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Reproducing Kernel Hilbert Spaces and the Associated Integral Equations

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Abstract. In this paper we consider various reproducing kernel Hilbert spaces and the classes of integral equations associated with them. We find solutions of these integral equations in the respective spaces.

Key Words and Phrases: Hardy space, Bergman space, Paley-Wiener space, symmetric measure, integral equations

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1. Reproducing kernel Hilbert spaces

Ahern, Flores and Rudin [1], Axler, Cuckovic [4] and Yi [16] studied the class of integral equations

$$g(x) = (1-x)^{n+1}\int_0^1 \frac{n+tx}{(1-tx)^{n+2}} g(t)t^{n-1}dt, n \geq 1 \tag{1}$$

associated with the invariant mean value property (see [1],[11], [2],[3]) of $\mathcal{M}$-harmonic functions. They proved that constants are the only solutions of these integral equations if $n \leq 11$ and this is not true if $n \geq 12$. Let $\mathbb{B}_n$ be the open unit ball of $\mathbb{C}^n$ [13], $n \geq 1, n \in \mathbb{Z}$, with respect to the Euclidean metric. The group of all one-to-one holomorphic maps of $\mathbb{B}_n$ onto $\mathbb{B}_n$ (the automorphisms of $\mathbb{B}_n$) will be denoted by $Aut(\mathbb{B}_n)$. It is generated by the unitary operators on $\mathbb{C}^n$ and the involutions ϕ_a of the form

$$\phi_a(z) = \frac{a - \mathcal{P}z - (1-|a|^2)^{\frac{1}{2}}Qz}{1-\langle z, a\rangle}, \tag{2}$$

where $a \in \mathbb{B}_n$, $\mathcal{P}$ is the orthogonal projection onto the space spanned by a, $Qz = z - \mathcal{P}z$,

$$\langle z, a\rangle = \sum_{i=1}^n z_i\overline{a_i}, \text{ and } |a|^2 = \langle a, a\rangle .$$

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The letter ν denotes the Lebesgue measure on $\mathbb{C}^n$, normalized so that $\nu(\mathbb{B}_n) = 1$ and for $1 \leq p \leq \infty$, the space $L^p(\mathbb{B}_n)$ refers to the usual Lebesgue spaces and the integration is with respect to the measure ν. For $n = 1$, $d\nu$ is equal to dA, the normalized area measure on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $L^p_a(\mathbb{B}_n), 1 \leq p < \infty$, denote the corresponding Bergman spaces of analytic functions in $\mathbb{B}_n$ that are also in $L^p(\mathbb{B}_n, d\nu)$. We now consider the Bergman space $L^2_a(\mathbb{B}_n)$ of holomorphic functions in $L^2(\mathbb{B}_n, d\nu)$. The reproducing kernel $K_{\mathbb{B}_n}(z, w)$ of $L^2_a(\mathbb{B}_n, d\nu)$ is holomorphic in z and antiholomorphic in w and

$$\int_{\mathbb{B}_n} |K_{\mathbb{B}_n}(z, w)|^2 d\nu(w) = K_{\mathbb{B}_n}(z, z) > 0 \tag{3}$$

for all $z \in \mathbb{B}_n$. Thus we define for each $\lambda \in \mathbb{B}_n$, a unit vector k_λ in $L^2_a(\mathbb{B}_n)$ by

$$k_\lambda(z) = \frac{K_{\mathbb{B}_n}(z, \lambda)}{\sqrt{K_{\mathbb{B}_n}(\lambda, \lambda)}}. \tag{4}$$

For $z, \lambda \in \mathbb{B}_n$,

$$K_{\mathbb{B}_n}(z, \lambda) = \frac{n!}{(1 - z \cdot \overline{\lambda})^{n+1}}, \tag{5}$$

where $z \cdot \overline{\lambda} = z_1\overline{\lambda}_1 + z_2\overline{\lambda}_2 + ... + z_n\overline{\lambda}_n$. For more details see [11]. If $f \in L^1(\mathbb{B}_n, d\nu)$, the Berezin transform of f is defined by

$$(Bf)(w) = \int_{\mathbb{B}_n} f(z)|k_w(z)|^2 d\nu(z). \tag{6}$$

Suppose f is radial and there is a function $g : [0, 1] \to \mathbb{C}$ such that $f(z) = g(|z|^2)$. It is not difficult to prove [1] that $Bf = f$ if and only if

$$g(x) = (1 - x)^{n+1} \int_0^1 \frac{n + tx}{(1 - tx)^{n+2}} g(t)t^{n-1} dt. \tag{7}$$

In 2003, Jevtic [10] studied the class of integral equations

$$g(x) = (1 - x)^\gamma \frac{\gamma}{2} \int_0^1 \frac{1 + tx}{(1 - tx)^{\gamma+1}} g(t)t^{\frac{\gamma}{2}-1} dt, \gamma \geq 2 \tag{8}$$

that arise naturally in the study of the invariant mean value properties of hyperbolically-harmonic functions and showed that the constants are the only solutions of the equation (8) if $2 \leq \gamma \leq 12$ but this is not true if $\gamma \geq 13$. The approach to the problem considered in [10] comes from Yi's work [16].

In this work we focus on various reproducing kernel Hilbert spaces like Hardy space, Bergman space, Paley-Wiener space $PW[-1, 1]$, the space $E^2(\mu_0)$, etc. In these spaces, solutions of integral equations similar to (1) are discussed. Characteristics of the nonconstant solutions of the integral equations (1), which are in $PW[-1, 1] \cap L^2[0, 1]$, are also discussed.

