Azerbaijan Journal of Mathematics
V. 5, No 1, 2015, January
ISSN 2218-6816

On Second Order Mixed Functional Integrodifferential Equations

K. D. Kucche*, M. B. Dhakne

Abstract. This paper is concerned with the existence, uniqueness and continuous dependence of mild solution of second order mixed Volterra-Fredholm functional integrodifferential equations. The results are established by an application of the topological transversality theorem, Pachpatte's inequality and the theory of strongly continuous cosine family of operators.

Key Words and Phrases: Mixed functional equations, existence, uniqueness, continuous dependence, fixed point theorem, integral inequality, cosine family of operators.

2010 Mathematics Subject Classifications: Primary: 45N05, Secondary: 47D09, 47H10.

1. Introduction

In this paper, we prove the existence, uniqueness and continuous dependence of mild solution of second order mixed Volterra-Fredholm functional integrodifferential equations of the form:

$$x''(t) = Ax(t) + f\left(t,\, x_t,\, x'(t), \int_0^t g(s, x_s, x'(s))ds, \int_0^b h(s, x_s, x'(s))ds\right),\ t \in J, \tag{1}$$

$$x_0 = \phi \in C,\ x'(0) = \xi \in X, \tag{2}$$

where $J = [0, b]$, A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in a Banach space X with the norm $\|\cdot\|$, and $f : J \times C \times X \times X \times X \to X$, $g,\ h : J \times C \times X \to X$ are given functions. Here $C = C([-r, 0], X),\ 0 < r < \infty$, is the Banach space of all continuous functions $\psi : [-r, 0] \to X$ endowed with supremum norm

$$\|\psi\|_C = \sup_{-r\le\theta\le 0} \|\psi(\theta)\|.$$

For any $x \in C([-r, b], X)$ and $t \in J$, x_t denotes the element of C given by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

*Corresponding author.

There have appeared a series of research results concerning the problem of existence for various special forms of (1)-(2), see for example, Balachandran et al. [1, 2], Ntouyas [14], R. Ye and G. Zhang [21], Tidke and Dhakne [17, 16] and the references cited therein.

Authors [4, 6, 13] have obtained the results pertaining to existence and qualitative properties of mild solutions for semilinear mixed functional integrodifferential equations of first and second order using different approaches. Further the controllability results for second order equations [11, 12] and non-densely defined systems [10] have been established.

Using the tool of Pachpatte's inequality given in the Lemma 3 we prove the existence, uniqueness and continuous dependence of mild solution of second order initial value problem (1)-(2) with minimum restrictions on the functions f, g and h.

The special forms of equations (1)-(2) serve as an abstract formulation of partial differential equations or partial integrodifferential equations which are appearing in different physical problems, such as the transverse motion of an extensible beam, the vibration of hinged bars, and many other physical phenomena [3, 9].

2. Preliminaries and Hypotheses

To study the system (1)-(2), we consider the space $B = C([-r, b], X) \cap C^1([0, b], X)$ with the norm

$$\|x\|_B = \max\{\|x\|_1, \|x\|_2\},\ x \in B,$$

where $\|x\|_1 = \sup\{\|x(t)\| : -r \le t \le b\}$ and $\|x\|_2 = \sup\{\|x'(t)\| : 0 \le t \le b\}$.

Here we introduce some definitions and preliminaries from [8, 18, 19] and hypotheses that will be used in our subsequent discussion.

Definition 1. *A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators mapping the Banach space X into itself is called strongly continuous cosine family if*

(a) $C(0) = I$ (I is the identity operator);

(b) $C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in X$;

(c) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

The associated strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x = \int_0^t C(s)xds,\ x \in X,\ t \in \mathbb{R}.$$

The infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2}C(t)x\Big|_{t=0},\ x \in D(A),$$

where

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}.$$

Throughout this paper, we assume that A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in a Banach space X, and the adjoint operator A^* is densely defined, that is, $D(A^*) = X^*$.

Lemma 1 ([19]). *Let $C(t)$, (resp. $S(t)$), $t \in \mathbb{R}$ be a strongly continuous cosine (resp. sine) family on X. Then:*

(*i*) *there exist constants $N \geq 1$ and $\omega \geq 0$ such that*

$$\|C(t)\| \leq Ne^{\omega|t|}, \ \textit{for } t \in \mathbb{R},$$

$$\|S(t_1) - S(t_2)\| \leq N \left| \int_{t_1}^{t_2} e^{\omega|s|} ds \right|, \ \textit{for } t_1, t_2 \in \mathbb{R}.$$

(*ii*) $\frac{d}{dt}C(t)x = AC(t)x$ *for* $x \in E = \{y \in X : C(t)y$ *is once continuously differentiable in* $t\}$.

Definition 2. *A mild solution of second-order abstract Cauchy problem (see[18, 19])*

$$x''(t) = Ax(t) + w(t), \ \sigma \leq t \leq \mu \tag{3}$$

$$x(\sigma) = z_1, \ x'(\sigma) = z_2, \tag{4}$$

where $w : [\sigma, \mu] \to X$ is an integrable function, is the function $x(.)$ given by

$$x(t) = C(t-\sigma)z_1 + S(t-\sigma)z_2 + \int_\sigma^t S(t-s)w(s)ds, \ \sigma \leq t \leq \mu.$$

Comparing with the abstract Cauchy problem (3)-(4), we have the following definition of mild solution for the initial value problem (1)-(2).

