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A Characterization of S-Essential Spectrum by Means of Measure of Non-Strict-Singularity and Application

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Abstract. In the present paper, we investigate the S-essential spectrum of a closed densely defined linear operator. Our approach consists principally in considering the notion of measure of non-strict-singularity. Furthermore, we apply the results to study the S-essential spectrum of 2×2 matrix operator acting on a Banach space.

Key Words and Phrases: S-essential spectrum, measure of non-strict-singularity, matrix operator.

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1. Introduction

Let X and Y be two infinite-dimensional Banach spaces. By an operator A from X to Y we mean a linear operator with domain $\mathcal{D}(A) \subset X$ and range $R(A) \subset Y$. We denote by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$) the set of all closed, densely defined linear operators (resp. the Banach algebra of all bounded linear operators) from X into Y and we denote by $\mathcal{K}(X,Y)$ the subspace of all compact operators from X into Y. We denote by $\sigma(A)$ and $\rho(A)$, respectively, the spectrum and the resolvent set of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y.

Let A and S be two operators on X such that S is nonzero and bounded and A is closed. We define the S-resolvent set by:

$$\rho_S(A) := \big\{\lambda \in \mathbb{C} \text{ such that } \lambda S - A \text{ has a bounded inverse}\big\}.$$

The S-spectrum of an operator A acting on a Banach space X is usually defined as

$$\sigma_S(A) := \mathbb{C}\backslash\ \rho_S(A).$$

Subsequently, the operator S should be taken as non invertible. Because otherwise the S-resolvent coincides with usual resolvent of the operator $S^{-1}A$, and this analysis is meaningless. If $\rho_S(A)$ is not empty, then A is closed. Indeed, let $x_n \in \mathcal{D}(A)$ be

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such that $x_n \longrightarrow x$ and $Ax_n \longrightarrow y$. Since $\rho_S(A) \neq \emptyset$, then there exists $\lambda_0 \in \rho_S(A)$ such that $(A-\lambda_0 S)^{-1} \in \mathcal{L}(X)$. As $S \in \mathcal{L}(X)$, then $(A-\lambda_0 S)x_n \longrightarrow y - \lambda_0 Sx$, thus $x_n \longrightarrow (A-\lambda_0 S)^{-1}(y-\lambda_0 Sx) = x$. We deduce that $Ax = y$ and $x \in \mathcal{D}(A)$.

Now, we introduce the following important operator classes: The set of upper semi-Fredholm operators is defined by

$$\Phi_+(X,Y) = \{A \in \mathcal{C}(X,Y) \text{ such that } \alpha(A) < \infty \text{ and } R(A) \text{ is closed in } Y\},$$

and the set of lower semi-Fredholm operators is defined by

$$\Phi_-(X,Y) = \{A \in \mathcal{C}(X,Y) \text{ such that } \beta(A) < \infty \text{ and } R(A) \text{ is closed in } Y\}.$$

The set of Fredholm operators from X into Y is defined by

$$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

The set of bounded upper (resp. lower) semi-Fredholm operators from X into Y is defined by

$$\Phi_+^b(X,Y) = \Phi_+(X,Y) \cap \mathcal{L}(X,Y) \quad (\text{resp. } \Phi_-(X,Y) \cap \mathcal{L}(X,Y)).$$

We denote by $\Phi^b(X,Y) = \Phi(X,Y) \cap \mathcal{L}(X,Y)$ the set of bounded Fredholm operators from X into Y. If A is a semi-Fredholm operator (either upper or lower), the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. It is clear that if $A \in \Phi(X,Y)$, then $i(A) < \infty$. If $A \in \Phi_+(X,Y) \setminus \Phi(X,Y)$, then $i(A) = -\infty$ and if $A \in \Phi_-(X,Y) \setminus \Phi(X,Y)$, then $i(A) = +\infty$. A complex number λ is in $\Phi_{+A,S}$, $\Phi_{-A,S}$ or $\Phi_{A,S}$ if $\lambda S - A$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$, respectively.

Lemma 1. (i) ([21, Lemma 3.1]) *Let L and M be densely defined operators on X. If M and LM are Fredholm operators, then the same is true of L.*
(ii) ([25, Theorem 3.8]) *Let X, Y, Z be Banach spaces and suppose $B \in \Phi^b(Y,Z)$. Assume that A is a closed, densely defined linear operator from X to Y such that $BA \in \Phi(X,Z)$. Then $A \in \Phi(X,Y)$.*
(iii) ([25, Theorem 3.1]) *If $A \in \Phi(X,Y)$ and $B \in \Phi(Y,Z)$, then $BA \in \Phi(X,Z)$ and $i(AB) = i(A) + i(B)$.*
(iv) ([25, Theorem 2.3]) *If A is a one-to-one closed linear operator from X to Y, then $R(A)$ is closed in Y if and only if A^{-1} is bounded linear operator from Y to X.*

There are several and in general non-equivalent definitions of the essential spectrum of a bounded linear operator on a Banach space. For a self-adjoint operator in a Hilbert space, there seems to be only one reasonable way to define the essential spectrum: The set of all points of the spectrum that are not isolated eigenvalues of finite algebraic multiplicity. Numerous mathematical and physical problems lead to operator pencils, $\lambda S - A$ (operator-valued functions of a complex argument) (see, for example, [15, 26]). Since recently, the spectral theory of operator pencils attracts an attention of many mathematicians. If X is a Banach space and $A \in \mathcal{C}(X)$, $S \in \mathcal{L}(X)$, various notions of essential spectrum appear in application of spectral theory.

