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Fixed Point Theorem in Partially Ordered Metric Spaces for Generalized Contraction Mappings

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Abstract. In this paper, we prove some fixed point results in the setting of two metric spaces endowed with a partial order satisfying a generalized contractive condition. The proved results generalize and extend some known results in the literature. As application, we establish an existence result for a nonlinear first order differential equation.

Key Words and Phrases: fixed point, generalized contraction mappings, ordered metric spaces.

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1. Introduction and Preliminaries

The Banach Contraction Mapping Principle is one of the pivotal results of analysis. It is a very popular tool for solving existence and uniqueness problems in different fields of mathematics. Due to its importance and applications potential, the Banach Contraction Mapping Principle has been investigated heavily by many authors. Consequently, a number of generalizations of this celebrated principle have appeared in the literature (see [1, 2, 3, 4, 5, 7, 6, 9, 8, 10, 11, 12] and references cited therein).

Ran and Reurings [11] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodŕiguez-López [10] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [2] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first order differential equation with periodic boundary conditions. Recently, many researchers have obtained fixed point, common fixed point results in metric spaces endowed with a partial order. The purpose of this paper is to establish some fixed point results in the setting of two metric spaces endowed with a partial order satisfying a generalized contractive condition. Also, an application to the study of the existence of solution to a nonlinear first order differential equation has been given.

In [12], Singh proved the following fixed point theorem.

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Theorem 1. *Let X be a metric space with metrics d and δ such that $d(x,y) \leq \delta(x,y)$ for each pair $x,y \in X$. If X is complete with respect to d, $T : X \to X$ is a function in (X,d) and T satisfies the following contractive condition:*

$$\delta(Tx,Ty) \quad \leq \quad \alpha\left(\delta(x,Ty)+\delta(y,Tx)\right) \tag{1}$$

for all $x,y \in X$, and for some $\alpha \in (0,\frac{1}{2})$, then T has a unique fixed point in X.

The aim of this paper is to give a version of Theorem 1 in a metric space endowed with a partial order.

2. Main Results

Definition 1. *Suppose $(X,\leq)$ is a partially ordered set and $T : X \to X$. T is said to be monotone nondecreasing if for all $x,y \in X$,*

$$x \leq y \text{ implies } Tx \leq Ty. \tag{2}$$

Theorem 2. *Let $(X,\leq)$ be a partially ordered set and suppose that there exist metrics d and δ on X such that (X,d) is a complete metric space and $d(x,y) \leq \delta(x,y)$. Suppose that T is a continuous self-mapping on X, T is monotone nondecreasing mapping and*

$$\delta(Tx,Ty) \quad \leq \quad \alpha\left(\delta(x,Tx)+\delta(y,Ty)\right) \tag{3}$$

for all $x,y \in X$, $x \geq y$ and for some $\alpha \in (0,\frac{1}{2})$.

If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. If $Tx_0 = x_0$, then we have the result. Suppose that $x_0 < Tx_0$. Since T is a monotone nondecreasing mapping, we obtain by induction that

$$x_0 < Tx_0 \leq T^2x_0 \leq \ldots \leq T^nx_0 \leq T^{n+1}x_0 \leq \ldots \tag{4}$$

By induction, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$, for every $n \geq 0$.

Since T is monotone nondecreasing mapping, we obtain

$$x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq x_{n+1} \leq \ldots.$$

If there exists $n \geq 1$ such that $x_{n+1} = x_n$, then from $x_{n+1} = Tx_n = x_n$ it follows that x_n is a fixed point and the proof is finished. Suppose that $x_{n+1} \neq x_n$ for all $n \geq 1$.