Definition 3. *Let $f \in L^1(0, b; X)$. Then we say that $x \in B = C([-r, b], X) \cap C^1([0, b], X)$ is a mild solution of the problem* (1)-(2) *if $x_0 = \phi$ and the integral equation*

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\xi \\ &+ \int_0^t S(t-s)f\left(s, \, x_s, \, x'(s), \int_0^s g(\tau, x_\tau, x'(\tau))d\tau, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right) ds, \ t \in J, \end{aligned}$$

is satisfied.

To establish our main results, we will use the following Lemmas.

Lemma 2 ([7], p. 61). *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let*

$$\varepsilon(F) = \{x \in S : x = \lambda Fx \ \textit{for some } 0 < \lambda < 1\}.$$

Then either $\varepsilon(F)$ is unbounded or F has a fixed point.

Lemma 3 ([15], p-47)**.** *Let $z(t),\ u(t),\ v(t),\ w(t) \in C([\alpha,\beta], R_+)$, $k \geq 0$ be a real constant and*

$$z(t) \leq k + \int_\alpha^t u(s)\left[z(s) + \int_\alpha^s v(\sigma)z(\sigma)d\sigma + \int_\alpha^\beta w(\sigma)z(\sigma)d\sigma\right] ds, \ \textit{for } t \in [\alpha,\beta].$$

If

$$r^* = \int_\alpha^\beta w(\sigma)\exp\left(\int_\alpha^\sigma [u(\tau)+v(\tau)]d\tau\right) d\sigma < 1,$$

then

$$z(t) \leq \frac{k}{1-r^*}\exp\left(\int_\alpha^t [u(s)+v(s)]ds\right), \ \textit{for } t \in [\alpha,\beta].$$

We list the following hypotheses.

(H_1) There exist continuous functions $k,\ l,\ m : J \to \mathbb{R}_+$ such that

(i) $\|f(t,\psi,x,y,z)\| \leq k(t)(\|\psi\|_C + \|x\| + \|y\| + \|z\|)$;

(ii) $\|g(t,\psi,x)\| \leq l(t)(\|\psi\|_C + \|x\|)$;

(iii) $\|h(t,\psi,x)\| \leq m(t)(\|\psi\|_C + \|x\|)$;

for every $t \in J,\ \psi \in C$ and $x,\ y,\ z \in X$.

(H_2) For each $t \in J$ the function $f(t,\cdot,\cdot,\cdot,\cdot) : C \times X \times X \times X \to X$ is continuous and for each $(\psi,\ x,\ y,\ z) \in C \times X \times X \times X$ the function $f(\cdot,\psi,\ x,\ y,\ z) : J \to X$ is strongly measurable.

(H_3) For each $t \in J$ the functions $g(t,\cdot,\cdot),\ h(t,\cdot,\cdot) : C \times X \to X$ are continuous and for each $(\psi,\ x) \in C \times X$ the functions $g(.,\psi,\ x), h(.,\psi,\ x) : C \times X \to X$ are strongly measurable.

(H_4) For each positive integer $m > 0$, there exists $\alpha_m \in L^1(J,\mathbb{R}_+)$ such that

$$\|f(t,\psi,\ x,\ y,\ z)\| \leq \alpha_m(t), \ \text{for all } \ \|\psi\|_C \leq m,\ \|x\| \leq m,\ \|y\| \leq m,\ \|z\| \leq m$$

and for almost all $t \in J$.

(H_5) $C(t),\ t > 0$ is compact.

3. Existence Results

Theorem 1. *If the hypotheses (H_1)-(H_5) are satisfied, then the system* (1)-(2) *has at least one mild solution on* $[-r, b]$, *provided*

$$r = \int_0^b m(\sigma) \exp\left(\int_0^\sigma [M(1+b)k(\tau) + l(\tau)]d\tau\right) d\sigma < 1, \tag{5}$$

where $M = \sup\{\|C(t)\| : t \in J\}$.

Proof. In view of Lemma 2, firstly we obtain the priori bounds for the mild solution of the equation

$$x(t) = \lambda F x(t),\ 0 < \lambda < 1, \tag{6}$$

where the operator $F : B \to B$ is defined by

$$(Fx)(t) = \begin{cases} \phi(t), \quad t \in [-r, 0], \\ C(t)\phi(0) + S(t)\xi + \int_0^t S(t-s) \times \\ f\left(s,\, x_s,\, x'(s), \int_0^s g(\tau, x_\tau, x'(\tau))d\tau, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right) ds,\ t \in J. \end{cases} \tag{7}$$