In this work, we are concerned with the following essential spectrum.

Definition 1. [1] *Let $A \in \mathcal{C}(X)$, $S \in \mathcal{L}(X)$. We define the S-essential spectrum of A by*

$$\sigma_{e,S}(A) := \bigcap_{K \in \mathcal{K}(X)} \sigma_S(A+K).$$

Note that, if $S = I$, we recover the usual definition of the essential spectra of a bounded linear operator A. The subset $\sigma_{e,I}(A)$ is the Schechter essential spectrum (see [9, 10, 21, 22, 23]). We mention that the modern name of the Schechter essential spectrum is the Weyl essential spectrum (see [2, 3]).

Lemma 2. [8, Lemma 2.1 (i)] *Let $A \in \mathcal{C}(X)$ and $S \in \mathcal{L}(X)$. If $\Phi_{A,S}$ is connected and $\rho_S(A)$ is not empty, then*

$$\sigma_{e,S}(A) = \left\{\lambda \in \mathbb{C} \text{ such that } A - \lambda S \notin \Phi(X)\right\} = \mathbb{C} \setminus \Phi_{A,S}.$$

Definition 2. *[5] Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$.*
F is called strictly singular, if for every infinite-dimensional closed subspace $\mathcal{M}$ of X, the restriction of F to $\mathcal{M}$ is not an homeomorphism.

Let $S(X,Y)$ denote the set of strictly singular operators from X into Y. If $X = Y$, the set of strictly singular operators on X will be denoted by $S(X)$.

The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [11] as a generalization of the notion of compact operators. For a detailed study of the properties of strictly singular operators, we refer to [6, 11]. Note that $S(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space, then $S(X) = \mathcal{K}(X)$.

Definition 3. *[18] Let X be a Banach space. For a bounded subset Ω of X we consider*

$$q(\Omega) := \inf\Big\{r > 0,\ \Omega \text{ can be covered by a finite set of open balls of radius } r\Big\}.$$

The Hausdorff measure of noncompactness of $A \in \mathcal{L}(X,Y)$ is defined by

$$q(A) = q[A(B_X)],$$

where B_X denotes the closed unit ball in X.

Definition 4. *[13] For $A \in \mathcal{L}(X,Y)$, set*

$$g_M(A) = \inf_{N \subset M} q(A_{|N}) \quad \text{and} \quad g(A) = \sup_{M \subset X} g_M(A),$$

where M, N represent infinite dimensional closed subspaces of X and $A_{|N}$ denotes the restriction of A to the subspace N.

The semi-norm g is a measure of non-strict-singularity, it was introduced by Schechter in [20]. We recall the following result established in [17].

Proposition 1. [17] *For $A \in \mathcal{L}(X, Y)$,*

(i) $A \in \mathcal{S}(X, Y)$ if and only if $g(A) = 0$.

(ii) $A \in \mathcal{S}(X, Y)$ if and only if $g(A + B) = g(B)$ for all $B \in \mathcal{L}(X, Y)$.

(iii) if Z is a Banach space and $B \in \mathcal{L}(Y, Z)$, then $g(BA) \leq g(B)g(A)$.

Proposition 2. [16, Proposition 2.3] *Let $A \in \mathcal{L}(X)$. If $g(A^n) < 1$ for some integer $n \geq 1$, then $I - A \in \Phi^b(X)$ with $i(I - A) = 0$.*

Definition 5. [21] *Let A and B be densely defined operators in a Banach space X with $\mathcal{D}(A) \subseteq \mathcal{D}(B)$.*
(i) The operator B is called A-bounded if

$$\|Bx\| \leq c\big(\|x\| + \|Ax\|\big) \text{ for all } x \in \mathcal{D}(A). \tag{1}$$

(ii) The operator B is called A-compact if for any sequence $x_n \in \mathcal{D}(A)$ satisfying

$$\|x_n\| + \|Ax_n\| \leq c, \tag{2}$$

the sequence Bx_n has a convergent subsequence.

Clearly, a compact operator is always A-compact, and an A-compact operator is always A-bounded. If A and B are closed, then B is A-bounded.

Definition 6. [21, Definition 2.2] *The operator B will be called A-pseudo-compact if*

$$\|x_n\| + \|Ax_n\| + \|Bx_n\| \leq c \tag{3}$$

for all $x_n \in \mathcal{D}(A)$ implies that Bx_n has a convergent subsequence.

Definition 7. [21, Definition 2.1] *The operator B will be called A-closed if $x_n \to x$, $Ax_n \to y$, $Bx_n \to z$ for $x_n \in \mathcal{D}(A)$ implies that $x \in \mathcal{D}(B)$ and $Bx = z$. It will be called A-closable if $x_n \to 0$, $Ax_n \to 0$, $Bx_n \to z$ implies $z = 0$.*

One of the central questions in the study of the S-essential spectra of closed densely defined linear operators consists in showing when different notions of essential spectrum coincide and we study the invariance by some class of perturbations. The purpose of this work is to generalize the notion of essential spectra and to extend many known results in the literature.

