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Necessary Condition for the Uniform Minimality of Kostyuchenko Type Systems

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Abstract. It is shown that, for non-real values of α, the systems of the form $\{\exp(i\alpha\mu_n t)\sin(\lambda_n t)\}_{n\geq 1}$ and $\{\exp(i\alpha\mu_n t)\cos(\lambda_n t)\}_{n\geq 1}$ are not uniformly minimal in $L_2(a,b)$ and thus do not form bases for $L_2(a,b)$, whatever the interval (a,b) is. Real sequences $\{\mu_n\}$ and $\{\lambda_n\}$ are such that the limits $\varliminf\limits_{n\to\infty}(\lambda_n/\mu_n)$, $\lim\limits_{n\to\infty}(\lambda_{n+1}-\lambda_n)$, $\lim\limits_{n\to\infty}(\mu_{n+1}-\mu_n)$ do not vanish.

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In the theory of quadratic pencils of second-order differential operators, there appears necessity to study functional properties of the systems of the form

$$\{\varphi_n\}:\ \varphi_n(t)=\exp(i\alpha\mu_n t)\sin(\lambda_n t),\qquad \{\psi_n\}:\ \psi_n(t)=\exp(i\alpha\mu_n t)\cos(\lambda_n t),\tag{1}$$

in Lebesgue spaces $L_p(a,b)$, $p\geq 1$, where $\{\mu_n\}$ and $\{\lambda_n\}$ are some number sequences, α is a complex constant.

The interest to these systems was invoked by A.G.Kostyuchenko who posed in 1969 the completeness problem for the first system in (1) with the simplest set of parameters: $\mu_n=\lambda_n=n$, $a=0$, $b=\pi$. This pattern system (we call it the Kostyuchenko system) has been studied intensively (see [1–4] for the overview of results). Particularly, it was shown that the Kostyuchenko system is complete in $L_2(0,\pi)$ for $\alpha\in\mathbb{C}\setminus\{(-\infty,-1]\cup[1,+\infty)\}$ and also complete in any $L_p(0,\pi)$, $p\geq 1$ for $\alpha\in(-1,1)$. For $\alpha\in(-1,1)$, this system forms the Riesz basis for $L_2(0,\pi)$ if and only if $|\alpha|<(\exp(\pi\sqrt{3}-1)/(\exp(\pi\sqrt{3}+1)$ [2]. In the case when α is not real, the Kostyuchenko system does not form a basis for $L_2(0,\pi)$ as it is not uniformly minimal [5]; its nonbasicity in $L_p(0,\pi)$ for any $p\geq 1$ follows from [6].

The general form of systems (1) gained less attention. E.g., Lubarskij [7] studied its completeness and minimality in $L_p(0,\pi)$ in the case when $\mu_n=i\lambda_n$ and $\{\lambda_n\}$ is a set of roots for the entire function of sine type. The conditions on the systems (1) to satisfy the Bessel-type inequality were discussed by Konashenko [8].

It is worth mentioning that the systems (1) also appear in other applications – in problems of optimal control [9], in boundary value problems for elliptic equations [1], etc.

In the present paper we consider a wide class of number sequences $\{\mu_n\}$, $\{\lambda_n\}$ and an arbitrary interval (a, b). We prove that the condition $\alpha \in \mathbb{R}$ is necessary for the uniform minimality in $L_2(a, b)$ and therefore, for the basicity in $L_2(a, b)$ of both systems in (1).

1. Preliminaries and main results

A system $\{e_n\}$ in a Banach space $\mathcal{B}$ with the norm $\|\cdot\|$ is called *uniformly minimal in* $\mathcal{B}$ if there exists $\varepsilon_0 > 0$ such that for any integer k: $\inf\limits_{f \in E_k} \|e_k - f\| \geq \varepsilon_0 \|e_k\|$, where E_k denotes the closure of span of all elements e_n with $n \neq k$. The system $\{e_n\}$ *forms a basis for* $\mathcal{B}$ if for any $f \in \mathcal{B}$ there exists a unique sequence of scalars $\{f_n\}$ such that $f = \sum_{n=1}^{\infty} f_n e_n$ with respect to the norm $\|\cdot\|$. It is known [10] that any basis in $\mathcal{B}$ is uniformly minimal.