Since $x_n > x_{n-1}$, for all $n \geq 1$, from (3) we have

$$\begin{aligned}
\delta(x_{n+2},x_{n+1}) &= \delta(Tx_{n+1},Tx_n) \\
&\leq \alpha\left(\delta(x_{n+1},Tx_{n+1})+\delta(x_n,Tx_n)\right) \\
&= \alpha\left(\delta(x_{n+1},x_{n+2})+\delta(x_n,x_{n+1})\right) \qquad (5)
\end{aligned}$$

which implies that

$$\delta(x_{n+2}, x_{n+1}) \quad \leq \quad \frac{\alpha}{1-\alpha}\delta(x_{n+1}, x_n). \tag{6}$$

Using mathematical induction, we have

$$\delta(x_{n+2}, x_{n+1}) \quad \leq \quad \left(\frac{\alpha}{1-\alpha}\right)^{n+1} \delta(x_1, x_0). \tag{7}$$

Put $k = \frac{\alpha}{1-\alpha} < 1$. Now we shall prove that $\{x_n\}$ is a Cauchy sequence. For $m \geq n$, we have

$$\begin{aligned} \delta(x_m, x_n) &\leq \delta(x_m, x_{m-1}) + \delta(x_{m-1}, x_{m-2}) + \ldots + \delta(x_{n+1}, x_n) \\ &\leq \left(k^{m-1} + k^{m-2} + \ldots + k^n\right) \delta(x_1, x_0) \\ &\leq \left(\frac{k^n}{1-k}\right) \delta(x_1, x_0), \end{aligned} \tag{8}$$

which implies that $\delta(x_m, x_n) \to 0$, as $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence with respect to δ, which further implies that it is a Cauchy sequence with respect to d for $d(x, y) \leq \delta(x, y)$, for all $x, y \in X$. Since (X, d) is a complete metric space, there exists $u \in X$ such that $\lim x_n = u$.

By the continuity of T, we have $Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u$. Hence u is a fixed point of T.◄

In what follows, we prove that Theorem 2 is still valid for T, not necessarily continuous, assuming the following hypothesis in X:

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to z$, then $x_n \leq z$.

Theorem 3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exist metrics d and δ on X and $d(x, y) \leq \delta(x, y)$. Suppose that T is a self-mapping on X, T is a monotone nondecreasing mapping and*

$$\delta(Tx, Ty) \quad \leq \quad \alpha\left(\delta(x, Tx) + \delta(y, Ty)\right) \tag{9}$$

for all $x, y \in X$, $x \geq y$ and for some $\alpha \in (0, \frac{1}{2})$.

Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to z$, then $x_n \leq z$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. Following the proof of Theorem 2, we have $\delta(T^n x, T^{n+1} x) \leq \left(\frac{\alpha}{1-\alpha}\right)^n \delta(x_1, x_0)$. Thus for $0 < \alpha < \frac{1}{2}$, $\delta(T^n x, T^{n+1} x) \to 0$.

Since $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to u$, then $x_n \leq u$ for all $n \in \mathbb{N}$.

Since T is a monotone nondecreasing mapping with $Tx_n \leq Tu$, for all $n \in \mathbb{N}$ or, equivalently, $x_{n+1} \leq Tu$, for all $n \in \mathbb{N}$. Consider

$$\delta(Tu, x_{n+1}) \quad = \quad \delta(Tu, Tx_n)$$

$$
\begin{aligned}
&\leq \alpha\left(\delta(u,Tu)+\delta(x_n,Tx_n)\right)\\
&= \alpha\left(\delta(u,Tu)+\delta(x_n,x_{n+1})\right).
\end{aligned}
$$

Letting $n\to\infty$, we have $\delta(Tu,u)\leq\alpha\delta(Tu,u)$, which implies that $Tu=u$. Hence u is a fixed point of T.◄

Now, we shall prove the uniqueness of the fixed point.

Theorem 4. *In addition to the hypotheses of Theorem 2 (or Theorem 3), suppose that for every $x,y\in X$, there exists $z\in X$ which is comparable to x and y. Then T has a unique fixed point.*

Proof. From Theorem 2 (or Theorem 3), the set of fixed points of T is non-empty. Suppose that $x,y\in X$ are two fixed points of T. We distinguish two cases:

Case 1. If x and y are comparable and $x\neq y$, then using (3), we have

$$
\begin{aligned}
\delta(x,y) &= \delta(Tx,Ty)\\
&\leq \alpha(\delta(x,Tx)+\delta(y,Ty)),
\end{aligned}
$$

which implies that $d(x,y)=0$. Hence $x=y$.