In the first part of this work we extend the analysis of [21] to closed linear operator $g(K^n) < 1$, where $g(.)$ is a measure of non-strict-singularity. More precisely, assume that $\lambda \in \rho_S(A) \cap \rho_S(A + B)$. If $\|x_n\| + \|Ax_n\| + \|Bx_n\| \leq c$, $\quad c \geq 0$, for all $x_n \in \mathcal{D}(A)$ implies that $(A - \lambda S)^{-1}Bx_n$ has a convergent subsequence, then $\sigma_{e,S}(A + B) = \sigma_{e,S}(A)$.

In the second part of this article we give a new characterization of the Schechter essential spectrum of closed densely defined linear operators. In fact, let $A \in \mathcal{C}(X)$, $S \in \mathcal{L}(X)$. Then $\sigma_{e,S}(A) = \sigma_1(A)$ (resp. $\sigma_{e,S}(A) = \sigma_2(A)$), where

$$\sigma_1(A) = \bigcap_{K \in \mathcal{S}^1_{A,S}(X)} \sigma_S(A+K),$$

$$\mathcal{S}^1_{A,S}(X) =$$

$$= \Big\{K \in \mathcal{L}(X) : g\Big([(\lambda S - A - K)^{-1}K]^n\Big) < 1 \text{ for some } n \in \mathbb{N} \text{ and for all } \lambda \in \rho_S(A+K)\Big\}$$

and

$$\sigma_2(A) = \bigcap_{K \in \mathcal{S}^2_{A,S}(X)} \sigma_S(A+K),$$

$$\mathcal{S}^2_{A,S}(X) =$$

$$= \Big\{K \in \mathcal{L}(X) : g\Big([K(\lambda S - A - K)^{-1}]^n\Big) < 1 \text{ for some } n \in \mathbb{N} \text{ and for all } \lambda \in \rho_S(A+K)\Big\}.$$

Finally, we generalize the results of N. Moalla in [16] where S-essential spectra of some 2×2 operator matrices on $X \times X$ are discussed with $M = I$.

We organize the paper in the following way: In the second section we study the stability of S-essential spectra of closed linear operators. Section 3 is dedicated to a new characterization of the S-essential spectrum of closed densely defined linear operators. Finally, in Section 4 we apply the obtained results to give a generalization of many known results on the S-essential spectrum of a 2×2 matrix operators by means of the measure of non-strict-singularity.

2. Invariance of the S-essential spectrum

The following result gives a characterization of the S-essential spectrum by means of Fredholm operators.

Proposition 3. *Let $S \in \mathcal{L}(X)$ and $A \in \mathcal{C}(X)$. Then*

$$\lambda \notin \sigma_{e,S}(A) \ \text{ if and only if } \ A - \lambda S \in \Phi(X) \text{ and } i(A - \lambda S) = 0.$$

Proof. Let $\lambda \notin \sigma_{e,S}(A)$. Then, there exists a compact operator K on X such that $\lambda \in \rho_S(A+K)$. Then

$$A + K - \lambda S \in \Phi(X) \ \text{ and } i(A + K - \lambda S) = 0.$$

Now, the operator $A - \lambda S$ can be written in the form

$$A - \lambda S = A + K - \lambda S - K.$$

By [25, Theorem 3.1] we have

$$A - \lambda S \in \Phi(X) \;\text{ and } i(A - \lambda S) = 0.$$

Conversely, we suppose that $(A - \lambda S) \in \Phi(X)$ and $i(A - \lambda S) = 0$.
Let $n = \alpha(A - \lambda S) = \beta(A - \lambda S)$, $\{x_1, ..., x_n\}$ be a basis for $N((A - \lambda S))$ and $\{y_1^{'}, ..., y_n^{'}\}$ be a basis for annihilator $R(A-\lambda S)^{\perp}$. By [25, Theorems 1.2.5, 1.2.6] there are functionals $x_1^{'}, ..., x_n^{'}$ in $X^{'}$ (the adjoint space of X) and elements $y_1, ..., y_n$ such that

$$x_j^{'}(x_k) = \delta_{jk} \;\text{ and } y_j^{\prime}(y_k) = \delta_{jk}, \quad 1 \leq j,\ k \leq n,$$

where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jk} = 1$ if $j = k$. The operator K is defined by

$$K : X \ni x \longrightarrow Kx := \sum_{i=1}^{n} x_i^{\prime}(x) y_i \in X.$$

Clearly K is a linear operator defined everywhere on X. It is bounded, since

$$\|Kx\| \leq \Big(\sum_{k=1}^{n} \|x_k^{'}\| \|y_k\| \Big) \|x\|.$$

Moreover the range of K is contained in a finite dimensional subspace of X. Then K is a finite rank operator in X (see [25, Lemma 1.3]). By [25, Lemma 2.7], K is a compact operator in X.
We prove that

$$N(A - \lambda S) \cap N(K) = \{0\} \;\text{ and } R(A - \lambda S) \cap R(K) = \{0\}. \tag{4}$$

