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Darboux Slant Ruled Surfaces

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Abstract. In this study, we introduce Darboux slant ruled surface in the Euclidean 3-space E^3 which is defined by the property that the Darboux vector $\vec{W}$ of orthonormal frame of a ruled surface satisfies the condition $\left\langle \vec{W}, \vec{u} \right\rangle = \text{constant} \neq 0$ where $\vec{u}$ is a fixed non-zero direction in the space. We obtain characterizations for Darboux slant ruled surfaces according to conical curvature κ and give the relations between a Darboux slant ruled surface and other slant ruled surfaces.

Key Words and Phrases: Slant ruled surface, Darboux vector, conical curvature.

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1. Introduction

Special curves and special surfaces defined by some properties according to their curvatures are the most fascinating problems of the differential geometry. The well-known types of special curves are helices, involute-evolute curves, Bertrand curves and slant helices. Helix curve is defined by the property that the tangent line of the curve makes a constant angle with a fixed straight line (the axis of the general helix) [2]. The classical result for a helix was stated by Lancret in 1802 and first proved by B. de Saint Venant in 1845: A necessary and sufficient condition that a curve to be a general helix is that the ratio of the first curvature to the second curvature be constant, i.e., κ/τ is constant along the curve, where κ and τ denote the first and second curvatures of the curve, respectively [12].

Recently, Izumiya and Takeuchi have defined a new special curve called slant helix for which the principal normal lines of the curve make a constant angle with a fixed non-zero constant direction and they have given a characterization for slant helix in the Euclidean 3-space E^3 [4]. The spherical images of the Frenet vectors of a slant helix have been studied by Kula and Yaylı and they have obtained that the spherical images of a slant helix are spherical helices [6]. Later, Kula et al have obtained some new characterizations for slant helices in E^3 [7]. Monterde has shown that for a curve with constant curvature and non-constant torsion, the principal normal vector makes a constant angle with a fixed constant direction, i.e., the curve is a slant helix [8]. Ali has considered the position

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vectors of slant helices and obtained some new properties of these curves [1]. Analogue to the definition of slant helix, Önder et al have defined B_2-slant helix in the Euclidean 4-space E^4 by the property that the second binormal vector B_2 of a space curve makes a constant angle with a fixed direction and they have given some characterizations for B_2-slant helix in the Euclidean 4-space E^4 [11]. Then, Gök, Camcı and Hacısalihoğlu have considered the notion of V_n-slant helix in the Euclidean n-space E^n [3]. Later, Zıplar, Şenol and Yaylı have defined a new type of helices called Darboux helices and they have given some characterizations for these curves [13].

Moreover, Önder has considered the notion of "slant helix" for ruled surfaces and defined slant ruled surfaces in the Euclidean 3-space E^3 by the property that the vectors of orthonormal frame of a ruled surface make constant angles with fixed non-zero directions [9]. He has shown that the striction curves of developable slant ruled surfaces are helices or slant helices.

In this work, we introduce Darboux slant ruled surfaces in E^3. We give characterizations of these special surfaces and obtain the relations between slant ruled surfaces and Darboux slant ruled surfaces.

2. Ruled Surfaces in the Euclidean 3-space E^3

In this section, we give a brief summary of ruled surfaces in E^3 presented in [5].

A ruled surface S is a special surface generated by a continuous movement of a line along a curve and has the parametrization

$$\vec{r}(u,v) = \vec{f}(u) + v\,\vec{q}(u), \tag{1}$$

where $\vec{f} = \vec{f}(u)$ is a regular curve in E^3 defined on an open interval $I \subset \mathrm{R}$ and $\vec{q} = \vec{q}(u)$ is a unit direction vector of an oriented line in E^3. The curve $\vec{f} = \vec{f}(u)$ is called base curve or generating curve of the surface and various positions of generating lines $\vec{q} = \vec{q}(u)$ are called rulings. In particular, if the direction of $\vec{q}$ is constant, then the ruled surface is said to be cylindrical, and non-cylindrical otherwise.

