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Multilinear Singular and Fractional Integral Operators on Generalized Weighted Morrey Spaces

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Abstract. In this paper, we study the boundedness of multilinear Calderón-Zygmund operators, multilinear fractional integral operators and their commutators on products of generalized weighted Morrey spaces with multiple weights.

Key Words and Phrases: Multilinear Calderón-Zygmund operators; multilinear fractional integrals; generalized weighted Morrey spaces; multiple weights
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1. Introduction

The classical Morrey spaces $L_{p,\lambda}$ were originally introduced by Morrey in [40] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [40, 45]. In [6], Chiarenza and Frasca showed the boundedness of the Hardy-Littlewood maximal operator, the Riesz potential and the Calderón-Zygmund singular integral operator in these spaces. The boundedness of the Riesz potential was originally studied by Adams [1].

On the other hand, in harmonic analysis it is very important to study weighted estimates for these operators. On the weighted L_p spaces, the boundedness of operators above was obtained by Muckenhoupt [41], Mukenhoupt and Wheeden [42], and Coifman and Fefferman [4]. Recently, Komori and Shirai [34] introduced the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator and the Calderón-Zygmund operator on these spaces. Also, Guliyev [22] first introduced the generalized weighted Morrey spaces $M_w^{p,\varphi}$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see also [23, 24, 25, 26, 43]). Note that Guliyev in [22] gave a concept of generalized weighted Morrey space which can be considered as an extension of both $M_w^{p,\varphi}$ and $L^{p,\kappa}(w)$.

Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The first work on the class of multilinear Calderón-Zygmund operators was done by Coifman and Meyer in [5]. Later this class was comprehensively studied by Grafakos and

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Torres in [14, 15, 16]. As is well known, multilinear fractional integral operators were first studied by Grafakos [12], followed by Kenig and Stein [33], Grafakos and Kalton [13]. In 2009, Moen [39] introduced weight function $A_{\vec{p},q}$ and obtained weighted inequalities for multilinear fractional integral operators. More results of the weighted inequalities for multilinear fractional integral and its commutators can be found in [2, 46, 47].

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space and $\left(\mathbb{R}^n\right)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be an m-fold product space ($m \in \mathbb{N}$). We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual space, the set of all tempered distributions on $\mathbb{R}^n$. Let $m \geq 2$ and T_m be an m-linear operator initially defined on the m-fold product of Schwartz spaces and taking values in the space of tempered distributions,

$$T_m : \mathcal{S}(\mathbb{R}^n) \times \ldots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x with a radius r, and by ${}^{\complement}B(x,r)$ we denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\overrightarrow{y} = (y_1, \ldots, y_n)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$. Following [14], for given $\overrightarrow{f}$ we say that T_m is an m-linear Calderón-Zygmund operator if for some $q_1, \ldots, q_m \in [1,\infty)$ and $q \in (0,\infty)$ with $1/q = \sum_{k=1}^m 1/q_k$, it extends to a bounded multilinear operator from $L_{q_1}(\mathbb{R}^n) \times \ldots \times L_{q_m}(\mathbb{R}^n)$ into $L_q(\mathbb{R}^n)$, and if there exists a kernel function $K(x, y_1, \ldots, y_m)$ in the class m-CZK(A,ε), defined away from the diagonal $x = y_1 = \ldots = y_m$ in $\left(\mathbb{R}^n\right)^{m+1}$ such that

$$\begin{aligned} T_m(\overrightarrow{f})(x) &\equiv T_m(f_1, f_2, \ldots, f_m)(x) \\ &= \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 dy_2 \ldots dy_m \end{aligned} \tag{1}$$

whenever $f_1, f_2, \ldots, f_m \in \mathcal{S}(\mathbb{R}^n)$ and $x \notin \cap_{k=1}^m \mathrm{supp} f_k$. We say that $K(x, y_1, \ldots, y_m)$ is a kernel in the class m-CZK(A,ε), if it satisfies the size condition