Let $x \in N(A - \lambda S)$. Then

$$x = \sum_{k=1}^{n} \alpha_k x_k,$$

therefore $x_j^{\prime}(x) = \alpha_j,\ 1 \leq j \leq n$. On the other hand, if $x \in N(K)$, then $x_j^{'}(x) = 0,\ 1 \leq j \leq n$. This proves the first relation in Eq. (4). The proof of the second inclusion is similar.
In fact, if $y \in R(K)$, then

$$y = \sum_{k=1}^{n} \alpha_k y_k,$$

and hence,

$$y_j^{'}(y) = \alpha_j,\ 1 \leq j \leq n.$$

But, if $y \in R(A - \lambda S)$, then,

$$y_j^{'}(y) = 0,\ 1 \leq j \leq n.$$

This gives the second relation in Eq. (4). On the other hand, K is a compact operator. We deduce from [25, Theorem 3.1] that $\lambda S - A \in \Phi(X)$ and $i(A - \lambda S + K) = 0$. If

$x \in N(A-\lambda S+K)$, then $(A-\lambda S)x$ is in $R(A-\lambda S)\cap R(K)$. This implies that $x \in N(A-\lambda S)\cap N(K)$, hence $x=0$. Thus $\alpha(A-\lambda S+K)=0$. In the same way, one proves that $R(A-\lambda S+K)=X$. Using Lemma 1 (iv), we get $\lambda \in \rho_S(A+K)$. Also, $\lambda \notin \bigcap_{K\in\ \mathcal{K}(X)} \sigma_S(A+K)$. So, $\lambda \notin \sigma_{e,S}(A)$.◄

Remark 1. *The Proposition* 3 *generalizes the [1, Corollary 2.1 (i)] with* $S \in \mathcal{L}(X)$ *and* $A \in \mathcal{L}(X)$.

Theorem 1. *Let* $S \in \mathcal{L}(X)$ *and* $\lambda \in \rho_S(A)\cap\rho_S(A+B)$. *If*

$$\|x_n\| + \|Ax_n\| + \|Bx_n\| \leq c,$$

for all $x_n \in\subset \mathcal{D}(A)$ *implies that* $(A-\lambda S)^{-1}Bx_n$ *has a convergent subsequence, then*

$$\sigma_{e,S}(A+B) = \sigma_{e,S}(A). \tag{5}$$

Proof. We use the identities

$$(A+B-\mu S)-(A-\mu S)(A-\lambda S)^{-1}(A+B-\lambda S) = (\mu-\lambda)S(A-\lambda S)^{-1}B. \tag{6}$$

Since $\rho_S(A)$ and $\rho_S(A+B)$ are not empty, then A and $A+B$ are closed, hence $A+B$ is A-bounded. This shows that the hypotheses imply that $(A-\lambda S)^{-1}B$ is A-compact and hence $(A+B)$-compact. Let $\mu \notin \sigma_{e,S}(A+B)$. Then from Proposition 3 we get

$$A+B-\mu S \in \Phi(X) \text{ and } i(A+B-\mu S)=0.$$

By Eq. (6) we have

$$(A-\mu S)(A-\lambda S)^{-1}(A+B-\lambda S) \in \Phi(X)$$

and

$$i\Big((A-\mu S)(A-\lambda S)^{-1}(A+B-\lambda S)\Big)=0.$$

Since $\lambda \in \rho_S(A+B)$, then by Proposition 3 we have

$$A+B-\lambda S \in \Phi(X) \text{ and } i(A+B-\lambda S)=0.$$

Using Lemma 1 (i), we get

$$(A-\mu S)(A-\lambda S)^{-1} \in \Phi(X) \text{ and } i\Big((A-\mu S)(A-\lambda S)^{-1}\Big)=0.$$

From this and the identity $(A-\mu S)=(A-\mu S)(A-\lambda S)^{-1}(A-\lambda S)$, we obtain

$$A-\mu S \in \Phi(X) \text{ and } i(A-\mu S)=0.$$

Thus, $\mu \notin \sigma_{e,S}(A)$. Hence,

$$\sigma_{e,S}(A) \subset \sigma_{e,S}(A+B).$$

Conversely, if $\mu \notin \sigma_{e,S}(A)$, then

$$A - \mu S \in \Phi(X) \text{ and } i(A - \mu S) = 0.$$

Since $\lambda \in \rho_S(A+B)$, then by Proposition 3 we have

$$A + B - \lambda S \in \Phi(X) \text{ and } i(A + B - \lambda S) = 0.$$

Thus

$$(A - \mu S)(A - \lambda S)^{-1}(A + B - \lambda S) \in \Phi(X)$$

and

$$i\Big((A - \mu S)(A - \lambda S)^{-1}(A + B - \lambda S)\Big) = 0.$$

By Eq. (6) we have

$$A + B - \mu S \in \Phi(X) \text{ and } i(A + B - \mu S) = 0.$$

Then, $\mu \notin \sigma_{e,S}(A+B)$. Hence

$$\sigma_{e,S}(A+B) \subset \sigma_{e,S}(A). \qquad \blacktriangleleft$$

Remark 2. *If A and B are bounded operators, Theorem 1 remains true if we replace $(A-\lambda S)^{-1}B$ by $B(A-\lambda S)^{-1}$. Indeed, it suffices to replace, Eq. (6) and Lemma 1. (i) by*

$$(A + B - \mu S) - (A + B - \lambda S)(A - \lambda S)^{-1}(A - \mu S) = (\mu - \lambda)B(A - \lambda S)^{-1}S.$$

and Lemma 1 (ii), respectively.

