $\Omega = L(it,\epsilon)$.*

(ii) Suppose Ω is an open non-empty subset of $\mathbb{K} = \mathbb{C}$ intersecting the imaginary axis, $f : \Omega \to E$ is an analytic mapping, $g : \Omega \to \mathbb{C} \setminus \{0\}$ is a scalar-valued mapping and

$Af(\lambda) = g(\lambda)f(\lambda)$, $\lambda \in \Omega$. Let $P_j(z)$ be non-zero complex polynomials ($j \in \mathbb{N}_n^0$), and let

$$\lambda^n P_n(g(\lambda)) + \sum_{j=0}^{n-1} \lambda^j P_j(g(\lambda)) = 0, \quad \lambda \in \Omega.$$

Then the equation (6) holds with $B := P_n(A)$ and $A_j := P_j(A)$, $j \in \mathbb{N}_{n-1}^0$. If, additionally, the presumption $\sum_{j=0}^{n-1} \langle x_j^*, \lambda^j f(\lambda)\rangle = 0$, $\lambda \in \Omega$ for some continuous linear functionals $x_j^* \in E^*$ given in advance ($j \in \mathbb{N}_{n-1}^0$) implies $x_j^* = 0$ for all $j \in \mathbb{N}_{n-1}^0$, then the space E_0 from Theorem 3 equals to E^n (cf. [11, Theorem 3.1] for a concrete example of this type with $n = 3$). Finally, note that Theorem 3 can be sucessfully applied in the analysis of topologically mixing properties of a wide class of partial differential equations in Fréchet function spaces (see e.g. [26, Example 3.1.29]).

Acknowledgements. The author was partially supported by the grant 174024 of the Ministry of Science and Technological Development, Republic of Serbia.

References

[1] A.B. Aliev, B.H. Lichaei, *Existence and non-existence of global solutions of the Cauchy problem for higher order semilinear pseudo-hyperbolic equations,* Nonlinear Anal., **72**, 2010, 3275-3288.

[2] A.R. Aliev, A.L. Elbably, *Well-posedness of a boundary value problem for a class of third-order operator-differential equations*, Boundary Value Problems, 2013, **2013(140)** doi:10.1186/1687-2770-2013-140.

[3] A.R. Aliev, A.L. Elbably, *Completeness of derivative chains for polynomial operator pencil of third order with multiple characteristics*, Azerbaijan J. Math., **4(2)**, 2014, 3-9.

[4] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser Verlag, Basel, 2001.

[5] J. Banasiak, M. Moszyński, *A generalization of Desch-Schappacher-Webb criteria for chaos*, Discrete Contin. Dyn. Syst., **12**, 2005, 959-972.

[6] F. Bayart, E. Matheron, *Dynamics of Linear Operators,* Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, UK, **179(1)**, 2009.

[7] E. Bazhlekova, *Fractional Evolution Equations in Banach Spaces,* PhD Thesis, Eindhoven University of Technology, Eindhoven, 2001.

[8] J. Bès, J.A. Conejero, *An extension of hypercyclicity for N-linear operators*, Abstr. Appl. Anal., Article ID 609873, 2014, 11 pp.

[9] J. Bonet, L. Frerick, A. Peris, J. Wengenroth, *Transitive and hypercyclic operators on locally convex spaces*, Bull. London Math. Soc., **37**, 2005, 254-264.

[10] R.W. Carroll, R.W. Showalter, *Singular and Degenerate Cauchy Problems*, Academic Press, New York, 1976.

[11] J.A. Conejero, C. Lizama, F. Rodenas, *Chaotic behaviour of the solutions of the Moore-Gibson-Thompson equation*, Appl. Math. Inf. Sci., **9**, 2015, 1-16.

[12] J.A. Conejero, F. Ródenas, M. Trujillo, *Chaos for the hyperbolic bioheat equation*, Discrete Contin. Dyn. Syst. Ser. A, **35**, 2015, 653-668.

[13] J.A. Conejero, E. Mangino, *Hypercyclic semigroups generated by Ornstein-Uhlenbeck operators*, Mediterr. J. Math., **7**, 2010, 101-109.

[14] R.A.A. Cuello, *Traslaciones y Semigrupos C0 Topolgicamente Mezclantes*, PhD Thesis, Universidad de Puerto Rico, Recinto Universitario de Mayagëz, 2008.

[15] W. Desch, W. Schappacher, G.F. Webb, *Hypercyclic and chaotic semigroups of linear operators*, Ergodic Theory Dynam. Systems, **17**, 1997, 1-27.

[16] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010.

[17] S. El Mourchid, *The imaginary point spectrum and hypercyclicity*, Semigroup Forum, **76**, 2006, 313-316.

[18] K.-G. Grosse-Erdmann, S.G. Kim, *Bihypercyclic bilinear mappings*, J. Math. Anal. Appl., **399**, 2013, 701-708.

[19] K.-G. Grosse-Erdmann, A. Peris, *Linear Chaos*, Springer-Verlag, London, 2011.

[20] A. Favini, A. Yagi, *Degenerate Differential Equations in Banach Spaces*, Chapman and Hall/CRC Pure and Applied Mathematics, New York, 1998.

[21] V.E. Fedorov, *A generalization of the Hille-Yosida theorem to the case of degenerate semigroups in locally convex spaces*, Siberian Math. J., **46**, 2005, 333-350 (Russian).