In section 2 we consider integral equations of the type

$$\int_0^1 \frac{g(t)}{1-tz}dt = \lambda g(z); \tag{9}$$

$$\int_0^1 \frac{h(t)}{1-tz}dt = \lambda h(z) + 1, \tag{10}$$

where $0 \le \lambda \le \pi$, $g(z) = \sum_{n=0}^{\infty} g_n z^n$, $h(z) = \sum_{n=0}^{\infty} h_n z^n$, $|z| \le 1$ and $\sum_{n=0}^{\infty} g_n^2 < \infty$, $\sum_{n=0}^{\infty} h_n^2 < \infty$, $g_n, h_n \in \mathbb{R}$, $n = 0, 1, 2, \cdots$. We showed that if $0 \le \lambda \le \pi$, then neither of the integral equations (9) and (10) has a solution, except for the solution $g(z) \equiv 0$ of (9). In section 3 we introduce the reproducing kernel Hilbert space $E^2(\mu_0)$ and show that the integral equation

$$F(z) = 2(1-z)^2 \int_{\mathbb{D}} \frac{1+\overline{z}w}{(1-\overline{z}w)^3} F(w) log \frac{1}{|w|} dA(w), z \in \mathbb{D},$$

has no non-zero solution in the space $E^2(\mu_0)$. In section 4 we introduce the Paley-Wiener space $PW[-1,1]$ which is also a reproducing kernel Hilbert space and show that if there exists a nonconstant solution f of the integral equation (1) in $PW[-1,1] \cap L^2[0,1]$, then there exists no sequence $(t_k)_{k=-\infty}^{\infty} \subset [0,1]$ that is uniformly discrete and

$$\sum_{k=-\infty}^{\infty} |f(t_k) - f(t_{k-1})| = \infty.$$

2. Hilbert matrix and the associated integral equations

In this section we focus on the Hardy space of the open unit disk $\mathbb{D}$ and consider integral equations (9) and (10). Wilhelm Magnus [9] arrived at these integral equations while looking at the spectrum of the Hilbert matrix. Magnus [9] showed that the spectrum of Hilbert's matrix is purely continuous and every real value λ for which $0 \le \lambda \le \pi$ belongs to the spectrum. In this section we simplified the proof of Magnus and showed that if $0 \le \lambda \le \pi$, then neither of the integral equations has a solution, except for the solution $g(z) \equiv 0$ of (9). We also showed that the spectrum of the Hilbert matrix contains the closed interval $[0, \pi]$.

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ and $d\theta$ be the arc-length measure on $\mathbb{T}$. For $1 \le p \le +\infty$, $L^p(\mathbb{T})$ will denote the Lebesgue space of $\mathbb{T}$ induced by $\frac{d\theta}{2\pi}$. Given $f \in L^1(\mathbb{T})$, the Fourier coefficients of f are

$$a_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta, \quad n \in \mathbb{Z}, \tag{11}$$

where $\mathbb{Z}$ is the set of all integers. Let $\mathbb{Z}_+$ denote the set of nonnegative integers. For $1 \le p \le +\infty$, the Hardy space of $\mathbb{T}$ denoted by $\mathbb{H}^p$, is the subspace of $L^p(\mathbb{T})$ consisting of functions f with $a_n(f) = 0$ for all negative integers n. We shall let $\mathbb{H}^p(\mathbb{D})$ denote the space of analytic functions on $\mathbb{D}$ which are harmonic extensions of functions in $\mathbb{H}^p$. The Hardy space $\mathbb{H}^2(\mathbb{D})$ is a reproducing kernel Hilbert space and the reproducing kernel (called the Cauchy or Szego kernel) $K_w(z) = \dfrac{1}{1-\bar{w}z}$, for $z, w \in \mathbb{D}$. It is not so important to distinguish $\mathbb{H}^p(\mathbb{D})$ from $\mathbb{H}^p$. Let $\mathbb{P}$ denote the orthogonal projection from $L^2(\mathbb{T})$ onto $\mathbb{H}^2(\mathbb{T})$. The sequence of functions $\{e^{in\theta}\}_{n\in\mathbb{Z}_+}$ forms an orthonormal basis for $\mathbb{H}^2(\mathbb{T})$.

It is well known [7] that the Hilbert matrix $H = \left(\dfrac{1}{i+j+1}\right), i, j = 0, 1, 2, \cdots$ acting by multiplication on sequences, induces a bounded linear operator $Ha = b, b_n = \displaystyle\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}$, on the l^2 space with norm $\|H\|_{l^2\to l^2} = \pi$. It also induces an operator $\mathcal{H}$ on the Hardy space $\mathbb{H}^2(\mathbb{D})$. To study the effect of the Hilbert matrix on Hardy spaces, let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ and assume $f \in \mathbb{H}^1(\mathbb{D})$. Hardy's inequality [7] says $\displaystyle\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \le \pi\|f\|_1$ and it follows that the power series

$$F(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n$$

has bounded coefficients, hence its radius of convergence is ≥ 1. In this way we obtain a well defined analytic function $F = \mathcal{H}(f)$ on the disk for each $f \in \mathbb{H}^1(\mathbb{D})$. A calculation shows that we can write

$$\mathcal{H}(f)(z) = \int_0^1 \frac{f(t)}{1-tz} dt, \tag{12}$$

where the convergence of the integral is guaranteed by Fejer-Riesz inequality [7] and the fact that $\dfrac{1}{1-tz}$ is bounded in t for each $z \in \mathbb{D}$.