Assume the real number sequences $\{\mu_n\}$ and $\{\lambda_n\}$ in (1) satisfy the conditions

$$\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} \equiv L \neq 0, \tag{2}$$

$$\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) \equiv L_1 \neq 0, \quad \lim_{n\to\infty} (\mu_{n+1} - \mu_n) \equiv L_2 \neq 0. \tag{3}$$

Theorem. *Let the conditions (2) and (3) hold. If* $\operatorname{Im} \alpha \neq 0$*, then neither system in (1) is uniformly minimal in* $L_2(a, b)$ *whatever the interval* (a, b) *is.*

Note that the assumption (3) implies both sequences $\{\mu_n\}$ and $\{\lambda_n\}$ are infinitely large, monotone (starting with some n), and without loss of generality we may put $L_1 = L_2 = 1$. In fact, instead of these sequences one may consider the sequences $\{\mu_n / L_2\}$ and $\{\lambda_n / L_1\}$ and introduce the new parameter $\widetilde{\alpha} = \alpha L_2 / L_1$ and new variable $\widetilde{t} = t L_1$. Besides, we may restrict exposition to the case when $\operatorname{Im} \alpha > 0$ and $L > 0$. If $\operatorname{Im} \alpha L < 0$, then it is possible to substitute the variable t by $(-t)$ and take into account the oddness of sine and evenness of cosine.

Therefore, in the sequel we assume that $\operatorname{Im} \alpha > 0$ and the sequences $\{\mu_n\}$ and $\{\lambda_n\}$ are infinitely large, monotone increasing and satisfy the conditions

$$\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} \equiv L > 0, \quad \lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = \lim_{n\to\infty} (\mu_{n+1} - \mu_n) = 1. \tag{3$'$}$$

We will consider only the first system in (1) as the second one is treated similarly.

The statement of the theorem follows immediately from the following

Lemma. *Under the assumptions of the Theorem, for any* $\varepsilon > 0$ *there exist integers* k, n_0 *such that the inequality**

$$\left\| \frac{\varphi_n}{\|\varphi_n\|_2} - \frac{\varphi_{n+k}}{\|\varphi_{n+k}\|_2} \right\|_2 < \varepsilon \tag{4}$$

*Here and below in the text $\|\cdot\|_2$ denotes the norm in $L_2(a, b)$.

holds for any $n \geq n_0$.

The inequality (4) with an arbitrarily small ε obviously contradicts the uniform minimality of the system $\{\varphi_n\}$ in $L_2(a, b)$.

Matching of indices in (4) qualifies the degree of nonbasicity of the system (1) in the case of a non-real α.

Corollary. *Under the conditions (2) and (3) and for* $\operatorname{Im}\alpha \neq 0$*:*

1) the system $\{\varphi_n\}$ *cannot form a basis for* $L_2(a, b)$ *even after removal of any finite number of its functions;*

2) one can remove an infinite subsystem of $\{\varphi_n\}$ *to leave the system still not uniformly minimal in* $L_2(a, b)$.

These features of the system $\{\varphi_n\}$ with a non-real α resemble those of non-minimal systems of powers $\{t^{\mu_n}\}_{n\geq 1}$ for which one can easily verify that the left-hand side of (4) vanishes as $n \to \infty$ for any fixed k and any interval (a, b) of the positive axis.

2. Proof of the lemma

Using the identity

$$\left\| \frac{\varphi_n}{\|\varphi_n\|_2} - \frac{\varphi_{n+k}}{\|\varphi_{n+k}\|_2} \right\|_2^2 = 2 - \frac{2}{\|\varphi_n\|_2 \cdot \|\varphi_{n+k}\|_2} \operatorname{Re} \int_a^b \varphi_n(t) \cdot \overline{\varphi_{n+k}(t)}\, dt, \tag{5}$$

we will estimate the asymptotic behavior of the factors in the right-hand side for the large values of $n \in \mathbb{N}$ and a specially chosen value of $k \in \mathbb{N}$.

For brevity, we denote $\operatorname{Re}\alpha = \alpha_1$, $\operatorname{Im}\alpha = \alpha_2$ and assume $\alpha_2 > 0$. We also introduce the function $K(\xi, \varkappa) = \dfrac{1}{1+\varkappa^2}(\cos\xi - \varkappa \sin\xi)$ which for any $\xi, \varkappa \in \mathbb{R}$, clearly satisfies the inequality

$$|K(\xi, \varkappa)| \leq \frac{1}{\sqrt{1+\varkappa^2}}. \tag{6}$$

The direct calculation yields

$$\|\varphi_n\|_2^2 = \frac{\exp(-2\alpha_2\mu_n a)}{4\alpha_2\mu_n} \left[1 - K\Big(2\lambda_n a, \frac{\lambda_n}{\alpha_2\mu_n}\Big) - \exp(-2\alpha_2\mu_n(b-a)) \left(1 - K\Big(2\lambda_n b, \frac{\lambda_n}{\alpha_2\mu_n}\Big) \right) \right]. \tag{7}$$

Here the factor $\exp(-2\alpha_2\mu_n(b-a))$ is infinitely small as $n \to \infty$. We denote this fact by $\exp(-2\alpha_2\mu_n(b-a)) = o(1)$, where $o(1)$ stands for any quantity which is infinitesimal as $n \to \infty$ when all its other parameters are fixed.