Using the hypothesis (H_1), from the equation (6), for each $t \in J$, we have

$$\begin{aligned} \|x(t)\| \leq & \|C(t)\|\|\phi(0)\| + \|S(t)\|\|\xi\| + \int_0^t \|S(t-s)\| \times \\ & \left\| f\left(s,\, x_s,\, x'(s), \int_0^s g(\tau, x_\tau, x'(\tau))d\tau, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right)\right\| ds \\ \leq & M\|\phi\|_C + Mb\|\xi\| + \int_0^t Mb\, k(s) \times \\ & \left[\|x_s\|_C + \|x'(s)\| + \int_0^s l(\tau)(\|x_\tau\|_C + \|x'(\tau)\|)d\tau + \int_0^b m(\tau)(\|x_\tau\|_C + \|x'(\tau)\|)d\tau\right] ds. \end{aligned}$$

Consider the function μ_1 given by

$$\mu_1(t) = \sup\{\|x(s)\| : -r \leq s \leq t\},\ t \in J.$$

Let $t^* \in [-r, t]$ be such that $\mu_1(t) = \|x(t^*)\|$. If $t^* \in [0, t]$, by the previous inequality we have

$$\begin{aligned} \mu_1(t) \leq & M\|\phi\|_C + Mb\|\xi\| + \int_0^t Mb\, k(s) \times \\ & \left[\mu_1(s) + \|x'(s)\| + \int_0^s l(\tau)(\mu_1(\tau) + \|x'(\tau)\|)d\tau + \int_0^b m(\tau)(\mu_1(\tau) + \|x'(\tau)\|)d\tau\right] ds \\ \leq & M\|\phi\|_C + Mb\|\xi\| + \int_0^t Mb\, k(s) \times \end{aligned}$$

$$\left[\mu_1(s)+\mu_2(s)+\int_0^s l(\tau)(\mu_1(\tau)+\mu_2(\tau))d\tau+\int_0^b m(\tau)(\mu_1(\tau)+\mu_2(\tau))d\tau\right]ds, \quad (8)$$

where $\mu_2(t)=\sup\{\|x'(s)\|:0\le s\le t\}$, $t\in J$. If $t^*\in[-r,0]$ then $\mu_1(t)\le\|\phi\|_C$ and the previous inequality obviously holds since $M\ge 1$. But for $t\in J$, using the equation (6) and (7) we have

$$\begin{aligned}x'(t)=AS(t)\phi(0)+C(t)\xi+\int_0^t C(t-s)\times\\ f\left(s,\,x_s,\,x'(s),\int_0^s g(\tau,x_\tau,x'(\tau))d\tau,\int_0^b h(\tau,x_\tau,x'(\tau))d\tau\right)ds,\end{aligned}$$

and using hypothesis (H_1), we obtain

$$\begin{aligned}&\|x'(t)\|\le\|AS(t)\|\|\phi(0)\|+\|C(t)\|\|\xi\|+\int_0^t\|C(t-s)\|k(s)\times\\ &\left[\|x_s\|_C+\|x'(s)\|+\int_0^s l(\tau)(\|x_\tau\|_C+\|x'(\tau)\|)d\tau+\int_0^b m(\tau)(\|x_\tau\|_C+\|x'(\tau)\|)d\tau\right]ds.\end{aligned}$$

Let $t^{**}\in[0,t]$ be such that $\mu_2(t)=\|x(t^{**})\|$, and let $L=\sup\{\|AS(t)\|:t\in J\}$. Then we have

$$\begin{aligned}&\mu_2(t)\le L\|\phi\|_C+M\|\xi\|+\int_0^t Mk(s)\times\\ &\left[\mu_1(s)+\mu_2(s)+\int_0^s l(\tau)(\mu_1(\tau)+\mu_2(\tau))d\tau+\int_0^b m(\tau)(\mu_1(\tau)+\mu_2(\tau))d\tau\right]ds. \quad (9)\end{aligned}$$

From (8) and (9), we obtain

$$\begin{aligned}&\mu_1(t)+\mu_2(t)\\ &\le(L+M)\|\phi\|_C+M(1+b)\|\xi\|+M(1+b)\int_0^t k(s)\times\\ &\left[\mu_1(s)+\mu_2(s)+\int_0^s l(\tau)(\mu_1(\tau)+\mu_2(\tau))d\tau+\int_0^b m(\tau)(\mu_1(\tau)+\mu_2(\tau))d\tau\right]ds. \quad (10)\end{aligned}$$

By applying Pachpatte's inequality given in Lemma 3 with $z(t)=\mu_1(t)+\mu_2(t)$ and using condition (5), we obtain

$$\begin{aligned}\mu_1(t)+\mu_2(t)&\le\frac{(L+M)\|\phi\|_C+M(1+b)\|\xi\|}{1-r}\exp\left(\int_0^t[M(1+b)k(s)+l(s)]ds\right)\\ &\le\frac{(L+M)\|\phi\|_C+M(1+b)\|\xi\|}{1-r}\exp\left(\int_0^b[M(1+b)k(s)+l(s)]ds\right):=K.\end{aligned}$$

Therefore $\|x(t)\|_1\le\mu_1(t)\le K$ and $\|x'(t)\|_2\le\mu_2(t)\le K$, $t\in J$, and hence $\|x\|_B\le K$.