Case 2. If x is not comparable to y, there exists $z\in X$ that is comparable to x and y. Monotonicity implies that that T^nz is comparable to $T^nx=x$ and $T^ny=y$ for $n=0,1,2,\ldots$. If there exists $n_0\geq 1$ such that $T^{n_0}z=x$, then as x is a fixed point, the sequence $\{T^nz: n\geq n_0\}$ is constant, and, consequently, $lim_{n\to\infty}T^nz=x$. On the other hand, if $T^nz\neq x$ for $n\geq 1$, using the contractive condition, we obtain, for $n\geq 2$,

$$
\begin{aligned}
\delta(T^nz,x) &= \delta(T^nz,T^nx)\\
&\leq \alpha(\delta(T^{n-1}x,T^nx)+\delta(T^{n-1}z,T^nz))\\
&= \alpha(\delta(x,x)+\delta(T^{n-1}z,T^nz))\\
&\leq \alpha(\delta(x,T^nz)+\delta(x,T^{n-1}z)),
\end{aligned}
$$

which implies that $\delta(T^nz,x)\leq\frac{\alpha}{1-\alpha}\delta(T^{n-1}z,x)$. Using mathematical induction, we have $\delta(T^nz,x)\leq\left(\frac{\alpha}{1-\alpha}\right)^n\delta(z,x)$, for $n\geq 2$, and as $\frac{\alpha}{1-\alpha}<1$, we have $\lim_{n\to\infty}T^nz=x$.

Using a similar argument, we can prove that $\lim_{n\to\infty}T^nz=y$. Now, the uniqueness of the limit implies $x=y$. Hence T has a unique fixed point.◄

Corollary 1. *Let $(X,\leq)$ be a partially ordered set and suppose that there exist metrics d and δ on X such that (X,d) is a complete metric space and $d(x,y)\leq\delta(x,y)$. Suppose that the mapping $T:X\to X$ satisfies*

$$\delta(Tx,Ty)\leq\alpha\left(\delta(x,Tx)+\delta(y,Ty)\right)$$

for all $x,y\in X$, $x\geq y$ and for some $\alpha\in(0,\frac{1}{2})$. Suppose that

$$Tx\leq T(Tx),\quad \textit{for all}\quad x\in X.$$

Also suppose that either

(a) if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$, then $x_n \preceq z$ for every n; or

(b) T is continuous.

Then T has a fixed point.

2.1. Application to nonlinear first order ordinary differential equation

Consider the nonlinear differential equation

$$\begin{cases} x'(t) & = & f(t, x(t)), \quad t \in I, \\ x(t_0) & = & x_0, \end{cases} \tag{10}$$

where $t_0 \in \mathbb{R}$, $I = [t_0, t_0 + a]$, $a > 0$, and $f : I \times \mathbb{R} \to \mathbb{R}$.

Let $X = C(I, \mathbb{R})$ denote the space of all continuous $\mathbb{R}$-valued functions on I. We endow this space with the metrics d and δ given by

$$d(u, v) = \frac{1}{2} \sup_{t \in I} |u(t) - v(t)|, \quad \text{for all} \quad u, v \in X.$$

$$\delta(u, v) = \sup_{t \in I} |u(t) - v(t)|, \quad \text{for all} \quad u, v \in X.$$

It is well known that (X, d) is a complete metric space. We define an order relation $\leq$ on X by

$$u, v \in X, \quad u \leq v \Longleftrightarrow u(t) \leq v(t), \text{ for all } t \in I.$$

We consider the following assumptions:

(H1) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous;

(H2) for all $t \in I$, $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is nondecreasing;

(H3) we have
$$f(t, z) \leq f(t, f(t, z)), \quad \text{for all} \quad t \in I, \ z \in \mathbb{R};$$

