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Existence of Best Proximity Points in Regular Cone Metric Spaces

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Abstract. In this paper we have established some conditions which guarantee the existence of the distance between two subsets of a regular cone metric space. Under these conditions we have given a main result which guarantee the existence of best proximity points for cyclic contraction mappings in regular cone metric space, which extends the earlier result of Haghi et al(2011).

Key Words and Phrases: Cone L-function, Cyclic contraction map, Lower bound, Regular cone metric space.

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1. Introduction

Consider a self map T defined on the union of two subsets A, B of a metric space X. A mapping $T : A \cup B \longrightarrow A \cup B$ is said to be cyclic provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. Let T be a cyclic map. If there exists a point $x \in A \cup B$ such that $d(x, Tx) = d(A, B)$, then x is a best proximity point with regard to T, where $dist(A, B) := inf\{d(x, y) : (x, y) \in A \times B\}$.

In 2003, Kirk et al. [9] proved the following extension of the Banach contraction principle for cyclic mappings.

Theorem 1. [9] *Let A, B be two non empty closed subsets of a complete metric space (X, d). Suppose that T is a cyclic mapping such that $d(Tx, Ty) \leq kd(x, y)$, for some $k \in (0, 1)$ and for all $(x, y) \in (A \times B)$. Then T has a unique fixed point in $A \cap B$.*

Further, in 2006, Eldered et al.[4] introduced the class of cyclic contractions and obtained best proximity point results for cyclic contraction mappings.

Definition 1. [4] Let A, B be two non empty subsets of a metric space (X, d). A mapping $T : A \cup B \longrightarrow A \cup B$ is said to be cyclic contraction if T is cyclic and $d(Tx, Ty) \leq kd(x, y) + (1 - k)dist(A, B)$, for some $k \in (0, 1)$ and for all $(x, y) \in A \times B$.

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There are further works about best proximity points for certain contraction mappings, some of them are noted in [3], [11], [14] and [16].
On the other hand, in 2007, Haung et al.[7] introduced cone metric spaces and proved the Banach contraction principle for such spaces. Cone metric space is a generalisation of metric space in which the set of real numbers is replaced by a real Banach space. Several fixed point results in cone metric space by different authors are noted in [2], [5], [8], [10], [12], [13], [14] and [15]. Before coming to our main results we give some preliminaries about cone metric spaces.

Definition 2. [7] Let E be a real Banach space and θ be the zero of the Banach space E. Let P be a subset of E. P is called a cone if

(i) P is closed, non-empty and $P \neq \{\theta\}$

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b

(iii) $P \bigcap (-P) = \{\theta\}$

For a given cone P we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. Here $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$; where $intP$ denotes the interior of P. $x \leq y$ is same as $y \geq x$ and $x \ll y$ is same as $y \gg x$. A cone P is called normal if there is a real number $K > 0$ such that for all $x, y \in E$,

$$\theta \leq x \leq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of cone P. The cone P is called regular if every increasing and bounded above sequence $\{x_n\}$ in E is convergent. Equivalently, the cone P is regular if and only if every decreasing and bounded below sequence is convergent. It is well known that a regular cone is a normal cone. The category of regular cone metric spaces is bigger than the category of metric spaces (see [6])
Let E be a real Banach space with cone P in E, int$P \neq \emptyset$, and $\leq$ be the partial ordering with respect to P.

Definition 3. [7] Let X be a non-empty set. Suppose the mapping

$$d : X \times X \to E \quad \text{satisfies :}$$

(i) $\theta \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.
As the Banach space E is only partially ordered (not well ordered), so maximum (or minimum) of two elements does not exist necessarily in cone metric spaces. Thus, there is no guarantee for existence of distance between two subsets of a cone metric space.
A subset A of a cone metric space X is said to be bounded whenever there exists $e >> \theta$ such that $d(x,y) \leq e$ for all $x, y \in A$.
Now for the rest of this paper, E is a real Banach space, (X,d) is a regular cone metric space, $\leq$ is the partial ordering with respect to P and A, B are non-empty subsets of X.