3. A Characterisation of the S-Essential Spectrum

In this section we will give a fine description of the S-essential spectrum of a closed densely defined linear operator by means of the measure of non-strict-singularity.

Remark 3. *Let $A \in \mathcal{C}(X)$ and $S,\ K \in \mathcal{L}(X)$.*

(i) If $(\lambda S - A)^{-1}K \in \mathcal{S}(X)$ (resp. $K(\lambda S - A)^{-1} \in \mathcal{S}(X)$) for some $\lambda \in \rho_S(A)$, then $(\lambda S - A)^{-1}K \in \mathcal{S}(X)$ (resp. $K(\lambda S - A)^{-1} \in \mathcal{S}(X)$) for all $\lambda \in \rho_S(A)$ we have $(\lambda S - A)^{-1}K \in \mathcal{S}(X)$. Indeed, for all $\lambda,\ \mu \in \rho_S(A)$ we have

$$(\lambda S - A)^{-1}K - (\mu S - A)^{-1}K = (\mu - \lambda)(\mu S - A)^{-1}S(\lambda S - A)^{-1}K$$

$$\Big(\textit{resp. } K(\lambda S - A)^{-1} - K(\mu S - A)^{-1} = (\mu - \lambda)K(\mu S - A)^{-1}S(\lambda S - A)^{-1}\Big).$$

(ii) Now, if we consider the sets

$$\mathcal{H}_{A,S}(X) = \Big\{K \in \mathcal{L}(X) : (\lambda S - A)^{-1}K \in \mathcal{S}(X) \textit{ for some (hence for all) } \lambda \in \rho_S(A)\Big\},$$

and

$$\mathcal{F}_{A,S}(X) = \Big\{K \in \mathcal{L}(X) : K(\lambda S - A)^{-1} \in \mathcal{S}(X) \textit{ for some (hence for all) } \lambda \in \rho_S(A)\Big\},$$

then

$$\mathcal{H}_{A,S}(X) \subset \mathcal{S}^1_{A,S}(X) \textit{ and } \mathcal{F}_{A,S}(X) \subset \mathcal{S}^2_{A,S}(X).$$

Indeed, let $K \in \mathcal{H}_{A,S}(X)$. Then there exists $\lambda \in \rho_S(A)$ such that $(\lambda S - A)^{-1}K$ is strictly singular. For $\mu \in \rho_S(A+K)$, we have

$$(\mu S - A - K)^{-1}K = \Big[I + (\mu S - A - K)^{-1}((\lambda - \mu)S + K)\Big]\Big[(\lambda S - A)^{-1}K\Big].$$

By the ideal propriety of $S(X)$, we deduce that $(\mu S - A - K)^{-1}K$ is strictly singular. Then, $g(\mu S - A - K)^{-1}K) = 0$. Therefore $K \in \mathcal{S}^1_{A,S}(X)$. So, $\mathcal{H}_{A,S}(X) \subset \mathcal{S}^1_{A,S}(X)$.

A similar reasoning allows us to deduce that $\mathcal{F}_{A,S}(X) \subset \mathcal{S}^2_{A,S}(X)$.

We begin with the following theorem which gives a refinement of the definition of S-Schechter essential spectrum.

Theorem 2. *Let $A \in \mathcal{C}(X)$, $S \in \mathcal{L}(X)$. Then*

$$\sigma_{e,S}(A) = \sigma_1(A).$$

Proof. We first claim that $\sigma_{e,S}(A) \subset \sigma_1(A)$. Indeed, if $\lambda \notin \sigma_1(A)$, then there exists $K \in \mathcal{S}^1_{A,S}(X)$ such that $\lambda \notin \sigma_S(A+K)$. So, $g([(\lambda S - A - K)^{-1}K]^n) < 1$ for some $n \in \mathbb{N}$. Hence, by Proposition 2, we get

$$I + (\lambda S - A - K)^{-1}K \in \Phi^b(X) \text{ and } i\Big(I + (\lambda S - A - K)^{-1}K\Big) = 0.$$

Writing

$$\lambda S - A = \Big(\lambda S - A - K\Big)\Big[I + (\lambda S - A - K)^{-1}K\Big],$$

we can deduce that

$$\lambda S - A \in \Phi(X) \quad \text{and} \quad i(\lambda S - A) = 0.$$

This shows that $\lambda \notin \sigma_{e,S}(A)$. Conversely, since $\mathcal{K}(X) \subset \mathcal{S}^1_{A,S}(X)$, then

$$\sigma_1(A) \subset \sigma_{e,S}(A).$$

Hence,

$$\sigma_{e,S}(A) = \sigma_1(A). \qquad \blacktriangleleft$$

Theorem 3. *Let $A \in \mathcal{C}(X)$, $S \in \mathcal{L}(X)$. Then*

$$\sigma_{e,S}(A) = \sigma_2(A).$$

Proof. Since $\mathcal{K}(X) \subset \mathcal{S}^2_{A,S}(X)$, then $\sigma_2(A) \subset \sigma_{e,S}(A)$. Now we prove that $\sigma_{e,S}(A) \subset \sigma_2(A)$. Indeed, if $\lambda \notin \sigma_2(A)$, then there exists $K \in \mathcal{S}^2_{A,S}(X)$ such that $\lambda \notin \sigma_S(A+K)$. So, $g([K(\lambda S - A - K)^{-1}]^n) < 1$ for some $n \in \mathbb{N}$. Hence, by applying Proposition 2, we get

$$I + K(\lambda S - A - K)^{-1} \in \Phi^b(X) \text{ and } i(I + K(\lambda S - A - K)^{-1}) = 0.$$