The correspondence $f \to \mathcal{H}(f)$ is clearly linear and we consider the restriction of this mapping to the space $\mathbb{H}^2(\mathbb{D})$. The isometric identification of $\mathbb{H}^2(\mathbb{D})$ with l^2 gives $\|\mathcal{H}\|_{\mathbb{H}^2(\mathbb{D})\to\mathbb{H}^2(\mathbb{D})} = \pi$. Let $g(z) = \displaystyle\sum_{m=0}^{\infty} g_m z^m$ and $h(z) = \displaystyle\sum_{m=0}^{\infty} h_m z^m$ belong to $\mathbb{H}^2(\mathbb{D})$ where $|z| < 1$. This implies $(g_m) \in l^2$ and $(h_m) \in l^2$. Now $Hg = \lambda g$ implies

$$\sum_{n=0}^{\infty} \langle Hg, z^n \rangle z^n = \lambda \sum_{n=0}^{\infty} g_n z^n. \tag{13}$$

Hence

$$\sum_{m,n=0}^{\infty} g_m \langle Hz^m, z^n \rangle z^n = \lambda \sum_{n=0}^{\infty} g_n z^n.$$

Therefore

$$\sum_{m,n=0}^{\infty} \frac{g_m}{m+n+1} z^n = \lambda \sum_{n=0}^{\infty} g_n z^n. \tag{14}$$

Since

$$\int_0^1 g(t)t^n dt = \sum_{m=0}^{\infty} \frac{g_m}{m+n+1},$$

we obtain

$$\sum_{n=0}^{\infty} \left(\int_0^1 g(t)t^n dt \right) z^n = \lambda \sum_{n=0}^{\infty} g_n z^n. \tag{15}$$

Hence

$$\int_0^1 \frac{g(t)}{1-tz} dt = \lambda g(z). \tag{16}$$

Thus, if for $0 \le \lambda \le \pi$ the above integral equation has only the trivial solution $g(z) \equiv 0$, then λ cannot be an eigenvalue of the Hilbert matrix H. The equation

$$\int_0^1 \frac{h(t)}{1-tz} dt = \lambda h(z) + 1$$

also follows from similar considerations. That is, if we define

$$(Vh)(z) = \int_0^1 \frac{h(t)}{1-tz} dt, \tag{17}$$

our question is whether the constant 1 belongs to the image of $V - \lambda$. Notice that if $1 \notin Ran(V - \lambda)$, then constants do not belong to the range of $V - \lambda$. If (10) has no solution and (9) has only the trivial solution, then it will mean that λ belongs to the spectrum of the Hilbert matrix H.

Suppose the real numbers $g_n, h_n, n = 0, 1, 2 \cdots$ satisfy

$$\sum_{n=0}^{\infty} g_n^2 = M < \infty, \sum_{n=0}^{\infty} h_n^2 = N < \infty. \tag{18}$$

Hence the power series $g(z) = \sum_{n=0}^{\infty} g_n z^n, h(z) = \sum_{n=0}^{\infty} h_n z^n$ converges for $|z| < 1$. The following holds:

Theorem 1. *Suppose λ is a real number such that $0 \le \lambda \le \pi$. Then neither of the following integral equations*

$$\int_0^1 \frac{g(t)}{1-tz} dt = \lambda g(z), \tag{19}$$

$$\int_0^1 \frac{h(t)}{1-tz} dt = \lambda h(z) + 1 \tag{20}$$

has a solution, except for the solution $g(z) \equiv 0$ of (19).

Proof. From Cauchy-Schwarz inequality it follows that for $|z| < 1$,

$$|g(z)| \leq \frac{\sqrt{M}}{\sqrt{|1-|z|^2|}}, |h(z)| \leq \frac{\sqrt{N}}{\sqrt{|1-|z|^2|}}.$$

We can iterate (19) and (20) to get

$$\int_0^1 \frac{g(t)}{t-z} \log\left(\frac{1-z}{1-t}\right) dt = \lambda^2 g(z);$$

$$\int_0^1 \frac{h(t)}{t-z} \log\left(\frac{1-z}{1-t}\right) dt = \lambda^2 h(z) + \lambda - \frac{1}{2}\log(1-z).$$

For $f \in \mathbb{H}^1(\mathbb{D})$, the Fejer-Riesz theorem [7], which guarantees convergence along with analyticity, shows that the integral in (12) is independent of the path of integration. For $z \in \mathbb{D}$, we can choose the path

$$\zeta(t) = \zeta_z(t) = \frac{t}{(t-1)z+1}, 0 \leq t \leq 1,$$

i.e., a circular arc in $\mathbb{D}$ joining 0 to 1. A change of variable in (12) gives

$$\mathcal{H}(f)(z) = \int_0^1 \frac{1}{(t-1)z+1} f\left(\frac{t}{(t-1)z+1}\right) dt = \int_0^1 (T_t f)(z) dt, \tag{21}$$

where $T_t(f)(z) = w_t(z) f(\varphi_t(z))$, $w_t(z) = \frac{1}{(t-1)z+1}$ and $\varphi_t(z) = \frac{t}{(t-1)z+1}$. It is easy to see that φ_t is a self map of the disk, hence $f \mapsto f \circ \varphi_t$ is bounded on $\mathbb{H}^2(\mathbb{D})$ (see [17]) and for each $0 < t < 1, w_t(z)$ is a bounded analytic function. Thus $T_t : \mathbb{H}^2(\mathbb{D}) \to \mathbb{H}^2(\mathbb{D})$ is bounded for $0 < t < 1$. More information about these operators can be found in [6].