Definition 4. [6] An element $p \in P$ is said to be a lower bound of $A \times B$ whenever $p \leq d(a,b)$ for all $(a,b) \in A \times B$. Moreover, if $p \geq q$ for all lower bounds q of $A \times B$, then p is called the greatest lower bound of $A \times B$. In this case, we denote it by $dis(A,B)$.
Clearly in above definition, $dis(A,B)$ is a unique vector in P if it exists. Also, θ is always a lower bound of $A \times B$.
Now we give some results of Haghi et al.[6], which are the motivation for our main results.

Theorem 2. [6] *Let $T : A \cup B \longrightarrow A \cup B$ be a map such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $d(Tx,Ty) \leq kd(x,y) + (1-k)d(a,b)$, for all (a,b), $(x,y) \in A \times B$, where $k \in [0,1)$ is a constant. Then $dis(A,B)$ exists.*

Theorem 3. [6] *Let $\phi : P \longrightarrow P$ be a strictly increasing map, $T : A \cup B \longrightarrow A \cup B$ be a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $d(Tx,Ty) \leq d(x,y) - \phi(d(x,y)) + \phi(p)$, for all $(x,y) \in A \times B$, where p is a lower bound for $A \times B$. Then $dis(A,B) = p$.*

Remark 1. If we put $\phi(x) = (1-k)x, k \in [0,1)$, then the inequality condition reduces to $d(Tx,Ty) \leq kd(x,y) + (1-k)p$ and in this case also we have $dis(A,B) = p$.

Theorem 4. [6] *Let $\phi : P \longrightarrow P$ be a strictly increasing sub-additive map such that $\phi(\theta) = \theta$ and $I - \phi$ be also strictly increasing map. Also, let $T : A \cup B \longrightarrow A \cup B$ be a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $d(Tx,Ty) \leq \phi(d(x,y)) + (I-\phi)(p)$, for all $(x,y) \in A \times B$, where p is a lower bound for $A \times B$ and I is the identity map. Then $dis(A,B) = p$.*

Definition 5. [6] A map $\psi : P \longrightarrow P$ is called cone L-function whenever $\psi(\theta) = \theta, \psi(s) > \theta$, for all $s \in P$ with $s \neq \theta$ and there exists $\delta_s >> \theta$ such that $\psi(t) \leq s$ for all $s \leq t \leq s+\delta_s$.

It is obvious that $\psi(s) \leq s$ for all $s \in P$ with $s \neq \theta$ whenever ψ is a cone L-function.

Theorem 5. [6] *Let $\psi : P \longrightarrow P$ be a cone L-function and $T : A \cup B \longrightarrow A \cup B$ be a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $d(Tx,Ty) - p < \psi(d(x,y)) - p)$, for all $(x,y) \in A \times B$ with $p < d(x,y)$, where p is a lower bound for $A \times B$. Then $dis(A,B) = p$.*

Theorem 6. [6] *Suppose that the conditions of the Theorems 2, 3 and 5 hold, $x_0 \in A$ and $x_n = Tx_{n-1}$ for all $n \geq 1$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x,Tx) = dis(A,B)$.*

2. Main results

Now we give our main results, which extend the results of Haghi et al.[6].

Theorem 7. *Let $T : A \cup B \longrightarrow A \cup B$ be a map such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $d(Tx, Ty) \leq \frac{k}{2}\{d(Tx, x) + d(Ty, y)\} + (1-k)d(a, b)$, for all (a, b), $(x, y) \in A \times B$, where $k \in [0, 1)$ is a constant. Then $dis(A, B)$ exists.*

Proof. Let $x_0 \in A \cup B$ and set $x_n = Tx_{n-1}$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$.Then for all $(a, b) \in A \times B$,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{k}{2}\{d(x_{n+1}, x_n) + d(x_n, x_{n-1})\} + (1-k)d(a, b) \\ \Longrightarrow\ d_{n+1} &\leq \frac{k}{(2-k)}d_n + \frac{2(1-k)}{(2-k)}d(a, b) \\ \Longrightarrow\ d_{n+1} &\leq \frac{k}{(2-k)}d_n + (1 - \frac{k}{(2-k)})d(a, b) \\ \Longrightarrow\ d_{n+1} &\leq hd_n + (1-h)d(a, b), \quad \text{where,} \quad h = \frac{k}{(2-k)} \in [0, 1). \end{aligned}$$

It follows that $d_{n+1} \leq d_n$ for all $n \geq 1$. By the regularity of P, there exists $p \in P$ such that $lim_{n\to\infty}d_n = p$. Therefore, $p \leq d(a, b)$ holds for any $(a, b) \in A \times B$. Now if q is a lower bound for $A \times B$, then $q \leq d_n$ for all $n \geq 1$. Hence, $q \leq p$. Therefore $dis(A, B) = p$.