Writing

$$\lambda S - A = [I + K(\lambda S - A - K)^{-1}](\lambda S - A - K)$$

we can deduce that

$$\lambda S - A \in \Phi(X) \quad \text{and} \quad i(\lambda S - A) = 0.$$

This shows that $\lambda \notin \sigma_{e,S}(A)$. Then

$$\sigma_{e,S}(A) \subset \sigma_2(A).$$

Hence,

$$\sigma_{e,S}(A) = \sigma_2(A).$$ ◄

Corollary 1. *Let $A \in \mathcal{C}(X)$ and $\mathcal{M}(X)$ be any subset of $\mathcal{L}(X)$ satisfying $\mathcal{K}(X) \subset \mathcal{M}(X) \subset \mathcal{S}^1_{A,S}(X)$ or $\mathcal{K}(X) \subset \mathcal{M}(X) \subset \mathcal{S}^2_{A,S}(X)$. Then*

$$\sigma_{e,S}(A) = \bigcap_{K \in \mathcal{M}(X)} \sigma_S(A+K).$$

4. The S-essential spectra of 2×2 matrix operator

During the last years, e.g. the papers [4, 12, 14, 19] were dedicated to the study of the $\mathcal{I}$- essential spectra of operators defined by a 2×2 block operator matrix

$$\mathcal{L}_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{7}$$

which act on the product $X \times Y$ of Banach spaces, where $\mathcal{I}$ is the identity operator defined on the product space $X \times Y$ by

$$\mathcal{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

An account of the research and a wide panorama of methods to investigate the spectrum of block operator matrices are presented by C. Tretter in [27]. In general, the operators occurring in $\mathcal{L}_0$ are unbounded and $\mathcal{L}_0$ need not be a closed nor a closable operator, even if its entries are closed. However, under some conditions $\mathcal{L}_0$ is closable and its closure $\mathcal{L}$ can be determined. The aim of this section is to generalize the previous results.

Let $\mathcal{M}$ is a bounded operator formally defined on the product space $X \times Y$ by

$$\mathcal{M} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where operator M_1 acts on X everywhere defined and the intertwining operator M_2 (resp. M_3) acts on the Banach space Y (resp. on X) everywhere defined and is strictly singular. The operator M_4 acts on Y everywhere defined and $\mathcal{L}_0$ is given by Eq. (7), where the operator A acts on X and has the domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space Y and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp. $\mathcal{D}(C)$) and acts on $Y \longrightarrow X$ (resp. $Y \longrightarrow Y$). The purpose of this section is to discuss the $\mathcal{M}$-essential spectra of the 2×2 matrix operator $\mathcal{L}_0$.

In what follows, we will assume that the following conditions, introduced by M. Faierman, R. Mennicken and M. Mller in [7], hold:

$(\mathcal{H}_1)$ A is a closed, densely defined linear operator on X with non empty M_1-resolvent set $\rho_{M_1}(A)$.

$(\mathcal{H}_2)$ B is a densely defined linear operator on X and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $(A - \mu M_1)^{-1} B$ is closable.

$(\mathcal{H}_3)$ The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $C(A - \mu M_1)^{-1}$ is bounded (in particular, if C is closable, then $C(A - \mu M_1)^{-1}$ is bounded).

$(\mathcal{H}_4)$ The lineal $\mathcal{D}(B) \cap \mathcal{D}(D)$ is dense in Y, and for some (hence for all) $\mu \in \rho_{M_1}(A)$, the operator $D - C(A - \mu M_1)^{-1} B$ is closable. We will denote by $S(\mu)$ the closure of the operator $D - (C - \mu M_3)(A - \mu M_1)^{-1}(B - \mu M_2)$.

Remark 4. *(i) It follows from the closed graph theorem that the operator*

$$G(\mu) = \overline{(A - \mu M_1)^{-1}(B - \mu M_2)}$$

is bounded on Y.

(ii) We emphasize that neither the domain of $S(\mu)$ nor the property of being closable depend on μ. Indeed, consider λ, $\mu \in \rho_{M_1}(A)$. Then we have:

$$S(\lambda) - S(\mu) = (\lambda - \mu)\Big[M_3 G(\mu) + F(\lambda) M_2 + F(\lambda) M_1 G(\mu)\Big],$$

where $F(\lambda) = (C - \lambda M_3)(A - \lambda M_1)^{-1}$. Since the operators $F(\lambda)$ and $G(\mu)$ are bounded (see the condition $(\mathcal{H}_3)$ and the remark (i), respectively), then the difference $S(\lambda) - S(\mu)$ is bounded. Therefore neither the domain of $S(\mu)$ nor the property of being closable depend on μ.

We recall the following result which describes the closure of the operator $\mathcal{L}_0$.