By (6) and the condition (2), the term $A_n \equiv 1 - K\Big(2\lambda_n a, \dfrac{\lambda_n}{\alpha_2\mu_n}\Big)$ satisfies the estimate

$$1 - \frac{1}{\sqrt{1 + L^2/4}} \leq A_n \leq 2 \tag{8}$$

for sufficiently large values of n.

Thus, the relation (7) can be rewritten in the form

$$\|\varphi_n\|_2^2 = \frac{\exp(-2\alpha_2\mu_n a)}{4\alpha_2\mu_n}\Big(A_n + o(1)\Big). \tag{9}$$

Let us consider now the norm of $\varphi_{n+k}(t)$ for any fixed value of $k \in \mathbb{N}$. The norm $\|\varphi_{n+k}\|_2^2$ satisfies (9) with n substituted by $n+k$. As (3′) implies

$$\lambda_{n+k} = \lambda_n + k + o(1), \quad \mu_{n+k} = \mu_n + k + o(1), \tag{10}$$

and, by (2), the relations

$$\begin{gathered} \lambda_{n+k} = \lambda_n(1+o(1)), \quad \mu_{n+k} = \mu_n(1+o(1)), \quad \frac{\lambda_{n+k}}{\mu_{n+k}} = \frac{\lambda_n}{\mu_n}(1+o(1)), \\ \Big[1+\Big(\frac{\lambda_{n+k}}{\alpha_2\mu_{n+k}}\Big)^2\Big]^{-1} - \Big[1+\Big(\frac{\lambda_n}{\alpha_2\mu_n}\Big)^2\Big]^{-1} = \Big[1+\Big(\frac{\lambda_n}{\alpha_2\mu_n}\Big)^2\Big]^{-1} o(1) \end{gathered} \tag{11}$$

hold, we finally come to the estimate

$$\|\varphi_{n+k}\|_2^2 = \frac{\exp(-2\alpha_2\mu_{n+k}a)}{4\alpha_2\mu_n}\Big(A_n + [1](1-\cos 2ka) + [1]\sin 2ka + o(1)\Big). \tag{12}$$

In (12) we use the notation $[c_0]$. It denotes any quantity f that satisfies the estimate $|f(\xi_1,\dots,\xi_m)| \le c_0$ for all admissible values of its parameters $\xi_1,\dots,\xi_m$.

The integral factor in the right-hand side of (5) equals the sum of three integrals:

$$\begin{aligned} &\int_a^b \exp(-\alpha_2(\mu_n+\mu_{n+k})t)\cdot \sin^2\lambda_n t \cdot \cos(\lambda_{n+k}-\lambda_n)t\,dt + \\ &\quad + \frac{1}{4}\int_a^b \exp(-\alpha_2(\mu_n+\mu_{n+k})t)\cdot \sin 2\lambda_n t\cdot \sin(\lambda_{n+k}-\lambda_n+\alpha_1(\mu_{n+k}-\mu_n))t\,dt + \\ &\quad + \frac{1}{4}\int_a^b \exp(-\alpha_2(\mu_n+\mu_{n+k})t)\cdot \sin 2\lambda_n t\cdot \sin(\lambda_{n+k}-\lambda_n-\alpha_1(\mu_{n+k}-\mu_n))t\,dt \equiv \\ &\qquad\qquad \equiv I_1 + I_2 + I_3. \end{aligned} \tag{13}$$

For convenience, we put $\nu_{nk} = \alpha_2(\mu_n + \mu_{n+k})$ and have the formula for I_1:

$$\begin{aligned} I_1 = \frac{\exp(-\nu_{nk}a)}{2\nu_{nk}}\bigg[& K\Big((\lambda_{n+k}-\lambda_n)a, \frac{\lambda_{n+k}-\lambda_n}{\nu_{nk}}\Big) - \frac{1}{2}K\Big((\lambda_n+\lambda_{n+k})a, \frac{\lambda_n+\lambda_{n+k}}{\nu_{nk}}\Big) - \\ & - \frac{1}{2}K\Big((3\lambda_n-\lambda_{n+k})a, \frac{3\lambda_n-\lambda_{n+k}}{\nu_{nk}}\Big) - \exp(-\nu_{nk}(b-a))\bigg(K\Big((\lambda_{n+k}-\lambda_n)b, \frac{\lambda_{n+k}-\lambda_n}{\nu_{nk}}\Big) - \\ & - \frac{1}{2}K\Big((\lambda_n+\lambda_{n+k})b, \frac{\lambda_n+\lambda_{n+k}}{\nu_{nk}}\Big) - \frac{1}{2}K\Big((3\lambda_n-\lambda_{n+k})b, \frac{3\lambda_n-\lambda_{n+k}}{\nu_{nk}}\Big)\bigg)\bigg] \end{aligned} \tag{14}$$