Theorem 8. *Let $T : A \cup B \longrightarrow A \cup B$ be a map such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $d(Tx, Ty) \leq \frac{k}{2}\{d(Tx, y) + d(Ty, x)\} + (1-k)d(a, b)$, for all (a, b), $(x, y) \in A \times B$, where $k \in [0, 1)$ is a constant. Then $dis(A, B)$ exists.*

Proof. Let $x_0 \in A \cup B$ and set $x_n = Tx_{n-1}$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$.Then for all $(a, b) \in A \times B$,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{k}{2}\{d(x_{n+1}, x_{n-1}) + d(x_n, x_n)\} + (1-k)d(a, b) \\ \Longrightarrow\ d(x_{n+1}, x_n) &\leq \frac{k}{2}\{d(x_{n+1}, x_n) + d(x_n, x_{n-1})\} + (1-k)d(a, b) \\ \Longrightarrow\ d_{n+1} &\leq \frac{k}{(2-k)}d_n + \frac{2(1-k)}{(2-k)}d(a, b) \\ \Longrightarrow\ d_{n+1} &\leq \frac{k}{(2-k)}d_n + (1 - \frac{k}{(2-k)})d(a, b) \\ \Longrightarrow\ d_{n+1} &\leq hd_n + (1-h)d(a, b), \quad \text{where,} \quad h = \frac{k}{(2-k)} \in [0, 1). \end{aligned}$$

It follows that $d_{n+1} \leq d_n$ for all $n \geq 1$. By the regularity of P, there exists $p \in P$ such that $lim_{n\to\infty}d_n = p$. Therefore, $p \leq d(a, b)$ holds for any $(a, b) \in A \times B$. Now if q is a lower bound for $A \times B$, then $q \leq d_n$ for all $n \geq 1$. Hence, $q \leq p$. Therefore $dis(A, B) = p$.

Theorem 9. *Let $T : A \cup B \longrightarrow A \cup B$ be a map such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $d(Tx, Ty) \leq \frac{k}{4}\{d(Tx, x) + d(Ty, y) + d(Tx, y) + d(Ty, x)\} + (1-k)d(a, b)$, for all (a, b), $(x, y) \in A \times B$, where $k \in [0, 1)$ is a constant. Then $dis(A, B)$ exists.*

Proof. Let $x_0 \in A \cup B$ and set $x_n = Tx_{n-1}$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$.Then for all $(a, b) \in A \times B$,

$$\begin{aligned}
d(x_{n+1}, x_n) &\leq \frac{k}{4}\{d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + d(x_{n+1}, x_{n-1}) + d(x_n, x_n)\} \\
&\quad + (1-k)d(a, b) \\
\Longrightarrow \ d_{n+1} &\leq \frac{k}{4}\{d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + d(x_{n+1}, x_n) + d(x_n, x_{n-1})\} \\
&\quad + (1-k)d(a, b) \\
\Longrightarrow \ d_{n+1} &\leq \frac{k}{2}\{d_{n+1} + d_n\} + (1-k)d(a, b) \\
\Longrightarrow \ d_{n+1} &\leq \frac{k}{2}\{d_{n+1} + d_n\} + (1-k)d(a, b) \\
\Longrightarrow \ (2-k)d_{n+1} &\leq kd_n + 2(1-k)d(a, b) \\
\Longrightarrow \ d_{n+1} &\leq \frac{k}{(2-k)}d_n + (1 - \frac{k}{(2-k)})d(a, b) \\
\Longrightarrow \ d_{n+1} &\leq hd_n + (1-h)d(a, b), \quad \text{where,} \quad h = \frac{k}{(2-k)} \in [0, 1).
\end{aligned}$$

It follows that $d_{n+1} \leq d_n$ for all $n \geq 1$. By the regularity of P, there exists $p \in P$ such that $lim_{n\to\infty} d_n = p$. Therefore, $p \leq d(a, b)$ holds for any $(a, b) \in A \times B$. Now if q is a lower bound for $A \times B$, then $q \leq d_n$ for all $n \geq 1$. Hence, $q \leq p$. Therefore $dis(A, B) = p$.