Now we prove that the operator F defined in (7) is completely continuous. Let $B_k = \{x \in B : \|x\|_B \leq k,\ t \in J\}$ for $k \geq 1$. Firstly we show that F maps B_k into an equicontinuous family. Let $x \in B_k,\ t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq b$, from (7), using the hypothesis (H_1), we get

$$
\begin{aligned}
&\|(Fx)(t_1) - (Fx)(t_2)\| \\
&\leq \|[C(t_1) - C(t_2)]\phi(0)\| + \|[S(t_1) - S(t_2)]\xi\| \\
&\quad + \left\| \int_0^{t_1} [S(t_1 - s) - S(t_2 - s)] \right. \\
&\qquad \left. \times f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right) ds \right\| \\
&\quad + \left\| \int_{t_1}^{t_2} S(t_2 - s) f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right) ds \right\| \\
&\leq \|[C(t_1) - C(t_2)]\| \|\phi\|_C + \|S(t_1) - S(t_2)\| \|\xi\| \\
&\quad + \int_0^{t_1} \|S(t_1 - s) - S(t_2 - s)\| \alpha_{m_0}(s) ds + \int_{t_1}^{t_2} \|S(t_2 - s)\| \alpha_{m_0}(s) ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(Fx)'(t_1) - (Fx)'(t_2)\| \\
&\leq \|A[S(t_1) - S(t_2)]\phi(0)\| + \|[C(t_1) - C(t_2)]\xi\| \\
&\quad + \left\| \int_0^{t_1} [C(t_1 - s) - C(t_2 - s)] \right. \\
&\qquad \left. \times f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right) ds \right\| \\
&\quad + \left\| \int_{t_1}^{t_2} C(t_2 - s) f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right) ds \right\| \\
&\leq \|A[S(t_1) - S(t_2)]\| \|\phi\|_C + \|C(t_1) - C(t_2)\| \|\xi\| \\
&\quad + \int_0^{t_1} \|C(t_1 - s) - C(t_2 - s)\| \alpha_{m_0}(s) ds + \int_{t_1}^{t_2} \|C(t_2 - s)\| \alpha_{m_0}(s) ds,
\end{aligned}
$$

where

$$m_0 = 2k \max\{1 + l(t) + m(t) : t \in J\}.$$

We know $C(t), S(t)$ are uniformly continuous for $t \in J$ and the compactness of $C(t), S(t)$ for $t > 0$ implies the continuity in the uniform operator topology. The compactness of $S(t)$ follows from that of $C(t)$, Lemma 2.1 and Lemma 2.5 of [20]. Further the right hand sides of the above inequalities are independent of $x \in B_k$. Therefore $\|(Fx)(t_1) - (Fx)(t_2)\| \to 0$ and $\|(Fx)'(t_1) - (Fx)'(t_2)\| \to 0$ as $(t_2 - t_1) \to 0$. Thus F maps B_k into an equicontinuous family of functions. The equicontinuity for the cases $t_1 \leq t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ follows from the uniform continuity of ϕ on $[-r, 0]$ and from the relation

$$\|(Fy)(t_1) - (Fy)(t_2)\| \leq \|\phi(t_1) - \phi(0)\| + \|(Fy)(0) - (Fy)(t_2)\|,$$

respectively.

It is easy to verify that FB_k is uniformly bounded and hence the details are omitted. Since we have proved that FB_k is an equicontinuous family, to prove $\overline{FB_k}$ is compact, in view of the Arzela-Ascoli theorem, it suffices to show that the set $V(t) = \{(Fx)(t) : x \in B_k\}$ is precompact in X for each $t \in [-r, b]$. This is trivial for $t \in [-r, 0]$, since in this case $V(t) = \{\phi(t)\}$ is singleton set. So let $0 < t \leq b$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in B_k$, we define

$$(F_\epsilon x)(t) = C(t)\phi(0) + S(t)\xi \\ + \int_0^{t-\epsilon} S(t-s) f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right) ds, t \in J.$$

Since $C(t)$ and $S(t)$ are compact operators, the set $V\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_k\}$ is relatively compact in X for every $\epsilon,\ \ 0 < \epsilon < t$. Moreover, by making use of hypothesis (H_4), for every $x \in B_k$ we have

$$\begin{aligned} &\|(Fx)(t) - (F_\epsilon x)(t)\| \\ &= \int_{t-\epsilon}^t \left\| S(t-s) f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right)\right\| ds \\ &\leq \int_{t-\epsilon}^t \|S(t-s)\| \alpha_{m_0}(s) ds \end{aligned}$$

and

$$\begin{aligned} &\|(Fx)'(t) - (F_\epsilon x)'(t)\| \\ &= \int_{t-\epsilon}^t \left\| C(t-s) f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right)\right\| ds \\ &\leq \int_{t-\epsilon}^t \|C(t-s)\| \alpha_{m_0}(s) ds. \end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $V(t)$; hence the set $V(t)$ is also precompact in X. Finally we prove that $F : B \to B$ is continuous. Let $\{x_n\} \subseteq B$ with $x_n \to x$ in B. By using hypotheses (H_2) and (H_3), we have