Let $\vec{m}$ be the unit normal vector of the ruled surface S. Then if v decreases infinitely along a ruling $u = u_1$, the unit normal $\vec{m}$ approaches a limiting direction. This direction is called asymptotic normal (central tangent) direction and is defined by

$$\vec{a} = \lim_{v\to\pm\infty} \vec{m}(u_1, v) = \frac{\vec{q}\times\dot{\vec{q}}}{\left\|\dot{\vec{q}}\right\|}, \tag{2}$$

where $\dot{\vec{q}} = \frac{d\vec{q}}{du}$. The point at which the unit normal of S is perpendicular to $\vec{a}$ is called striction point (or central point) C and the set of striction points of all rulings is called striction curve of surface. The parametrization of striction curve $\vec{c} = \vec{c}(u)$ on a ruled surface is given by

$$\vec{c}(u) = \vec{f} - \frac{\left\langle \dot{\vec{q}}, \dot{\vec{f}} \right\rangle}{\left\langle \dot{\vec{q}}, \dot{\vec{q}} \right\rangle} \vec{q}. \tag{3}$$

A non-cylindrical ruled surface always has a parametrization of the form

$$\vec{r}(s,v) = \vec{c}(s) + v\,\vec{q}(s),$$

where $\|\vec{q}(s)\| = 1$, $\left\langle \frac{d\vec{c}}{ds}, \frac{d\vec{q}}{ds} \right\rangle = 0$. Furthermore, the striction curve does not depend on the choice of the base curve.

The vector $\vec{h}$ defined by $\vec{h} = \vec{a} \times \vec{q}$ is called central normal vector. Then the orthonormal system $\left\{C;\, \vec{q}, \vec{h}, \vec{a}\right\}$ is called Frenet frame of the ruled surface S where C is the central point and $\vec{q}$, $\vec{h}$, $\vec{a}$ are unit vectors of ruling, central normal and central tangent, respectively.

The set of all bound vectors $\vec{q}(s)$ at origin O constitutes a cone which is called directing cone of the ruled surface S. The end points of unit vectors $\vec{q}(s)$ drive a spherical curve k_1 on the unit sphere S^2 and this curve is called spherical image of ruled surface S, whose arc length parameter is denoted by s_1.

For the Frenet formulae of ruled surface S and of its directing cone with respect to the arc length parameter s_1 we have

$$\begin{bmatrix} d\vec{q}/ds_1 \\ d\vec{h}/ds_1 \\ d\vec{a}/ds_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa \\ 0 & -\kappa & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}, \tag{4}$$

where κ is called conical curvature of directing cone. The Frenet formulae can be interpreted kinematically as follows: If $\vec{q}$ traverses the directing cone in such a way that s_1 is the time parameter, then the moving frame $\left\{C;\, \vec{q}, \vec{h}, \vec{a}\right\}$ moves in accordance with (4). This motion contains, apart from an instantaneous translation, instantaneous rotation with angular velocity vector given by the Darboux vector

$$\vec{W} = \kappa\vec{q} + \vec{a}. \tag{5}$$

The direction of the Darboux vector is that of instantaneous axis of rotation, and its length $\left\|\vec{W}\right\| = \sqrt{1+\kappa^2}$ is the scalar angular velocity. Then, Frenet formulae (4) can be given as follows:

$$\vec{q}\,' = \vec{W} \times \vec{q}, \quad \vec{h}' = \vec{W} \times \vec{h}, \quad \vec{a}\,' = \vec{W} \times \vec{a}. \tag{6}$$

Definition 1. *([9]) Let S be a regular ruled surface in E^3 given by the parametrization*

$$\vec{r}(s,v) = \vec{c}(s) + v\,\vec{q}(s), \quad \|\vec{q}(s)\| = 1,$$

where $\vec{c}(s)$ is striction curve of S and s is arc length parameter of $\vec{c}(s)$. Let the Frenet frame of S be $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$. Then S is called a q-slant (h-slant or a-slant, respectively) ruled *surface if the ruling (the vector $\vec{h}$ or the vector $\vec{a}$, respectively) makes a constant angle θ with a fixed non-zero unit direction $\vec{u}$ in the space, i.e.,*

$$\langle \vec{q}, \vec{u} \rangle = \cos\theta = constant;\ \ \theta \neq \frac{\pi}{2}$$

$$(\left\langle \vec{h}, \vec{u} \right\rangle = \cos\theta = constant;\ \ \theta \neq \tfrac{\pi}{2} \text{ or } \langle \vec{a}, \vec{u} \rangle = \cos\theta = constant;\ \ \theta \neq \tfrac{\pi}{2}, \text{ respectively}).$$