With the help of a control function ψ, we give our next two results, which generalize Theorem 3 and Theorem 4, respectively.

Theorem 10. *Let $\psi, \phi : P \longrightarrow P$ be strictly increasing maps with ψ continuous. Let $T : A \cup B \longrightarrow A \cup B$ be a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) + \phi(p)$, for all $(x, y) \in A \times B$, where p is a lower bound for $A \times B$. Then $dis(A, B) = p$.*

Proof. Let $x_0 \in A \cup B$ and set $x_n = Tx_{n-1}$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$. As p is a lower bound for $A \times B$, so $p \leq d_n$ for all $n \geq 1 \implies \phi(p) \leq \phi(d_n) \implies \phi(p) - \phi(d_n) \leq \theta$ for all $n \geq 1$. Now from the inequality condition we have,

$$\psi(d_{n+1}) \leq \psi(d_n) - \phi(d_n) + \phi(p) \leq \psi(d_n).$$

As ψ is strictly increasing, $d_{n+1} \leq d_n$ for all $n \geq 1$. By the regularity of P, there exists $q \in P$ such that $lim_{n\to\infty} d_n = q$. Since $\phi(d_n) - \phi(p) \leq \psi(d_n) - \psi(d_{n+1})$ for all $n \geq 1$ and $lim_{n\to\infty}\psi(d_n) - \psi(d_{n+1}) = \theta \implies lim_{n\to\infty}\phi(d_n) = \phi(p)$. Since $p \leq d_n$ for all $n \geq 1$, $p \leq q$. Therefore, $\phi(p) \leq \phi(q) \leq \phi(d_n)$ for all $n \geq 1$. Hence, $\phi(p) = \phi(q) \implies p = q$ and so $dis(A, B) = p$.

Theorem 11. *Let $\psi, \phi : P \longrightarrow P$ be strictly increasing mappings such that $\psi(\theta) = \phi(\theta) = \theta$ and $\psi - \phi$ is also strictly increasing map. In addition let ϕ be subadditive, ψ be continuous and $\psi(c) \leq c$, for all $c \gg \theta$. Let $T : A \cup B \longrightarrow A \cup B$ be a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $\psi((Tx, Ty)) \leq \phi(d(x, y)) + (\psi - \phi)(p)$, for all $(x, y) \in A \times B$, where p is a lower bound for $A \times B$. Then $dis(A, B) = p$.*

Proof. Let $x_0 \in A \cup B$ and set $x_n = Tx_{n-1}$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$. As p is a lower bound for $A \times B$, so $p \leq d_n$ for all $n \geq 1 \implies (\psi - \phi)(p) \leq (\psi - \phi)(d_n)$. Then from inequality condition

$$\psi(d_{n+1}) \leq \phi(d_n) + (\psi - \phi)(p).$$

So $\psi(d_{n+1}) - \psi(d_n) \leq \phi(d_n) + (\psi - \phi)(p) - \psi(d_n) \leq (\psi - \phi)(p) - (\psi - \phi)(d_n) \leq 0 \implies d_{n+1} \leq d_n$ for all $n \geq 1$. By the regularity of P, there exists $q \in P$ such that $lim_{n\to\infty} d_n = q$. Now, let $\theta \ll c \in P$. Then, there exists a positive real number r such that $c + N_r(\theta) \subseteq P$, where

$$N_r(\theta) = \{t \in E : \parallel t - \theta \parallel < r\}.$$

Choose a natural number n_0 such that $\theta \leq d_n - q \leq c$ for all $n \geq n_0$. So

$$\theta \leq \phi(d_n) \leq \phi(q + c) \leq \phi(q) + \phi(c) \leq \phi(q) + \psi(c) \leq \phi(q) + c, \forall n \geq n_0.$$