If $0 < \lambda < \pi$ and $Hg = \lambda g$, then $(Hg)(1) = \lambda g(1)$. But $(Hg)(1) = \int_0^1 (T_t g)(1) dt = \int_0^1 w_t(1) g(\varphi_t(1)) dt = \int_0^1 \frac{1}{t} g(1) dt = g(1) \int_0^1 \frac{1}{t} dt$. Thus $g(1) = 0$. Further

$$\begin{aligned} \lambda g'(z) &= \int_0^1 (T_t g)'(z) dt \\ &= \frac{-(t-1)}{[(t-1)z+1]^2} g(\varphi_t(z)) + g'(\varphi_t(z)) \frac{1}{(t-1)z+1} \frac{1-(t-1)(t-z)}{[(t-1)z+1]^2}. \end{aligned}$$

Hence $\lambda g'(1) = g'(1)\left[\frac{2-t}{t^3}\right]$. Thus $g'(1) = 0$. Similarly one can show that $h(1) = h'(1) = 0$.

Thus we have shown that if $0 < \lambda < \pi$, then

$$g(1) = g'(1) = 0, |g'(z)| \leq M', \tag{22}$$

where M' is a constant and $0 \leq z \leq 1$. Similarly, one can verify that

$$h(1) = h'(1) = 0, |h'(z)| \leq N'. \tag{23}$$

Differentiating (19), (20) with respect to z, we obtain from (22) and (23) that

$$\lambda g'(z) = \int_0^1 \frac{tg(t)}{(1-tz)^2}dt = -\frac{1}{z}\int_0^1 \frac{tg'(t)+g(t)}{1-tz}dt \tag{24}$$

and (24) is also valid if we substitute $h(z)$ for $g(z)$.
If we combine (24) and (19), we obtain

$$\lambda\{zg'(z)+g(z)\} = \int_0^1 \frac{tg'(t)}{1-tz}dt.$$

Since $|g'(t)| \leq M'$, this gives $\lambda(n+1)|g_n| \leq M' \int_0^1 t^{n+1}dt$, which also shows that

$$\sum_{n=0}^{\infty}(n+1)^2 g_n^2 < \infty. \tag{25}$$

Expanding $\frac{1}{1-tz}$ in a series of ascending powers of tz in the equation

$$\lambda g'(z) = -\frac{1}{z}\int_0^1 \frac{tg'(t)+g(t)}{1-tz}dt,$$

we obtain the infinite system of linear equations

$$\sum_{m=0}^{\infty}\frac{(m+1)g_m}{n+m+1} = -\lambda n g_n, n = 0,1,2,\cdots . \tag{26}$$

Putting $(m+1)g_m = x_m$, we find

$$\sum_{m,n=0}^{\infty}\frac{x_n x_m}{n+m+1} = -\lambda\sum_{n=0}^{\infty}\frac{n}{n+1}x_n^2. \tag{27}$$

Since (25) holds, using Hilbert inequality [7] we obtain a contradiction unless $x_1 = x_2 = \cdots = 0$. One can also obtain from (27) that $x_0 = 0$ and therefore $g(z) \equiv 0$. The proof for $h(z) \equiv 0$ (i.e., for the non-existence of $h(z)$) is precisely the same. Thus, if $0 < \lambda < \pi$, then the equations (19) and (20) has only trivial solution.

To complete the proof, it will be sufficient to show that $g(z) \equiv 0$ if $\lambda = 0$ or $\lambda = \pi$. Let $G(z) = \sum_{n=0}^{\infty}\frac{g_n}{n+1}z^{n+1}$. This is a continuous function of z for $0 \leq z \leq 1$ and from (19) we have in the case $\lambda = 0$,

$$\int_0^1 g(t)t^n dt = G(1) - n\int_0^1 G(t)t^{n-1}dt = 0$$

for $n = 0,1,2\cdots$. For $n = 0$, this gives $G(1) = 0$, and for $n = 1,2,\cdots$, we find that all the moments of $G(t)$ vanish, i.e., $G(t) \equiv 0$. Since the norm of the operator H is equal to π and the spectrum of H is a closed set, the equation (19) has only trivial solution if $\lambda = \pi$.
◀

3. The spaces $E^2(\mu)$

In this section we introduce the reproducing kernel Hilbert space $E^2(\mu_0)$ and show that the integral equation

$$F(z) = 2(1-z)^2 \int_{\mathbb{D}} \frac{1+\overline{z}w}{(1-\overline{z}w)^3} F(w) log \frac{1}{|w|} dA(w),$$

$z \in \mathbb{D}$, has no non-zero solution in the space $E^2(\mu_0)$. Let ν be a finite positive Borel measure defined on the closed unit interval [0,1] such that $\nu(\{0\}) = 0$ and $\nu([a,1]) > 0$ for every $0 \leq a < 1$. Then by the Riesz representation theorem [5], [15] there is a finite positive Borel measure μ supported on the closed unit disk $\overline{\mathbb{D}}$ such that

$$\int_{\overline{\mathbb{D}}} f(z) d\mu(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^1 f(re^{i\theta}) d\nu(r) \tag{28}$$

holds for all continuous functions $f(z)$ on $\overline{\mathbb{D}}$. Notice that the condition $\nu(\{0\}) = 0$ is needed to insure that (28) is always meaningful. Symbolically we write $d\mu(r,\theta) = \frac{1}{2\pi} d\nu(r) d\theta$. Any measure μ defined in this way is said to be a symmetric measure on $\overline{\mathbb{D}}$. The measure ν is called the radial component of μ. In case ν is a probability measure, we say that μ is normalized.