Theorem 1. *([10]) Let S be a regular ruled surface in E^3 with Frenet frame $\left\{ \vec{q}, \vec{h}, \vec{a} \right\}$ and conical curvature $\kappa \neq 0$. Then S is a q-slant ruled surface if and only if the function κ is constant.*

In this paper, we will study the Darboux slant ruled surfaces by considering Eq. (5) and give the relationships between slant ruled surfaces and Darboux slant ruled surfaces. First we prove the following theorem for h-slant ruled surfaces.

Theorem 2. *Let S be a regular ruled surface in E^3 with Frenet frame $\left\{ \vec{q}, \vec{h}, \vec{a} \right\}$ and conical curvature $\kappa \neq 0$. Then S is an h-slant ruled surface if and only if the function*

$$\frac{\kappa'}{(1+\kappa^2)^{3/2}} \tag{7}$$

is a non-zero constant.

Proof. Assume that S is an h-slant ruled surface in E^3. So, for a non-zero constant $c \in \mathrm{R}$ we can write

$$\left\langle \vec{h},\, \vec{u} \right\rangle = c,$$

where $\vec{u}$ is a non-zero fixed direction. Then, for the vector $\vec{u}$ we have

$$\vec{u} = b_1(s_1)\vec{q}(s_1) + c\vec{h}(s_1) + b_2(s_1)\vec{a}(s_1), \tag{8}$$

where $b_1 = b_1(s_1)$ and $b_2 = b_2(s_1)$ are smooth functions of arc length parameter s_1. On the other hand, since $\vec{u}$ is a fixed direction, that is $\vec{u}' = 0$ and Frenet frame $\left\{ \vec{q}, \vec{h}, \vec{a} \right\}$ is linearly independent, differentiation of (8) gives

$$b_1' - c = 0,\ \ b_1 - \kappa b_2 = 0,\ \ b_2' + c\kappa = 0. \tag{9}$$

From the second equation of system (9) we have

$$b_1 = \kappa b_2. \tag{10}$$

Moreover, since $\vec{u}$ is a fixed direction we have $\|\vec{u}\|$ is constant. Then it follows

$$b_1^2 + c^2 + b_2^2 = \text{constant}. \tag{11}$$

Substituting (10) in (11) gives

$$b_2^2\left(1+\kappa^2\right)=n^2=\text{constant}\quad (n\in \mathrm{R}). \tag{12}$$

If $n=0$, then $b_2=0$ and from (9) we have $b_1=0,\quad c=0$. This means that $\vec{u}=0$ which is a contradiction. So, $n\neq 0$ and from (12) it is obtained that

$$b_2=\pm\frac{n}{\sqrt{1+\kappa^2}}. \tag{13}$$

Considering the third equation of the system (9), from (13) we have

$$\frac{d}{ds_1}\left[\pm\frac{n}{\sqrt{1+\kappa^2}}\right]=-c\kappa. \tag{14}$$

For a non-constant $m\in \mathrm{R}$, from (14) the following desired equality is obtained:

$$\frac{\kappa'}{(1+\kappa^2)^{3/2}}=\frac{c}{n}=m=\text{constant}\neq 0.$$

Conversely, assume that the function in (7) is constant. Then, for a non-constant $m\in \mathrm{R}$, we have

$$\frac{\kappa'}{(1+\kappa^2)^{3/2}}=m=\text{constant}\neq 0.$$

We define the function

$$\vec{u}=\frac{\kappa}{\sqrt{1+\kappa^2}}\vec{q}+m\vec{h}+\frac{1}{\sqrt{1+\kappa^2}}\vec{a}. \tag{15}$$

From (15) and Frenet formulae (4), we have $\vec{u}'=0$, i.e., $\vec{u}$ is a constant vector. On the other hand,

$$\left\langle \vec{h},\vec{u}\right\rangle=m=\text{constant}\neq 0,$$

which gives that S is an h-slant ruled surface in E^3.◄

3. Darboux Slant Ruled Surfaces in the Euclidean 3-space E^3

In this section, we consider the notion of "slant" for Darboux vector $\vec{W}=\kappa\vec{q}+\vec{a}$ and give some theorems for Darboux slant ruled surfaces in the Euclidean 3-space. First, we give the following definition.