$$\begin{aligned} &f\left(t,\, x_{n_t},\, x_n'(t),\, \int_0^t g(s, x_{n_s} x_n'(s))ds,\, \int_0^b h(s, x_{n_s} x_n'(s))ds\right) \\ &\quad \to f\left(t,\, x_t,\, x'(t),\, \int_0^t g(s, x_s, x'(s))ds,\, \int_0^b h(s, x_s, x'(s))ds\right) \quad \text{as } n \to \infty, \end{aligned}$$

for each $t \in J$ and since

$$\left\| f\left(t,\, x_{n_t},\, x_n'(t),\, \int_0^t g(s, x_{n_s} x_n'(s))ds,\, \int_0^b h(s, x_{n_s} x_n'(s))ds\right)\right.$$

$$- f\left(t,\, x_t,\, x'(t),\, \int_0^t g(s, x_s, x'(s))ds,\, \int_0^b h(s, x_s, x'(s))ds\right)\Big\| \le 2\alpha_{m_0}(s),$$

where $m_0 = 2k \max\{1 + l(t) + m(t) : t \in J\}$, we have by dominated convergence theorem,

$$\begin{aligned}
&\|(Fx_n)(t) - (Fx)(t)\| \\
&\le \int_0^b \|S(t-s)\| \Big\| f\left(s,\, x_{n_s},\, x_n'(s),\, \int_0^s g(\tau, x_{n_\tau}, x_n'(\tau))d\tau,\, \int_0^b h(\tau, x_{n_\tau}, x_n'(\tau))d\tau\right) \\
&\quad - f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right)\Big\| ds \\
&\to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\|(Fx_n)'(t) - (Fx)'(t)\| \\
&\le \int_0^b \|C(t-s)\| \Big\| f\left(s,\, x_{n_s},\, x_n'(s),\, \int_0^s g(\tau, x_{n_\tau}, x_n'(\tau))d\tau,\, \int_0^b h(\tau, x_{n_\tau}, x_n'(\tau))d\tau\right) \\
&\quad - f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right)\Big\| ds \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, $\|Fx_n - Fx\|_B \to 0$ as $n \to \infty$ and hence F is continuous. This completes the proof that F is completely continuous operator.

Moreover, the set $\varepsilon(F) = \{y \in B : x = \lambda Fx,\, 0 < \lambda < 1\}$ is bounded, as we proved in the first part. Consequently, by Lemma 2, the operator F has a fixed point $\tilde{x}$ in B. This means that the initial value problem peoblem (1)-(2) has a mild solution on $[-r, b]$, completing the proof of the theorem. ◄

4. Continuous Dependence and Uniqueness

Theorem 2. *Suppose that the functions f, g and h in (1) satisfy the following conditions:*

(i) $\|f(t, \psi, x, y, z) - f(t, \varphi, u, v, w)\| \le p(t)(\|\psi - \varphi\|_C + \|x - u\| + \|y - v\| + \|z - w\|)$;

(ii) $\|g(t, \psi, x) - g(t, \varphi, u)\| \le q(t)(\|\psi - \varphi\|_C + \|x - u\|)$;

(iii) $\|h(t, \psi, x) - h(t, \varphi, u)\| \le r(t)(\|\psi - \varphi\|_C + \|x - u\|)$,

where $\psi, \varphi \in C$, $x, y, z, u, v, z \in X$ and p, q, $r : J \to [0, \infty)$ are continuous functions. Let x and y be the mild solutions of (1) corresponding to initial conditions $x_0 = \phi_x$, $x'(0) = \xi_x$ and $y_0 = \phi_y$, $y'(0) = \xi_y$, respectively, where $(\phi_x,\ \xi_x)$, $(\phi_y,\ \xi_y) \in C \times X$. Then

$$\|x - y\|_B \le Q\left[(L + M)\|\phi_x - \phi_y\|_C + M(1 + b)\|\xi_x - \xi_y\|\right], \tag{11}$$

where

$$Q = \frac{\exp\left(\int_0^b [M(1+b)p(s)+q(s)]ds\right)}{1-\zeta},$$

$$\zeta = \int_0^b r(\sigma)\exp\left(\int_0^\sigma [M(1+b)p(\tau)+q(\tau)]d\tau\right)d\sigma < 1, \tag{12}$$

$M = \sup\{\|C(t)\| : t \in J\}$ *and* $L = \sup\{\|AS(t)\| : t \in J\}$.