Let μ be a symmetric measure on $\overline{\mathbb{D}}$. For any $0 < p < \infty$ and any F in $L^p(\mu)$ we write

$$\|F\|_{p,\mu} = \left(\int_{\overline{\mathbb{D}}} |F(w)|^p d\mu(w) \right)^{\frac{1}{p}}. \tag{29}$$

If $f(z)$ is analytic in $\mathbb{D}$, we put

$$M_p(r,f,\mu) = \left(\int_{\overline{\mathbb{D}}} |f(rw)|^p d\mu(w) \right)^{\frac{1}{p}} \tag{30}$$

for $0 \leq r < 1$. If ν has no mass at 1, (30) is also meaningful for $r = 1, M_p(1,f,\mu)$ possibly being equal to $+\infty$. If ν is a single unit point mass at 1, the means $M_p(r,f,\mu)$ reduce to the standard $\mathbb{H}^p$ means

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}. \tag{31}$$

If μ is a symmetric measure on $\overline{\mathbb{D}}$ and $f(z)$ is an analytic function in $\mathbb{D}$, then for any $0 < p < \infty$ it is not so difficult to verify [7] that $M_p(r,f,\mu)$ is a nondecreasing function of r.

For $0 < p < \infty$, we denote by $E^p(\mu)$ the linear space of all functions $f(z)$ analytic in $\mathbb{D}$ such that

$$|||f|||_{p,\mu} = \sup_{r<1} M_p(r,f,\mu) < \infty. \tag{32}$$

Relation (32) determines a metric on $E^p(\mu)$ for all $0 < p < \infty$. For $p \geq 1, E^p(\mu)$ becomes a normed linear space with the above norm. Fix some $0 < p < \infty$. For any z in $\mathbb{D}$, we define the evaluation functional ϵ_z on $E^p(\mu)$ by

$$\epsilon_z : f \longrightarrow f(z) \tag{33}$$

for all f in $E^p(\mu)$. If μ is a symmetric measure on $\overline{\mathbb{D}}$, and $0 < p < \infty$, then $E^p(\mu)$ is complete and if K is a compact subset of $\mathbb{D}$, then $\{\epsilon_z : z \in K\}$ is a uniformly bounded set of linear functionals on $E^p(\mu)$. For proof see [8]. If $\nu(\{1\}) = 0$, we denote by $L^p_a(\mu, \mathbb{D})$ the Bergman space of functions analytic in $\mathbb{D}$ that are $L^p(\mu)$-integrable. It is known [8] that if $\nu(\{1\}) > 0$, then $E^p(\mu) = \mathbb{H}^p$ with an equivalent metric and if $\nu(\{1\}) = 0$, then $E^p(\mu) = L^p_a(\mu, \mathbb{D})$. Let $\mathbb{H}^p(\mu)$ be the $L^p(\mu)$-closure of the polynomials. It is shown in [8] that $\mathbb{H}^p(\mu)$ is isometrically isomorphic to $E^p(\mu)$ for all $0 < p < \infty$. We shall now focus our attention on the case $p = 2$. For $\alpha \geq 0$, we write

$$w(\alpha) = \int_0^1 t^\alpha d\nu(t). \tag{34}$$

The function $w(\alpha)$ is obviously a nonincreasing function of α. Since point evaluation is a bounded linear functional on the space $E^2(\mu)$, it is a functional Hilbert space. The sequence $e_n(z) = w(2n)^{-\frac{1}{2}} z^n, n \geq 0$ is easily seen to be a complete orthonormal set in $E^2(\mu)$. If $f \in E^2(\mu)$, it is not difficult to verify that $\langle f, e_n \rangle = a_n w(2n)^{\frac{1}{2}}$, where a_n is the nth Taylor coefficient of $f(z)$. The reproducing kernel of $E^2(\mu)$ is given by

$$K(w, z) = \sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)} = \sum_{n=0}^{\infty} w(2n)^{-1}(\overline{w}z)^n. \tag{35}$$

Consider the measure μ_0 defined on $\overline{\mathbb{D}}$ by

$$d\mu_0(r, \theta) = \frac{2}{\pi} r \log\left(\frac{1}{r}\right) dr d\theta. \tag{36}$$

Theorem 2. *(a)There exists no nonzero $F \in E^2(\mu_0)$ such that for all $z \in \mathbb{D}$,*

$$F(z) = 2(1-z)^2 \int_{\mathbb{D}} \frac{1 + \bar{z}w}{(1 - \bar{z}w)^3} F(w) \log \frac{1}{|w|} dA(w). \tag{37}$$

(b)Radial functions in $E^2(\mu_0)$ are constants.