Definition 2. *Let S be a regular ruled surface in E^3 given by the parametrization*

$$\vec{r}(s,v)=\vec{c}(s)+v\,\vec{q}(s),\quad \|\vec{q}(s)\|=1, \tag{16}$$

where $\vec{c}(s)$ is the striction curve of S and s is the arc length parameter of $\vec{c}(s)$. Let the orthonormal frame $\left\{ \vec{q}, \vec{h}, \vec{a} \right\}$ and the function $\kappa = \kappa(s)$ denote the Frenet frame and conical curvature of S, respectively. Then, S is called a Darboux slant ruled surface if its Darboux vector $\vec{W}$ and a non-zero fixed direction $\vec{u}$ satisfy

$$\left\langle \vec{W},\, \vec{u} \right\rangle = \textit{constant} \neq \textit{0}.$$

Then, we give the following characterizations for Darboux slant ruled surfaces. Whenever we talk about S, we will mean that the surface has the parametrization and Frenet elements as given in Definition 2.

Theorem 3. *Let S be a ruled surface in E^3 with non-zero conical curvature κ. If S is a Darboux slant ruled surface, then the conical curvature κ is constant.*

Proof. Let S be a Darboux slant ruled surface. Then for a non-zero fixed direction $\vec{u}$, we have

$$\left\langle \vec{W},\, \vec{u} \right\rangle = \text{constant}. \tag{17}$$

Taking the derivative of (17) gives

$$\left\langle \vec{W}',\, \vec{u} \right\rangle = 0. \tag{18}$$

From (18) and Frenet formulae (4), we can write

$$\kappa' \left\langle \vec{q},\, \vec{u} \right\rangle = 0. \tag{19}$$

From (19) we get two possibilities as follows:

$$\begin{cases} \kappa = \text{constant}, \\ \left\langle \vec{q},\ \vec{u} \right\rangle = 0. \end{cases} \tag{20}$$

If $\langle \vec{q}, \vec{u} \rangle = 0$, then $\vec{u}$ is perpendicular to the vector $\vec{q}$ and can be written as

$$\vec{u} = a_1 \vec{h} + a_2 \vec{a}, \tag{21}$$

where $a_1,\ a_2$ are smooth functions of s_1. By taking the derivative of (21) and using the Frenet formulae given in (4) we have

$$-a_1 \vec{q} + (a_1' - \kappa a_2)\vec{h} + (a_2' + \kappa a_1)\vec{a} = 0. \tag{22}$$

Considering that the Frenet frame $\left\{ \vec{q}, \vec{h}, \vec{a} \right\}$ is linearly independent, from (22) we obtain the following system:

$$\begin{cases} a_1 = 0, \\ a_1' - \kappa a_2 = 0, \\ a_2' + \kappa a_1 = 0. \end{cases} \tag{23}$$

Since $\kappa \neq 0$, from (23) we get that $a_1 = a_2 = 0$ which gives us that $\vec{u} = 0$ which is a contradiction, that is $\langle \vec{q}, \vec{u} \rangle \neq 0$. Therefore $\kappa =$ constant.◄

The converse of Theorem 3 is satisfied in the following special case:

Corollary 1. *Let S be a ruled surface in E^3 with constant curvature $\kappa \neq 0$. Then S is a Darboux slant ruled surface if and only if the angle ϕ between the vectors $\vec{W}$, $\vec{u}$ is constant.*

Proof. Let S be a Darboux slant ruled surface. Then we have that $\left\langle \vec{W}, \vec{u} \right\rangle = x =$ constant $\neq 0$. If the angle between the vectors $\vec{W}$, $\vec{u}$ is ϕ, then we can write

$$\left\| \vec{W} \right\| \|\vec{u}\| \cos\phi = x.$$

Since $\|\vec{u}\|$ and κ are non-zero constants, the last equality gives us $\cos\phi = \frac{x}{\|\vec{u}\|\sqrt{1+\kappa^2}}$ is a non-zero constant. It means that ϕ is a constant.