[22] V.E. Fedorov, *A class of second-order equations of Sobolev type and degenerate groups of operators*, Vestn. Chelyab. Gos. Univ. Mat. Mekh. Inform., **26(13)**, 2011, 59-75.

[23] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publ. Co., Singapore, 2000.

[24] T. Kalmes, *Hypercyclic, Mixing, and Chaotic C_0-Semigroups*, PhD Thesis, Universität Trier, 2006.

[25] M. Kostić, *Generalized Semigroups and Cosine Functions*, Mathematical Institute SANU, Belgrade, 2011.

[26] M. Kostić, *Abstract Volterra Integro-Differential Equations*, Taylor and Francis Group/CRC Press/Science Publishers, Boca Raton, New York, London, 2015.

[27] M. Kostić, *Degenerate abstract Volterra equations in locally convex spaces*, Filomat (in press).

[28] M. Kostić, *Degenerate abstract multi-term fractional differential equations in locally convex spaces*, Publications de l'Institut Mathématique, submitted.

[29] M. Kostić, *Hypercyclicity and mixing for abstract time-fractional equations*, Dyn. Syst., **27**, 2012, 213-221.

[30] M. Kostić, *Hypercyclic and topologically mixing properties of abstract time-fractional equations with discrete shifts*, Sarajevo J. Math., **9**, 2013, 1-13.

[31] M. Kostić, *Hypercyclic and topologically mixing properties of certain classes of abstract time-fractional equations*, In: Springer Proceedings in Mathematics, Discrete dynamical systems and applications, Edited by Lluis Alseda, J. Cushing, S. Elaydi and A. Pinto, forthcoming paper 2015.

[32] M. Kostić, *Hypercyclic and topologically mixing properties of degenerate multi-term fractional differential equations*, Diff. Eqns. Dyn. Sys., in press.

[33] A.I. Kozhanov, *On boundary value problems for some classes of higher-order equations that are not solved with respect to the highest derivative*, Siberian Math. J., **35**, 1994, 324-340.

[34] S.G. Krein, *Linear Differential Equations in Banach Space*, Translations of Mathematical Monographs, AMS Providence, 1972.

[35] L. Ji, A. Weber, *Dynamics of the heat semigroup on symmetric spaces*, Ergodic Theory Dynam. Systems, **30**, 2010, 457-468.

[36] I.V. Melnikova, A.I. Filinkov, *Abstract Cauchy Problems: Three Approaches,* Chapman Hall/CRC Press, Boca Raton, 2001.

[37] I.V. Melnikova, *The Cauchy problem for differential inclusion in Banach space and distribution spaces*, Siberian Math. J., **42**, 2001, 751-765.

[38] V.I. Nalimov, *Degenerate second-order differential equations in Hilbert spaces*, Siberian Math J., **35**, 1994, 566-576.

[39] V. Obukhovskii, P. Zecca, *On boundary value problems for degenerate differential inclusions in Banach spaces*, Abstr. Appl. Anal., **13**, 2003, 769-784.

[40] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[41] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[42] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser-Verlag, Basel, 1993.

[43] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Derivatives and Integrals: Theory and Applications*, Gordon and Breach, New York, 1993.

[44] D.E. Shafranov, *Cauchy problem for the equation of a free surface of filtered liquid, in space of k-forms defined on Riemannian manifolds without boundary*, Proceedings of the Chelyabinsk Scientific Centre, **3(37)**, 2007, (Russian).

[45] R. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, London, 1977.

[46] G.A. Sviridyuk, V.E. Fedorov, *Linear Sobolev Type Equations and Degenerate Semigroups of Operators*, Inverse and Ill-Posed Problems (Book 42), VSP, Utrecht, Boston, 2003.

[47] G.A. Sviridyuk, *Morphology of the phase space of a class of semilinear equations of Sobolev type*, Vestnik Chelyabinsk. Univ. Ser. 3 Mat. Mekh., **2(5)**, 1999, 68-86 (Russian).

[48] G.A. Sviridyuk, O.V. Vakarina, *Higher-order linear equations of Sobolev type*, Dokl. Akad. Nauk., **363**, 1998, 308-310 (Russian).

[49] G.A. Sviridyuk, O.V. Vakarina, *The Cauchy problem for a class of higher-order linear equations of Sobolev type*, Differ. Uravn., **33**, 1998, 1415-1424.

[50] G.A. Sviridyuk, A.A. Zamyshlyaeva, *The phase spaces of a class of higher-order linear equations of Sobolev type*, Differ. Uravn., **42**, 2006, 269-278.

[51] T.-J. Xiao, J. Liang, *The Cauchy Problem for Higher–Order Abstract Differential Equations,* Springer-Verlag, Berlin, 1998.

[52] T.-J. Xiao, J. Liang, *Abstract degenerate Cauchy problems in locally convex spaces,* J. Math. Anal. Appl., **259**, 2001, 398-412.

[53] T.-J. Xiao, J. Liang, *Higher order degenerate Cauchy problems in locally convex space*, Math. Comp. Modelling, **41**, 2005, 837-847.

Marko Kostić
Faculty of Technical Sciences, University of Novi Sad, Trg Dositeja Obradovića 6, 21125 Novi Sad, Serbia
E-mail: marco.s@verat.net

Received 17 January 2015
Accepted 19 March 2015