Proof. (a) In the space $E^2(\mu_0)$, we have

$$\begin{aligned} w(2n) &= \int_0^1 r^{2n} \left(4r \log\left(\frac{1}{r}\right)\right) dr \\ &= \int_0^\infty 4t e^{-(2n+2)t} dt \end{aligned}$$

$$= \frac{1}{(n+1)^2}.$$

It then follows from Parseval's identity that $f(z)=\sum_{n=0}^{\infty} a_n z^n$ belongs to $E^2(\mu_0)$ if and only if $\sum_{n=0}^{\infty}(n+1)^{-2}|a_n|^2<\infty$ and

$$|||f|||^2_{2,\mu_0}=\sum_{n=0}^{\infty}\frac{|a_n|^2}{(n+1)^2}.$$

The reproducing kernel of $E^2(\mu_0)$ is given by

$$K(w,z)=\sum_{n=0}^{\infty}(n+1)^2(\overline{w}z)^n=\frac{1+\overline{w}z}{(1-\overline{w}z)^3}.$$

Suppose there exists a function $F\in E^2(\mu_0)$ such that for all $z\in\mathbb{D}$,

$$F(z)=2(1-z)^2\int_{\mathbb{D}}\frac{1+\bar{z}w}{(1-\bar{z}w)^3}F(w)\log\frac{1}{|w|}dA(w). \tag{38}$$

Then for all $z\in\mathbb{D}$, $F(z)=(1-z)^2F(z)$ as $\frac{1+\bar{z}w}{(1-\bar{z}w)^3}$ is the reproducing kernel for the space $E^2(\mu_0)$. Thus $F\equiv 0$.

(b) The function $F(w)\in E^2(\mu_0)$ where $w=se^{i\theta}$ if and only if

$$\int_{\mathbb{D}}|F(w)|^2d\mu_0(w)<\infty.$$

That is, if and only if

$$\int_0^{2\pi}d\theta\int_0^1|F(se^{i\theta})|^2\frac{2}{\pi}s\log\frac{1}{s}ds<\infty. \tag{39}$$

This is true if and only if

$$4\int_0^1|F(se^{i\theta})|^2s\log\frac{1}{s}ds<\infty.$$

Let $f(s)=f(|w|)=F(se^{i\theta})=F(w)$. Then $4\int_0^1|f(s)|^2s\log\frac{1}{s}ds<\infty$. Thus, if $f(|w|)=F(w)$, then $F\in E^2(\mu_0)$ if and only if $f\in L^2([0,1],4s\log\frac{1}{s}ds)$. To prove (b), let F be a radial function in $E^2(\mu_0)$ and assume $f(|z|)=F(z)$. Then $f\in L^2([0,1],4s\log\frac{1}{s}ds)$. Let $t=|z|$. Hence

$$f(t) = f(|z|)=F(z)=\int_{\mathbb{D}}K(z,w)F(w)d\mu_0(w)$$

$$
\begin{aligned}
&= \int_0^{2\pi} d\theta \int_0^1 \frac{1+tse^{i\theta}}{(1-tse^{i\theta})^3} f(s)\frac{2}{\pi} s \log\frac{1}{s} ds \\
&= \frac{2}{\pi}\int_0^{2\pi} K(t, se^{i\theta}) d\theta \int_0^1 f(s) s\log\frac{1}{s} ds \\
&= 4\int_0^1 f(s) s \log\frac{1}{s} ds.
\end{aligned}
$$

Thus F is a constant. ◄

Notice that $f \in E^2(\mu_0)$ if and only if $f(z) = g'(z)$ for some $g \in \mathbb{H}^2$. Moreover, $\frac{d}{dz}$ is an isometric isomorphism from $\mathbb{H}^2_0$ onto $E^2(\mu_0)$, where $\mathbb{H}^2_0$ is the collection of all $\mathbb{H}^2$ functions that vanish at 0. This is immediate, since $\frac{d}{dz}$ carries the orthonormal basis $\{z^n\}_{n=1}^{\infty}$ for $\mathbb{H}^2_0$ onto the orthonormal basis $\{(n+1)z^n\}_{n=0}^{\infty}$ for $E^2(\mu_0)$. One of the measure that is equivalent to the measure μ_0 is given by

$$d\mu_1(r,\theta) = r(1-r)drd\theta.$$

4. The Paley-Wiener spaces

Let K be a compact subset of $\mathbb{R}$. The Paley-Wiener space $PW(K)$ is the space of functions f whose Fourier transforms $\check{\mathbf{f}}$ are supported on K. That is,

$$PW(K) = \left\{ f \in L^2(\mathbb{R}) : f(t) = \frac{1}{2\pi}\int_K \check{\mathbf{f}}(x) e^{ixt} dx, \text{ where } \check{\mathbf{f}}(x) \in L^2(K) \right\}.$$

Notice that

$$
\begin{aligned}
f(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \check{\mathbf{f}}(x) e^{ixs}\chi_K(x) dx \\
&= \frac{1}{2\pi}\langle \check{\mathbf{f}}(x), e^{-ixs}\chi_K(x)\rangle \\
&= \langle f, K_s\rangle,
\end{aligned}
$$

where the reproducing kernel K_s satisfies $\check{\mathbf{K}}_{\mathbf{s}}(x) = e^{-ixs}\chi_K(x)$ or $K_s(t) = \frac{1}{2\pi}\int_K e^{ix(t-s)} dx$.