Proof. Using the conditions (i)-(iii), for $t \in J$ we get

$$\begin{aligned}
\|x(t)-y(t)\| \leq & \|C(t)\|\|\phi_x(0)-\phi_y(0)\| + \|S(t)\|\|\xi_x-\xi_y\| + \int_0^t \|S(t-s)\|\times \\
& \left(\left\|f\left(s,\, x_s,\, x'(s), \int_0^s g(\tau,x_\tau,x'(\tau))d\tau, \int_0^b h(\tau,x_\tau,x'(\tau))d\tau\right)\right\|\right. \\
& \left. - \left\|f\left(s,\, y_s,\, y'(s), \int_0^s g(\tau,y_\tau,y'(\tau))d\tau, \int_0^b h(\tau,y_\tau,y'(\tau))d\tau\right)\right\|\right) ds \\
\leq & M\|\phi_x-\phi_y\|_C + Mb\|\xi_x-\xi_y\| \\
& + \int_0^t Mb\,p(s)\Big[\|x_s-y_s\|_C + \|x'(t)-y'(t)\| \\
& + \int_0^s q(\tau)(\|x_\tau-y_\tau\|_C + \|x'(\tau)-y'(\tau)\|)d\tau \\
& + \int_0^b r(\tau)(\|x_\tau-y_\tau\|_C + \|x'(\tau)-y'(\tau)\|)d\tau\Big] ds.
\end{aligned}$$

Define $\vartheta_1(t) = \sup\{\|(x-y)(s)\| : -r \leq s \leq t\}$, $t \in J$ and let $\sigma \in [-r,t]$ be such that $\vartheta_1(t) = \|(x-y)(\sigma)\|$. If $\sigma \in [0,t]$, by the previous inequality we have

$$\begin{aligned}
\vartheta_1(t) \leq\ & M\|\phi_x-\phi_y\|_C + Mb\|\xi_x-\xi_y\| \\
& + \int_0^t Mb\,p(s)\Big[\vartheta_1(s) + \|(x-y)'(t)\| \\
& + \int_0^s q(\tau)(\vartheta_1(\tau) + \|(x-y)'(\tau)\|)d\tau \\
& + \int_0^b r(\tau)(\vartheta_1(\tau) + \|(x-y)'(\tau)\|)d\tau\Big] ds \\
\leq\ & M\|\phi_x-\phi_y\|_C + Mb\|\xi_x-\xi_y\| + \int_0^t Mb\,p(s)\times \\
& \left[\vartheta_1(s) + \int_0^s q(\tau)(\vartheta_1(\tau)+\vartheta_2(\tau))d\tau + \int_0^b r(\tau)(\vartheta_1(\tau)+\vartheta_2(\tau))d\tau\right] ds,
\end{aligned} \tag{13}$$

where $\vartheta_2(t) = \sup\{\|(x-y)'(s)\| : 0 \leq s \leq t\}$, $t \in J$. If $\sigma \in [-r,0]$, then $\vartheta_1(t) \leq \|\phi_x-\phi_y\|_C$ and the previous inequality obviously holds since $M \geq 1$.

But we see that for $t \in J$ we have

$$\|(x-y)'(t)\| \le \|AS(t)\|\|\phi_x(0)-\phi_y(0)\| + \|C(t)\|\|\xi_x-\xi_y\| + \int_0^t \|C(t-s)\|\times$$
$$\left(\left\|f\left(s,\, x_s,\, x'(s),\, \int_0^s g(\tau, x_\tau, x'(\tau))d\tau,\, \int_0^b h(\tau, x_\tau, x'(\tau))d\tau\right)\right\|\right.$$
$$\left.-\left\|f\left(s,\, y_s,\, y'(s),\, \int_0^s g(\tau, y_\tau, y'(\tau))d\tau,\, \int_0^b h(\tau, y_\tau, y'(\tau))d\tau\right)\right\|\right) ds$$
$$\le L\|\phi_x-\phi_y\|_C + M\|\xi_x-\xi_y\| + \int_0^t M\, p(s)\Big[\|x_s-y_s\|_C + \|x'(t)-y'(t)\|$$
$$+\int_0^s q(\tau)(\|x_\tau-y_\tau\|_C + \|x'(\tau)-y'(\tau)\|)d\tau$$
$$\left.+\int_0^b r(\tau)(\|x_\tau-y_\tau\|_C + \|x'(\tau)-y'(\tau)\|)d\tau\right] ds.$$

If $\sigma^* \in [0,t]$ is such that $\vartheta_2(t) = \|x(\sigma^*)\|$, then we get

$$\vartheta_2(t) \le L\|\phi_x-\phi_y\|_C + M\|\xi_x-\xi_y\| + \int_0^t M\, p(s)\times$$
$$\left[\vartheta_1(s)+\vartheta_2(s) + \int_0^s q(\tau)(\vartheta_1(\tau)+\vartheta_2(\tau))d\tau + \int_0^b r(\tau)(\vartheta_1(\tau)+\vartheta_2(\tau))d\tau\right] ds. \tag{14}$$

From equations (13) and (14), we get

$$\vartheta_1(t)+\vartheta_2(t)$$
$$\le (L+M)\|\phi_x-\phi_y\|_C + M(1+b)\|\xi_x-\xi_y\| + M(1+b)\int_0^t p(s)\times$$
$$\left[\vartheta_1(s)+\vartheta_2(s) + \int_0^s q(\tau)(\vartheta_1(\tau)\| + \vartheta_2(\tau))d\tau + \int_0^b r(\tau)(\vartheta_1(\tau)\| + \vartheta_2(\tau))d\tau\right] ds. \tag{15}$$