Conversely, assume that the angle ϕ between the vectors $\vec{W}$, $\vec{u}$ is a constant. Then we have

$$\cos\phi = \frac{\left\langle \vec{W}, \vec{u} \right\rangle}{c\sqrt{1+\kappa^2}} = \text{constant},$$

where $\|\vec{u}\| = c$ is a non-zero constant. From last equality we have

$$\left\langle \vec{W}, \vec{u} \right\rangle = y\sqrt{1+\kappa^2},$$

where $y = c\cos\phi$ is a non-zero constant. Since $\kappa =$ constant, we have $\left\langle \vec{W}, \vec{u} \right\rangle =$ constant, i.e., S is a Darboux slant ruled surface.◄

Moreover, from Theorem 3 we have the following corollaries:

Corollary 2. *Let S be a ruled surface in E^3 with conical curvature $\kappa \neq 0$. If S is a Darboux slant ruled surface, then* $\det(\vec{W},\ \vec{W}',\ \vec{W}'') = 0$ *holds.*

Proof. From Darboux vector and its derivatives we have

$$\begin{aligned} &\vec{W} = \kappa\vec{q} + \vec{a}, \\ &\vec{W}' = \kappa'\vec{q}, \\ &\vec{W}'' = \kappa''\vec{q} + \kappa'\vec{h}. \end{aligned}$$

Then we obtain that

$$\det(\vec{W},\ \vec{W}',\ \vec{W}'') = \left(\kappa'\right)^2.$$

If S is a Darboux slant ruled surface, from Theorem 3 we have that κ is a non-zero constant which gives that $\det(\vec{W},\ \vec{W}',\ \vec{W}'') = 0$.◄

Corollary 3. *Every Darboux slant ruled surface S with conical curvature $\kappa \neq 0$ is also a q-slant ruled surface.*

Proof. The proof is clear from Theorem 1 and Theorem 3.◄

Theorem 4. *An h-slant ruled surface S with $\kappa \neq 0$ cannot be a Darboux slant ruled surface with the same axis.*

Proof. Let S be an h-slant ruled surface with axis $\vec{u}$. Then we have

$$\left\langle \vec{h}, \vec{u} \right\rangle = \text{constant} \neq 0. \tag{24}$$

From Theorem 2 we know that the axis of h-slant ruled surface is given by

$$\vec{u} = \frac{\kappa}{\sqrt{1+\kappa^2}}\vec{q} + m\vec{h} + \frac{1}{\sqrt{1+\kappa^2}}\vec{a}, \tag{25}$$

where $m = \frac{\kappa'}{(1+\kappa^2)^{3/2}}$ is a non-zero constant. By using (25) and considering Darboux vector $\vec{W} = \kappa\vec{q} + \vec{a}$ we can write

$$\vec{u} = \frac{\vec{W}}{\left\|\vec{W}\right\|} + m\vec{h}. \tag{26}$$

Then, from (26) we obtain

$$\left\langle \vec{W}, \vec{u} \right\rangle = \left\|\vec{W}\right\| = \sqrt{1+\kappa^2}. \tag{27}$$

From (27) it is clear that S is a Darboux slant ruled surface, i.e., $\left\langle \vec{W}, \vec{u} \right\rangle = \text{constant}$ if and only if $\kappa = \text{constant}$. But if $\kappa = \text{constant}$, from (25) we have that $\left\langle \vec{h}, \vec{u} \right\rangle = 0$. On the other hand, from Definition 1 we know that the angle θ between the vectors $\vec{h}$, $\vec{u}$ satisfies $\theta \neq \frac{\pi}{2}$. So, the result is a contradiction. Then, an h-slant ruled surface S with $\kappa \neq 0$ cannot be a Darboux slant ruled surface with the same axis.◄

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