and the twin formulas for I_2, I_3:

$$I_{2,3} = \frac{\exp(-\nu_{nk}a)}{8\nu_{nk}}\Bigg[K\Big((2\lambda_n - \gamma_{nk}^{\pm})a, \frac{2\lambda_n - \gamma_{nk}^{\pm}}{\nu_{nk}}\Big) - K\Big((2\lambda_n + \gamma_{nk}^{\pm})a, \frac{2\lambda_n + \gamma_{nk}^{\pm}}{\nu_{nk}}\Big) - \\ - \exp(-\nu_{nk}(b-a))\Bigg(K\Big((2\lambda_n - \gamma_{nk}^{\pm})b, \frac{2\lambda_n - \gamma_{nk}^{\pm}}{\nu_{nk}}\Big) - K\Big((2\lambda_n + \gamma_{nk}^{\pm})b, \frac{2\lambda_n + \gamma_{nk}^{\pm}}{\nu_{nk}}\Big)\Bigg)\Bigg] \tag{15}$$

where $\gamma_{nk}^{\pm} = \lambda_{n+k} - \lambda_n \pm \alpha_1(\mu_{n+k} - \mu_n)$.

By (8) and (11), the relations (14), (15) yield the estimates

$$I_1 = \frac{\exp(-\alpha_2(\mu_n + \mu_{n+k})a)}{4\alpha_2\mu_n}\Big(A_n + [1](1 - \cos ka) + [1]\sin ka + [4]\sin(ka/2) + o(1)\Big), \tag{16}$$

$$I_{2,3} = \frac{\exp(-\alpha_2(\mu_n + \mu_{n+k})a)}{4\alpha_2\mu_n}\Big([2]\sin((1 \pm \alpha_1)ka/2) + o(1)\Big). \tag{17}$$

Applying (9), (12), (16) and (17), we rewrite (5) in the form

$$\left\|\frac{\varphi_n}{\|\varphi_n\|_2} - \frac{\varphi_{n+k}}{\|\varphi_{n+k}\|_2}\right\|_2^2 = 2 - 2\frac{A_n + \Delta_1(k) + o(1)}{\sqrt{A_n + o(1)}\sqrt{A_n + \Delta_2(k) + o(1)}} \tag{18}$$

where

$$\begin{gathered} |\Delta_1(k)| \le 8|\sin(ka/2)| + 2|\sin((1+\alpha_1)ka/2)| + 2|\sin((1-\alpha_1)ka/2)|, \\ |\Delta_2(k)| \le 8|\sin(ka/2)|. \end{gathered} \tag{19}$$

Now let $\delta > 0$ be any small number. By the theorem on approximation of real numbers by rational fractions, we choose $k \in \mathbb{N}$ to satisfy three inequalities

$$|\sin(ka/2)| \le \delta, \quad |\sin((1+\alpha_1)ka/2)| \le \delta, \quad |\sin((1-\alpha_1)ka/2)| \le \delta. \tag{20}$$

In fact, the system (20) is equivalent to the inequalities

$$\Big|\frac{a}{4\pi} - \frac{N_1}{k}\Big| \le \frac{\arcsin\delta}{2\pi k}, \quad \Big|\frac{(1 \pm \alpha_1)a}{4\pi} - \frac{N_{2,3}}{k}\Big| \le \frac{\arcsin\delta}{2\pi k}$$

with some integers N_1, N_2, N_3 while it is known [11, p.31] that the set

$$\Big|\frac{a}{4\pi} - \frac{N_1}{k}\Big| \le k^{-4/3}, \quad \Big|\frac{(1 \pm \alpha_1)a}{4\pi} - \frac{N_{2,3}}{k}\Big| \le k^{-4/3}$$

has infinitely many integer solutions with respect to N_1, N_2, N_3 and k.

Let us fix the chosen integer k and take $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, all the quantities $o(1)$ in the right-hand side of (18) satisfy the relation

$$|o(1)| \le \delta. \tag{21}$$

Since δ in (20) and (21) is arbitrarily small, (8) and (19) imply that the right-hand side of (5) can be made less than any given positive ε. Lemma is proved.

In conclusion, we remark that *the statement of the main theorem holds true also in the case when the sequences* $\{\mu_n\}$ *and* $\{\mu_n\}$ *are complex and satisfy the following assumptions:*

a) the value of $\lim\limits_{n\to\infty}(\lambda_{n+1}-\lambda_n)=L_1$ *is a non-zero real number;*

b) the value of $\lim\limits_{n\to\infty}(\mu_{n+1}-\mu_n)=L_2$ *is a non-zero complex number;*

c) $\varliminf\limits_{n\to\infty}|\lambda_n/\mu_n|>0$ *and* $\mathrm{Im}\,(\alpha L_2)\neq 0$.

The proof of this fact mimics the above reasoning with minor changes.

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