An application of Pachpatte's inequality given in Lemma 3 to (15) with $z(t) = \vartheta_1(t)+\vartheta_2(t)$ and condition (12) yields

$$\vartheta_1(t)+\vartheta_2(t) \le \frac{(L+M)\|\phi_x-\phi_y\|_C + M(1+b)\|\xi_x-\xi_y\|}{1-\zeta}\exp\left(\int_0^t [M(1+b)p(s)+q(s)]ds\right)$$
$$\le Q\left[(L+M)\|\phi_x-\phi_y\|_C + M(1+b)\|\xi_x-\xi_y\|\right].$$

This implies

$$\|(x-y)(t)\|_1 \le \vartheta_1(t) \le Q\left[(L+M)\|\phi_x-\phi_y\|_C + M(1+b)\|\xi_x-\xi_y\|\right]$$

and

$$\|(x-y)'(t)\|_2 \le \vartheta_2(t) \le Q\left[(L+M)\|\phi_x-\phi_y\|_C + M(1+b)\|\xi_x-\xi_y\|\right],$$

and hence inequality (11) holds.◄

Remark 1. *Note that the Theorem 2 not only gives the continuous dependence of mild solutions of initial value problem* (1)-(2) *on initial data, but also gives the uniqueness of mild solutions, as* $(\phi_x,\ \xi_x) = (\phi_y,\ \xi_y) \in C \times X$ *in* (11) *gives* $\|x-y\|_B = 0$.

5. Example

Consider the following nonlinear mixed partial integrodifferential equation:

$$\frac{\partial^2}{\partial t^2} z(y,t) = \frac{\partial^2}{\partial y^2} z(y,t) + F\left(t, z(y,t-r), \frac{\partial}{\partial t} z(y,t), \int_0^t G\left(s, z(y,s-r), \frac{\partial}{\partial s} z(y,s)\right) ds,\right.$$
$$\left.\int_0^b H\left(s, z(y,s-r), \frac{\partial}{\partial s} z(y,s)\right)\right) ds,\ t \in [0,b],\ y \in (0,\pi), \tag{16}$$

$$z(0,t) = z(\pi,t) = 0,\ t \in J, \tag{17}$$

$$z(y,t) = \phi(y,t),\ y \in (0,\pi),\ t \in [-r,0], \tag{18}$$

$$\frac{\partial}{\partial t}(y,0) = z_0(y),\ y \in (0,\pi), \tag{19}$$

where $\phi : [-r,0] \times (0,\pi) \to (0,\pi)$ is continuous, and the functions F, G and H are specified below.

Consider the space $X = L^2[0,\pi]$ with usual norm $|\cdot|_{L^2}$ and the operator $A : X \to X$ defined by $Aw = w''$, where $w \in D(A) = \{w \in X : w,\ w'$ are absolutely continuous, $w'' \in X$ and $w(0) = w(\pi) = 0\}$. It is well known that $Aw = \sum_{n=1}^{\infty} -n^2(w, w_n)w_n,\quad w \in D(A)$, where $w_n(s) = (\sqrt{2/\pi})\sin ns,\ n = 1,2,3,...$ is the orthogonal set of eigenvectors of A. Then A is the infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in \mathbb{R}$, in X which is given by (see[18])

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w, w_n)w_n,\ w \in X,$$

and the associated sine family is given by $S(t)w = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt(w, w_n)w_n,\ w \in X$.

To formulate the above partial differential equations (16)-(19) as an abstract form (1)-(2), we define the function $f : J \times C \times X \times X \times X \to X$ by

$$f(t,\psi,\mu,x,y)(v) = F(t, \psi(-r)(v), \mu(v), x(v), y(v)),\ v \in (0,\pi),$$

where $F : [0,b] \times (0,\pi) \times (0,\pi) \times (0,\pi) \times (0,\pi) \to (0,\pi)$ is continuous and strongly measurable.

Further we define $g, h : J \times C \times X \to X$, by

$$g(t,\psi,\mu)(v) = G(t,\psi(-r)(v),\mu(v)),$$
$$h(t,\psi,\mu)(v) = H(t,\psi(-r)(v),\mu(v)),$$

where $G, H : [0,b] \times (0,\pi) \times (0,\pi) \to (0,\pi)$ are continuous and strongly measurable.

We assume that the functions $F : [0,b] \times (0,\pi) \times (0,\pi) \times (0,\pi) \times (0,\pi) \to (0,\pi)$ and $G, H : [0,b] \times (0,\pi) \times (0,\pi) \to (0,\pi)$ satisfy the following assumptions:

(A_1) There exist continuous functions $k_1,\ l_1,\ m_1 : [0,b] \to (0,\pi)$ such that

(i) $|F(t,\sigma,x,y,z)| \le k_1(t)(|\sigma|+|x|+|y|+|z|)$;

(ii) $\|G(t,x,y)\| \le l_1(t)(|x|+|y|)$;

(iii) $\|H(t,x,y)\| \le m_1(t)(|x|+|y|)$;

for every $t \in [0,b]$ and $\sigma, x, y, z \in (0,\pi)$.