Theorem 4. [7] *Let the conditions $(\mathcal{H}_1)-(\mathcal{H}_3)$ be satisfied and the lineal $\mathcal{D}(B)\cap\mathcal{D}(D)$ be dense in Y. Then the operator $\mathcal{L}_0$ is closable if and only if the operator $D-C(A-\mu M_1)^{-1}B$ is closable in Y, for some $\mu\in\rho_{M_1}(A)$. Moreover, the closure $\mathcal{L}$ of $\mathcal{L}_0$ is given by*

$$\mathcal{L}=\mu\mathcal{M}+\begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}\begin{pmatrix} A-\mu M_1 & 0 \\ 0 & S(\mu)-\mu M_4 \end{pmatrix}\begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}. \tag{8}$$

Lemma 3. [16] *For all bounded operators $T=\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ on $X\times Y$, we consider*

$$g(T)=\max\Big\{g(T_1)+g(T_2),\ g(T_3)+g(T_4)\Big\}.$$

Then g defines a measure of non-strict-singularity on the space $\mathcal{L}(XY)$.

For $n\in\mathbb{N}$, let

$$\mathcal{I}_n=\Big\{K\in\mathcal{L}(X)\text{ satisfing }g\Big((KB)^n\Big)<1\text{ for all }B\in\mathcal{L}(X)\Big\}.$$

We have the following inclusion:

$$\mathcal{S}(X)\subset\mathcal{I}_n(X).$$

Theorem 5. *Let $A\in\Phi(X)$. Then for all $K\in\mathcal{I}_n(X)$ we have $A+K\in\Phi(X)$ and $i(A+K)=i(A)$.*

Proof. Let $A\in\Phi(X)$. Then by [24, Theorem 1.1, p. 162] there exist $F\in\mathcal{K}(X)$ and $A_0\in\mathcal{L}(X)$ such that

$$AA_0=I-F\quad\text{on}\quad X.$$

Thus,

$$(A+K)A_0=I-F+KA_0.$$

Since $K\in\mathcal{I}_n(X)$, then $g(KA_0)^n<1$. By applying Proposition 2 we get $I+KA_0\in\Phi(X)$ and $i(I+KA_0)=0$. Since F is a compact operator, then $(A+K)A_0\in\Phi(X)$ and $i((A+K)A_0)=0$. Using the fact that $A\in\Phi(X)$ and $i(A_0)=-i(A)$, we can deduce that $A+K\in\Phi(X)$ and $i(A+K)=i(A)$.◀

Remark 5. *(i) If $K\in\mathcal{I}_n(X)$ and $A\in\mathcal{L}(X)$, then $KA\in\mathcal{I}_n(X)$.*

(ii) If $K\in\mathcal{I}_n(X)$ and $S\in\mathcal{S}(X)$, then $K+S\in\mathcal{I}_n(X)$. Indeed, for all $B\in\mathcal{L}(X)$, $((K+S)B)^n=(KB)^n+T$, where T is a strictly singular operator.

So, $g\big(((K+S)B)^n\big)=g((KB)^n)<1$.

In all that follows, we will make the following assumption:

$$(\mathcal{A}): \quad \begin{cases} g\Big(M_1G(\mu)HM_1G(\mu)K\Big) < \frac{1}{4}, \;\; g\Big(F(\mu)M_1HF(\mu)M_1K\Big) < \frac{1}{4} \\ \\ g\Big(M_1G(\mu)HF(\mu)M_1K\Big) < \frac{1}{4}, \;\; g\Big(F(\mu)M_1HM_1G(\mu)K\Big) < \frac{1}{4} \\ \text{for some } \; \mu \in \rho_{M_1}(A) \text{ and for all bounded operators } H \text{ and } K. \end{cases}$$

Remark 6. (i) *Note that if* $G(\mu)$ *and* $F(\mu)$ *are strictly singular operators, then hypothesis* $(\mathcal{A})$ *is satisfied.*

(ii) *If* $g\Big(F(\mu)M_1HM_1G(\mu)K\Big) < \frac{1}{4}$ *for all bounded operators* H *and* K, *then* $F(\mu)M_1G(\mu)$ *is strictly singular.*

Indeed, since $g\Big(F(\mu)M_1HM_1G(\mu)K\Big) < \frac{1}{4}$ *for all bounded operators* K *and* H, *we can consider* $K = n^2I_X$ *and* $HM_1 = I_Y$ *(where* $n \in \mathbb{N}^*$, I_X *and* I_Y *denote the identity operator). We obtain*

$$g\Big(F(\mu)M_1G(\mu)\Big) < \frac{1}{4n^2}.$$

So, $g\Big(F(\mu)M_1G(\mu)\Big) = 0$, *hence* $F(\mu)M_1G(\mu)$ *is strictly singular.*

Theorem 6. *Let the matrix operator* $\mathcal{L}_0$ *satisfy the conditions* $(\mathcal{H}_1)$- $(\mathcal{H}_4)$ *and assume that hypothesis* $(\mathcal{A})$ *is satisfied. Then*

$$\sigma_{e,\mathcal{M}}(\mathcal{L}) \subseteq \sigma_{e,M_1}(A) \cup \sigma_{e,M_4}(S(\mu)).$$