And since $\phi(q) \leq \phi(d_n), \forall n \geq n_0 \implies lim_{n\to\infty} \phi(d_n) = \phi(q)$. Again

$$\begin{aligned} &\psi(d_{n+1}) \leq \phi(d_n) + (\psi - \phi)(p) \\ &\implies lim_{n\to\infty} \psi(d_{n+1}) - lim_{n\to\infty} \phi(d_n) \leq (\psi - \phi)(p) \\ &\implies (\psi - \phi)(q) \leq (\psi - \phi)(p). \end{aligned}$$

As $(\psi - \phi)$ is strictly increasing, $q \leq p$ and since $p \leq d_n$ for all $n \geq 1$, we have $p \leq q \implies p = q$. This implies that $dis(A, B) = p$.

Further we give two results, which extend Theorem 5.

Theorem 12. *Let $\psi : P \longrightarrow P$ be a cone L-function and $T : A \cup B \longrightarrow A \cup B$ be a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $d(Tx, Ty) - p < \psi(\frac{1}{2}(d(Tx, x) + d(Ty, y)) - p)$, for all $(x, y) \in A \times B$, where p is a lower bound for $A \times B$. Then $dis(A, B) = p$.*

Proof. Let $x_0 \in A \cup B$ and set $x_n = Tx_{n-1}$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$. Then from inequality condition, we have

$$\begin{aligned} &d_{n+1} - p < \psi(\frac{1}{2}(d_{n+1} + d_n) - p) \\ \implies \quad &d_{n+1} - p \leq \frac{1}{2}(d_{n+1} + d_n) - p \quad (\because \psi(s) \leq s \ \forall s \in P \quad \text{with} \quad s \neq \theta) \\ \implies &\frac{1}{2} d_{n+1} \leq \frac{1}{2} d_n \implies d_{n+1} \leq d_n. \end{aligned}$$

Let $s_n = d_n - p$. Then $s_{n+1} \leq s_n$. Now,

$$s_{n+1} < \psi(\frac{1}{2}(d_{n+1} - p) + \frac{1}{2}(d_n - p))$$
$$\Longrightarrow \; s_{n+1} < \psi(\frac{1}{2}(s_{n+1} + s_n)) \leq \frac{1}{2}(s_{n+1} + s_n) \leq s_n.$$

Since s_n is convergent, we have, $lim_{n\to\infty}\psi(\frac{1}{2}(s_{n+1} + s_n)) = lim_{n\to\infty}s_n = q$. We claim that $q = \theta$. If not, then since s_n is a strictly decreasing sequence $q < s_n, \forall n$. As ψ is a cone L-function and $q \neq \theta$, there exists $\delta_q \gg \theta$ such that $\psi(t) \leq q, \forall q \leq t \leq q + \delta_q$. Since $lim_{n\to\infty}s_n = q$, there exists a natural number n_0 such that $q < s_n < q + \delta_q, \forall n \geq n_0$. Then $\psi(\frac{1}{2}(s_{n+1} + s_n)) \leq q, \forall n \geq n_0$, which contradicts that $q \leq s_{n+1} < \psi(\frac{1}{2}(s_{n+1} + s_n))$. Hence, $q = \theta$ implies $lim_{n\to\infty}d_n = p$ and so $dis(A, B) = p$.

Theorem 13. *Let $\psi : P \longrightarrow P$ be a cone L-function and $T : A \cup B \longrightarrow A \cup B$ be a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and $d(Tx, Ty) - p < \psi(\frac{1}{2}(d(Tx, y) + d(Ty, x)) - p)$, for all $(x, y) \in A \times B$, where p is a lower bound for $A \times B$. Then $dis(A, B) = p$.*