(A_2) For each positive integer $m_1 > 0$, there exists $\alpha_{m_1} \in L^1(0,b)$ such that

$$|F(t,\sigma,x,y,z)| \le \alpha_{m_1}(t), \text{ for all } |\sigma| \le m_1,\ |x| \le m_1,\ |y| \le m_1,\ |z| \le m_1$$

and for almost all $t \in [0,b]$.

With the functions f, g, h and the operator A chosen above, the problem (1)-(2) is an abstract formulation of a(16)-(19). Note that all the assumptions of Theorem 3.1 are satisfied and hence (16)-(19) has a solution on $[-r,b]$.

Acknowledgments

The authors express their gratitude to the anonymous referee for his/her valuable comments and suggestions.

References

[1] K. Balachandran and S.M. Anthoni, Existence of solutions of second-order Volterra integrodifferential equations in Banach spaces. Nonlinear functional Analysis and Applications, 2000, **5**, 113–121.

[2] K. Balachandran, D.G. Park and S.M. Anthoni, Existence of solutions of abstract nonlinear second order neutral functional integrodifferential equations. Comput. Math. Appl., 2003, **46**, 1313–1324.

[3] J. Ball, Initial boundary value problems for an extensible beam. Journal of Mathematical Analysis and Applications, 1973, **42**, 61–90.

[4] M. B. Dhakne and K. D. Kucche, Global existence for abstract nonlinear Volterra-Fredholm functional integrodifferential equation. Demonstratio Mathematica, 2012, **45(1)**, 117–127.

[5] M. B. Dhakne and K. D. Kucche, On mild solution of second order functional integrodifferential equation with nonlocal condition. Communications on Applied Nonlinear Analysis, 2011, **18(3)**, 79–88.

[6] M. B. Dhakne and K. D. Kucche, Second order Volterra-Fredholm functional Integrodifferential equations. Malaya Journal of Mathematics, 2012, **1(1)**, 1–8.

[7] J. Dugundji and A. Granas, Fixed Point Theory. Vol. I., Monographie Matematycane, 1982, PNW Warsawa.

[8] H. O. Fattorini, Second order linear differential equations in Banach spaces. North-Holland Mathematics Studies, Vol. 108, 1985, North-Holland, Amsterdam.

[9] W. E. Fitzgibbon, Global existence and boundedness of solutions to the extensible beam equation. Journal of Mathematical Analysis, 1982, **13**, 739–745.

[10] K. D. Kucche and M. B. Dhakne, Controllability of non-densely defined abstract mixed Volterra-Fredholm neutral functional integrodifferential equations. Asian-European Journal of Mathematics, 2013, **6(2)**, 1350025 (17Pages).

[11] K. D. Kucche and M. B. Dhakne, Controllability results for Second Order Abstract Mixed Volterra-Fredholm Functional integrodifferential Equations. Pan American Mathematical Journal, 2012, **22(4)**, 109–121.

[12] K. D. Kucche and M. B. Dhakne, Controllability result for Second Order Nonlinear Mixed Neutral Functional Integrodifferential Equations in Abstract spaces. International Journal of mathematical archive, 2013, **10(4)**, 1–10.

[13] K. D. Kucche and M. B. Dhakne, Existence results and qualitative properties of mild solution of semilinear mixed Volterra-Fredholm functional integrodifferential equations in Banach spaces. Applied Mathematics and Computation, 2013, **219**, 10806–10816.

[14] S. K. Ntouyas, Global existence for neutral functional integrodifferential equations. Nonlinear Analysis: Theory, Methods and Applications, 1997, **30**, 2133–2142.

[15] B. G. Pachpatte, Integral and finite difference inequalities and applications. North-holland Mathematics Studies, Vol.205, 2006, Elsevier Science B.V., Amsterdam.

[16] H. L. Tidke and M. B. Dhakne, Existence and uniqueness of solutions of certain second order nonlinear equations. Note de Mathematica, 2010, **30(2)**, 73–81.

[17] H. L. Tidke and M. B. Dhakne, Global existence of mild solutions of second order nonlinear Volterra integrodifferential equations in Banach spaces. Differential equations and Dynamical systems, 2009, **17(2)**, 331–342.

[18] C. C. Travis and G. F. Webb, Compactness, regularity and uniform continuity properties of strongly continuous cosine families. Houston Journal of Mathematics, 1978, **3**, 555–567.

[19] C. C. Travis and G. F. Webb Cosine families and abstract nonlinear second order differential equations. Acta Mathematica Academiae Scientiarum Hungaricae, 1978, **32**, 75–96.

[20] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations. Trans.Amer. Math. Soc. 1974, **200**, 395–418.

[21] R. Ye and G. Zhang, Neutral functional differential equations of second-order with infinite delays. Electronic Journal of Differential Equations, 2010, **2010(36)**, 1–12.

Kishor D. Kucche
Department of Mathematics,
Shivaji University,
Kolhapur-416 004, Maharashtra, India.
E-mail: kdkucche@gmail.com

M. B. Dhakne
Department of Mathematics,
Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad-431 004, Maharashtra, India.
E-mail: mbdhakne@yahoo.com

Received 24 February 2014
Accepted 03 June 2014