(H4) for all $t \in I$, for all $u \in C(I, \mathbb{R})$,
$$f(t, u(t)) \leq x_0 + \int_{t_0}^{t} f(\tau, x(\tau)) \, d\tau;$$

(H5) there exists $k \in [0, 1/2)$ such that for all $u, v \in C(I, \mathbb{R})$ with $u \leq v$, we have
$$\int_{t_0}^{t} [f(s, v(s)) - f(s, u(s))] \, ds \leq$$
$$k \left(\left| v(t) - x_0 - \int_{t_0}^{t} f(s, v(s)) \, ds \right| + \left| u(t) - x_0 - \int_{t_0}^{t} f(s, u(s)) \, ds \right| \right)$$
for all $t \in I$.

We have the following result.

Theorem 5. *Suppose that (H1)-(H5) hold. Then (10) has at least one solution $x^* \in C(I,\mathbb{R})$.*

Proof. Consider the mapping $T : C(I,\mathbb{R}) \to C(I,\mathbb{R})$ defined by

$$Tu(t) = x_0 + \int_{t_0}^{t} f(s,u(s))\, ds, \quad t \in I,$$

for all $u \in C(I,\mathbb{R})$. Clearly, $x^* \in C(I,\mathbb{R})$ is a solution of (10) if and only if x^* is a fixed point of T.

Let $x, y \in C(I,\mathbb{R})$ be such that $x \preceq y$. From (H5), we have

$$\int_{t_0}^{t} [f(s,y(s)) - f(s,x(s))]\, ds$$

$$\begin{aligned}
&\leq k\left(\left| y(t) - x_0 - \textstyle\int_{t_0}^{t} f(s,y(s))\, ds \right| + \left| x(t) - x_0 - \textstyle\int_{t_0}^{t} f(s,x(s))\, ds \right| \right) \\
&\leq k(|y(t) - Ty(t)| + |x(t) - Tx(t)|) \\
&\leq k(\delta(x,Tx) + \delta(y,Ty)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \left| \int_{t_0}^{t} [f(s,x(s)) - f(s,y(s))]\, ds \right| \\
&\leq \int_{t_0}^{t} |f(s,x(s)) - f(s,y(s))|\, ds \\
\text{(from (H2))} &= \int_{t_0}^{t} [f(s,y(s)) - f(s,x(s))]\, ds.
\end{aligned}$$

Then we have

$$|Tx(t) - Ty(t)| \leq k(\delta(x,Tx) + \delta(y,Ty)), \quad \text{for all} \quad t \in I.$$

This implies that

$$\delta(Tx,Ty) \leq k(\delta(x,Tx) + \delta(y,Ty)).$$

Let $x \in C(I,\mathbb{R})$. For all $t \in I$, we have

$$\begin{aligned}
Tx(t) &= x_0 + \int_{t_0}^{t} f(s,x(s))\, ds \\
\text{(from (H3))} &\leq x_0 + \int_{t_0}^{t} f(s,f(s,x(s)))\, ds
\end{aligned}$$

$$
\begin{aligned}
\text{(from (H2) and (H4))} \quad &\leq \quad x_0 + \int_{t_0}^{t} f\left(s, x_0 + \int_{t_0}^{s} f(\tau, x(\tau))\, d\tau\right)\, ds \\
&= \quad x_0 + \int_{t_0}^{t} f(s, Tx(s))\, ds \\
&= \quad T(Tx(t)).
\end{aligned}
$$

Thus we have

$$Tx \leq T(Tx), \quad \text{for all} \quad x \in C(I, \mathbb{R}).$$

Also, it is proved in [10] that if $\{x_n\} \subset C(I, \mathbb{R})$ is a nondecreasing sequence with $x_n \to z$, then $x_n \preceq z$ for every n.

Now, applying Corollary 1, we obtain that there exists $x^* \in C(I, \mathbb{R})$, a fixed point of T. ◄

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