Proof. Let $x_0 \in A \cup B$ and set $x_n = Tx_{n-1}$ and $d_{n+1} = d(x_{n+1}, x_n)$ for all $n \geq 1$. Then putting $x = x_n, y = x_{n-1}$ in the inequality condition, we have

$$\begin{aligned}
&d(x_{n+1}, x_n) - p < \psi(\frac{1}{2}(d(x_{n+1}, x_{n-1}) + d(x_n, x_n)) - p)\\
\Longrightarrow\; & d_{n+1} - p < \psi(\frac{1}{2}(d(x_{n+1}, x_n) + d(x_n, x_{n-1})) - p)\\
\Longrightarrow\; & d_{n+1} - p < \psi(\frac{1}{2}(d_{n+1} + d_n) - p)\\
\Longrightarrow\; & d_{n+1} - p \leq \frac{1}{2}(d_{n+1} + d_n) - p \quad (\because \psi(s) \leq s \; \forall s \in P \quad \text{with} \quad s \neq \theta)\\
\Longrightarrow\; & \frac{1}{2}d_{n+1} \leq \frac{1}{2}d_n \implies d_{n+1} \leq d_n.
\end{aligned}$$

The rest of the proof is the same as in Theorem 12, so we omit it here.

Now we give the result that guarantees the existence of best proximity point in regular cone metric space.

Theorem 14. *Suppose that the conditions of Theorems 7, 8, 10, 12 and 13 hold, $x_0 \in A$ and $x_n = Tx_{n-1}$ for all $n \geq 1$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x \in A$ such that $d(x, Tx) = dis(A, B)$.*

Proof. Let $\{x_{2n_k}\}$ be the convergent subsequence of $\{x_{2n}\}$ in A. Choose $x \in A$ such that $\lim_{n\to\infty}x_{2n_k} = x$. Note that the relation

$$p = dis(A, B) \leq d(x, x_{2n_k-1}) \leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1})$$

holds for all $k \geq 1$. Since $\{d(x_{2n_k}, x_{2n_k-1})\}$ is a subsequence of $\{d_n\}$, we have $d(x_{2n_k}, x_{2n_k-1}) \longrightarrow p$. Hence,

$$lim_{n\to\infty}d(x, x_{2n_k-1}) = p.$$

Firstly suppose that the conditions of Theorems 7 and 12 hold. Then in both cases, the inequality condition implies

$$d(Tx,Ty) \leq \frac{1}{2}\{d(Tx,x)+d(Ty,y)\}.$$

Now,

$$\begin{aligned} d(x,Tx) &\leq d(x,x_{2n_k})+d(x_{2n_k},Tx) \\ &\leq d(x,x_{2n_k})+\frac{1}{2}\{d(x_{2n_k},x_{2n_k-1})+d(Tx,x)\}. \end{aligned}$$

This implies

$$\begin{aligned} & d(x,Tx) \leq lim_{n\to\infty}\{d(x,x_{2n_k})+\frac{1}{2}\{d_{2n_k-1}+d(Tx,x)\}\} \\ & \implies d(x,Tx) \leq p. \end{aligned}$$

Now, suppose that the conditions of Theorems 8 and 13 hold. Then in both cases, the inequality condition implies

$$d(Tx,Ty) \leq \frac{1}{2}\{d(Tx,y)+d(Ty,x)\}.$$

Now,

$$\begin{aligned} d(x,Tx) &\leq d(x,x_{2n_k})+d(x_{2n_k},Tx) \\ &\leq d(x,x_{2n_k})+\frac{1}{2}\{d(x_{2n_k},x)+d(Tx,x_{2n_k-1})\} \\ &\leq \frac{3}{2}d(x,x_{2n_k})+\frac{1}{2}\{d(x,Tx)+d(x,x_{2n_k-1})\}. \end{aligned}$$

This implies

$$\begin{aligned} & d(x,Tx) \leq lim_{n\to\infty}\left\{\frac{3}{2}d(x,x_{2n_k})+\frac{1}{2}\left\{d(x,Tx)+d(x,x_{2n_k-1})\right\}\right\} \\ & \implies d(x,Tx) \leq \frac{1}{2}(d(x,Tx)+p) \\ & \implies d(x,Tx) \leq p. \end{aligned}$$

So, $d(x,Tx) \leq dis(A,B)$. As $p = dis(A,B)$, we have $p \leq d(x,Tx)$. Hence $d(x,Tx) = dis(A,B) = p$.

Now, suppose that the conditions of Theorem 10 hold. Then the proof is the same as that of Theorem 6.

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