Moreover, if Φ_{A,M_1} *is connected, then*

$$\sigma_{e,\mathcal{M}}(\mathcal{L}) = \sigma_{e,M_1}(A) \cup \sigma_{e,M_4}(S(\mu)).$$

Proof. Let $\mu \in \rho_{M_1}(A)$ be such that hypothesis $(\mathcal{A})$ is satisfied and let λ be a complex number. It follows from Eq. (8) that

$$\lambda\mathcal{M} - \mathcal{L} = UV(\lambda)W - (\lambda - \mu)\begin{pmatrix} 0 & M_1G(\mu) - M_2 \\ F(\mu)M_1 - M_3 & F(\mu)M_1G(\mu) \end{pmatrix},$$

where

$$U = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}, \; W = \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}$$

and

$$V(\lambda) = \begin{pmatrix} \lambda M_1 - A & 0 \\ 0 & \lambda M_4 - S(\mu) \end{pmatrix}.$$

Let $\mathcal{K} = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$ be a bounded operator on $X \times Y$. Then

$$\left[\begin{pmatrix} 0 & M_1G(\mu) \\ F(\mu)M_1 & 0 \end{pmatrix}\mathcal{K}\right]^2 =$$
$$\begin{pmatrix} (M_1G(\mu)K_3)^2 + M_1G(\mu)K_4F(\mu)M_1K_1 & M_1G(\mu)K_3M_1G(\mu)K_4 + M_1G(\mu)K_4F(\mu)M_1K_2 \\ F(\mu)M_1K_1M_1G(\mu)K_3 + F(\mu)M_1K_2F(\mu)M_1K_1 & F(\mu)M_1K_1M_1G(\mu)K_4 + (F(\mu)M_1K_2)^2 \end{pmatrix}.$$

It follows from hypothesis $(\mathcal{A})$ and Lemma 3 that

$$g\left((\lambda-\mu)^2\left[\begin{pmatrix} 0 & M_1G(\mu) \\ F(\mu)M_1 & 0 \end{pmatrix}\mathcal{K}\right]^2\right) < 1,$$

which implies that the operator

$$(\lambda-\mu)\begin{pmatrix} 0 & M_1G(\mu) \\ F(\mu)M_1 & 0 \end{pmatrix} \in \mathcal{I}_2(X\times Y).$$

Then we can deduce from Remark 5 (ii) and the fact that $F(\mu)M_1G(\mu)$, M_2 and M_3 are strictly singular that

$$(\lambda-\mu)\begin{pmatrix} 0 & M_1G(\mu)-M_2 \\ F(\mu)M_1-M_3 & F(\mu)M_1G(\mu) \end{pmatrix} \in \mathcal{I}_2(X\times Y).$$

Now by applying Theorem 5 we can conclude that the operator $\lambda M - \mathcal{L}$ is a Fredholm if and only if $UV(\lambda)W$ is a Fredholm operator. Now, observe that the operators U and W are bounded and have bounded inverse. Hence the operator $UV(\lambda)W$ is a Fredholm operator if and only if $V(\lambda)$ has this property if and only if $\lambda M_1 - A$ and $\lambda M_4 - S(\mu)$ are Fredholm operators. By Lemma 1 (iii) we have

$$\begin{aligned} i(\lambda\mathcal{M}-\mathcal{L}) &= i(U)+i(V(\lambda))+i(W) \\ &= 0+i(V(\lambda))+0. \end{aligned}$$

So,

$$i(\lambda\mathcal{M}-\mathcal{L}) = i(\lambda M_1 - A) + i(\lambda M_4 - S(\mu)). \tag{9}$$

Let $\lambda \notin \left(\sigma_{e,M_1}(A)\cup\sigma_{e,M_4}(S(\mu))\right)$. Using Proposition 3, we get $\lambda M_1 - A$ and $\lambda M_4 - S(\mu)$ are Fredholm operators and $i(\lambda M_1 - A) = i(\lambda M_4 - S(\mu)) = 0$. Then $\lambda\mathcal{M}-\mathcal{L}$ is a Fredholm operator and $i(\lambda\mathcal{M}-\mathcal{L}) = 0$. So, $\lambda \notin \sigma_{e,\mathcal{M}}(\mathcal{L})$. This shows that

$$\sigma_{e,\mathcal{M}}(\mathcal{L}) \subseteq \sigma_{e,M_1}(A)\cup\sigma_{e,M_4}(S(\mu)).$$

Now, let $\lambda \notin \sigma_{e,M}(\mathcal{L})$. Using Proposition 3, we get $\lambda M - \mathcal{L}$ is a Fredholm operator and $i(\lambda\mathcal{M}-\mathcal{L}) = 0$. Then $\lambda M_1 - A$ and $\lambda M_4 - S(\mu)$ are Fredholm operators. Since Φ_{A,M_1} is connected and $\rho_{M_1}(A) \neq \emptyset$ (see hypothesis $\mathcal{H}_1$) then, using Lemma 2 we get $i(\lambda M_1 - A) = 0$. By, Eq. (9) we have $i(\lambda M_4 - S(\mu)) = 0$. This shows that

$$\sigma_{e,\mathcal{M}}(\mathcal{L}) = \sigma_{e,M_1}(A)\cup\sigma_{e,M_4}(S(\mu)). \qquad \blacktriangleleft$$

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