In this section we show that if there exists a nonconstant solution f of the integral equation (1) in $PW[-1,1] \cap L^2[0,1]$, then there exists no sequence $(t_k)_{k=-\infty}^{\infty} \subset [0,1]$ that is uniformly discrete and

$$\sum_{k=-\infty}^{\infty} |f(t_k) - f(t_{k-1})| = \infty.$$

The reproducing kernel of $PW[-b,b]$ is given by $K_s(t) = K(t-s)$, where

$$K(t) = \begin{cases} \frac{\sin bt}{\pi t}, & \text{if } t \neq 0; \\ \frac{b}{\pi}, & \text{if } t = 0. \end{cases}$$

It is known [12] that if $f \in PW[-b,b]$, then

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{b}\right) \frac{\sin b\left(t-\frac{n\pi}{b}\right)}{\pi\left(t-\frac{n\pi}{b}\right)} \\ &= \sum_{n=-\infty}^{\infty} \frac{\pi}{b}\langle f, K_{\frac{n\pi}{b}}\rangle K_{\frac{n\pi}{b}}(t) \end{aligned}$$

converges uniformly and in the $L^2(\mathbb{R})$ norm. That is, f can be reconstructed from samples spaced at equal intervals $\frac{\pi}{b}$. Thus we see that the normalized reproducing kernel functions $k_{\frac{n\pi}{b}}(t) = \sqrt{\frac{\pi}{b}} K_{\frac{n\pi}{b}}(t), n \in \mathbb{Z}$ form an orthonormal basis for $PW[-b,b]$.

Theorem 3. *Suppose $V \in L^2[0,1] \cap PW([-1,1])$, V is an absolutely continuous function such that V' is in $L^2[0,1]$. Suppose V is a solution of the integral equation (1) and $\int_0^1 V(s)s^{n-1}ds = 0$. Then*

$$\|V\|_2 \leq \|V'\|_2 \leq \|V\|_\infty.$$

Proof. From Bernstein's inequality [12],[14] it follows that $\|V'\|_\infty \leq \|V\|_\infty$. Thus $\|V'\|_2 \leq \|V\|_\infty$. Now suppose V is a solution of the integral equation (1). Hence

$$V(t) = (1-t)^{n+1} \int_0^1 \frac{n+ts}{(1-st)^{n+2}} V(s)s^{n-1}ds. \tag{40}$$

Therefore $V(0) = n\int_0^1 V(s)s^{n-1}ds = 0$. Suppose $f, f' \in L^2[0,2\pi]$, $f(t) = \sum_{m=-\infty}^{\infty} a_m e^{imt}$, $f'(t) = \sum_{m=-\infty}^{\infty} ima_m e^{imt}$ are the corresponding Fourier series converging in L^2 norm. Suppose $a_0 = 0$. Then

$$\|f\|_2^2 = \sum_{m=-\infty}^{\infty} |a_m|^2 \leq \sum_{m=-\infty}^{\infty} m^2|a_m|^2 = \|f'\|_2^2.$$

Now for $g, g' \in L^2(0,\frac{\pi}{2})$, with $g(0) = 0$, we can extend g to a function f satisfying the hypotheses above by setting

$$f(t) = \begin{cases} g(\pi - t), & \text{for } \frac{\pi}{2} \leq t \leq \pi; \\ -g(t-\pi), & \text{for } \pi \leq t \leq 2\pi. \end{cases}$$

It is then clear that f and f' lie in $L^2[0,2\pi]$ and that $\int_0^{2\pi} f(t)dt = 0$. Consequently, $\|f\|_2 \leq \|f'\|_2$ and hence $\|g\|_2 \leq \|g'\|_2$. The inequality for the interval [0,1] is obtained by a change of variable $u = \dfrac{\pi}{2}t$.◀

Definition 1. *A set* $(t_k) \subset [0,1]$ *is said to be uniformly discrete if there is a constant* $\delta > 0$ *such that* $|t_j - t_k| \geq \delta$ *whenever* $j \neq k$.

Theorem 4. *Suppose* $f \in PW([-1,1]) \cap L^2[0,1]$ *and suppose* f *is a nonconstant solution of the integral equation (1). Then there exists no sequence* $(t_k)_{k=-\infty}^{\infty} \subset [0,1]$ *that is uniformly discrete and*

$$\sum_{k=-\infty}^{\infty} |f(t_k) - f(t_{k-1})| = \infty. \tag{41}$$

Proof. We shall first show that if f is a nonconstant fixed point of the integral operator

$$TV(t) = (1-t)^{n+1} \int_0^1 \frac{n+ts}{(1-ts)^{n+2}} V(s) s^{n-1} ds,$$

then f is a function of unbounded variation not attaining its supremum and infimum anywhere, but approaching both at 1. The points to note are:

(a) Constant functions are fixed points.

(b) If $u \geq 0$ on $[0,1), u \in C([0,1))$ and u is not identically zero, then $Tu > 0$ on $[0,1)$.

These statements can be verified easily (see [1] and [16]). Thus it follows that if $u \in L^\infty$, then $Tu \in L^\infty$ and $\|Tu\|_\infty \leq \|u\|_\infty$. Therefore $\|T\| \leq 1$. When u is a constant function, we have $\|Tu\|_\infty = \|u\|_\infty$. Hence $\|T\| = 1$ and the spectral radius of T is 1. From these, it also follows that if u is a nonconstant fixed point in $C([0,1))$, then

(i) $\inf_{t\in[0,1)} u < u(t)$ for all $t \in [0,1)$.

(ii) $\sup_{t\in[0,1)} u > u(t)$ for all $t \in [0,1)$.

(iii) $\liminf_{t\to 1} u(t) = \inf_{t\in[0,1)} u$.

(iv) $\limsup_{t\to 1} u(t) = \sup_{t\in[0,1)} u$.

If u is unbounded below, then (i) and (iii) are trivial. If u is bounded below, let $\alpha = \inf_{t\in[0,1)} u$. Now, $u(t) - \alpha \geq 0$ and is not identically zero since u is not constant. By (a) and (b), $u - \alpha = T(u-\alpha) > 0$ on $[0,1)$, proving (i). Again (iii) is now immediate by continuity. If u is unbounded above, then (ii) and (iv) are trivial. If u is bounded above, the same argument as above applied to $\sup\ u - u$ shows (ii) and (iv). What (i)-(iv) show is that nonconstant $C([0,1))$ fixed points of T look something like the plot in Figure 1 where either the infimum or supremum may be infinite. They oscillate infinitely many times, thus having unbounded variation on $[0,1)$, not attaining their supremum or infimum anywhere, but approaching both at 1.

Now let $h \in L^2(\mathbb{R})$ such that h is supported on $[-\frac{\delta}{2}, \frac{\delta}{2}]$ and $|\check{\mathbf{h}}(x)| \geq 1$ for $x \in [-1,1]$. For example, we could take

$$\check{\mathbf{h}}(x) = \frac{2\sin \epsilon x}{\epsilon x}$$

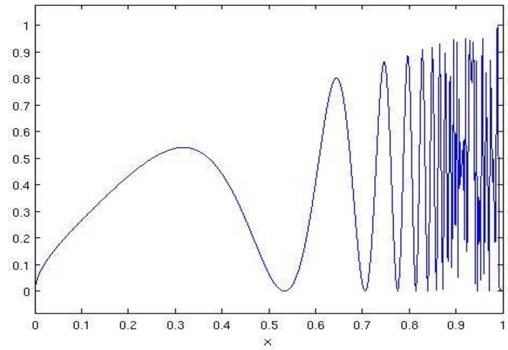

Figure 1: Unbounded variation of the fixed points of T

for sufficiently small $\epsilon > 0$. We write $\check{\mathbf{g}} = \dfrac{\check{\mathbf{f}}}{\check{\mathbf{h}}}$ so that $g \in PW([-1,1])$ as well and since $\check{\mathbf{f}} = \check{\mathbf{g}}\check{\mathbf{h}}$ we see that

$$\begin{aligned} f(t) &= \int_0^1 g(u)h(t-u)du \\ &= \int_{|t-u|\leq \frac{\delta}{2}} g(u)h(t-u)du. \end{aligned}$$

Suppose $(t_k)_{k=-\infty}^{\infty}$ is a sequence in $[0,1]$ which is uniformly discrete and

$$\sum_{k=-\infty}^{\infty} |f(t_k) - f(t_{k-1})| = \infty.$$

Thus

$$\begin{aligned} f(t_k) &= \int_{|t_k-u|\leq \frac{\delta}{2}} g(u)h(t_k-u)du, \\ f(t_{k-1}) &= \int_{|t_{k-1}-u|\leq \frac{\delta}{2}} g(u)h(t_{k-1}-u)du. \end{aligned}$$

Hence

$$\begin{aligned} |f(t_k) - f(t_{k-1})| &= \left| \int_{|t_k-u|\leq \frac{\delta}{2}} g(u)h(t_k-u)du - \int_{|t_{k-1}-u|\leq \frac{\delta}{2}} g(u)h(t_{k-1}-u)du \right| \\ &\leq \left| \int_{|t_k-u|\leq \frac{\delta}{2}} g(u)h(t_k-u)du \right| + \left| \int_{|t_{k-1}-u|\leq \frac{\delta}{2}} g(u)h(t_{k-1}-u)du \right| \\ &\leq \|h\|_2 \left(\int_{|t_k-u|\leq \frac{\delta}{2}} |g(u)|^2 du \right)^{\frac{1}{2}} + \|h\|_2 \left(\int_{|t_{k-1}-u|\leq \frac{\delta}{2}} |g(u)|^2 du \right)^{\frac{1}{2}}. \end{aligned}$$

Thus

$$\infty = \sum_{k=-\infty}^{\infty} |f(t_k) - f(t_{k-1})| \leq 2\|h\|_2\|g\|_2.$$

But this is not possible as $g, h \in L^2(\mathbb{R})$. ◄

It is known [12] that if $f \in PW[-b,b]$ and $s \in R$, then the value of the derivative $f'(s)$ is given by

$$f'(s) = \int_{-\infty}^{\infty} f(t)h_s(t)dt,$$

where $h_s(t) = h(t-s)$ with

$$h(t) = \begin{cases} \frac{\sin bt}{\pi t^2} - \frac{b\cos bt}{\pi t}, & \text{if } t \neq 0; \\ 0, & \text{if } \ t = 0. \end{cases}$$

Now suppose $n \in Z, 1 \leq n \leq 11$ and $V \in PW([-1,1]) \bigcap L^2[0,1]$ is a solution of the integral equation (1). Then by [12], it follows that

$$V'(s) = \int_0^1 V(t)h_s(t)dt = 0$$

for all $s \in [0,1]$ as constants are the only solutions of (1).

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