$$|K(x, y_1, \ldots, y_m)| \leq A\Big(\sum_{k=1}^m |x - y_k|\Big)^{-mn},$$

for some $A > 0$ and all $(x, y_1, \ldots, y_m) \in \left(\mathbb{R}^n\right)^{m+1}$ with $x \neq y_k$ for some $1 \leq k \leq m$. Moreover, for some $\varepsilon > 0$, it satisfies the regularity condition

$$|K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \leq \frac{A \cdot |x - x'|^{\varepsilon}}{\Big(\sum\limits_{k=1}^m |x - y_k|\Big)^{mn+\varepsilon}}$$

whenever $2|x - x'| \leq \max\limits_{1 \leq k \leq m} |x - y_k|$, and also for each fixed k with $1 \leq k \leq m$,

$$|K(x, y_1, \ldots, y_k, \ldots, y_m) - K(x, y_1, \ldots, y_k', \ldots, y_m)| \leq \frac{A \cdot |y_k - y_k'|^{\varepsilon}}{\Big(\sum\limits_{k=1}^m |x - y_k|\Big)^{mn+\varepsilon}},$$

whenever $2|y_k - y'_k| \le \max_{1\le k\le m} |x - y_k|$. In recent years, many authors have been interested in studying the boundedness of these operators on function spaces, see e.g. [13, 29, 37, 38]. In 2009, the weighted strong and weak type estimates of multilinear Calderón-Zygmund singular integral operators were established in [36] by Lerner et al. New, more refined multilinear maximal function was defined in [21] to characterize the class of multiple $A_{\overrightarrow{P}}$ weights.

Theorem A ([36]) *Let $m \ge 2$ and T_m be an m-linear Calderón-Zygmund operator. If $p_1, \dots, p_m \in (1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, and $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies the $A_{\overrightarrow{P}}$ condition, then there exists a constant $C > 0$ independent of $\overrightarrow{f} = (f_1, f_2, \dots, f_m)$ such that*

$$\|T_m(\overrightarrow{f})\|_{L_p(\nu_{\overrightarrow{w}})} \le C \prod_{i=1}^m \|f_i\|_{L_{p_i}(w_i)},$$

where $\nu_{\overrightarrow{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

Theorem B ([36]) *Let $m \ge 2$ and T_m be an m-linear Calderón-Zygmund operator. If $p_1, \dots, p_m \in [1, \infty)$, $\min\{p_1, \dots, p_m\} = 1$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, and $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies the $A_{\overrightarrow{P}}$ condition, then there exists a constant $C > 0$ independent of $\overrightarrow{f} = (f_1, f_2, \dots, f_m)$ such that*

$$\|T_m(\overrightarrow{f})\|_{WL_p(\nu_{\overrightarrow{w}})} \le C \prod_{i=1}^m \|f_i\|_{L_{p_i}(w_i)},$$

where $\nu_{\overrightarrow{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

The multilinear theory has been well developed in the past twenty years. In 1992, Grafakos [11] studied the following multilinear integrals:

$$I_\alpha^m(\overrightarrow{f})(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} f_1(x - \theta_1 y) \dots f_m(x - \theta_m y)\, dy,$$

where $\theta_i (i = 1, \dots, m)$ are fixed distinct and nonzero real numbers and $0 < \beta < n$. He proved that the operator I_α^m is bounded from $L_{p_1}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $0 < 1/q = 1/p_1 + \dots + 1/p_m - \beta/n < 1$, which can be considered as an extension result for the classical fractional integrals on Lebesgue spaces. In [20, 21] some O'Neil type inequality was proved for dilated multi-linear convolution operators, including permutations of functions. This inequality was used to extend Grafakos's result [11] to more general multi-linear operators of potential type and the relevant maximal operators.

Let $\overrightarrow{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \dots \times L_{p_m}^{loc}(\mathbb{R}^n)$. The multi-sublinear maximal operator M_m and multi-sublinear fractional maximal operator $M_{\alpha,m}$ are defined by

$$M_m(\overrightarrow{f})(x) = \sup_{r>0} \prod_{i=1}^m \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_i(y_i)| dy_i,$$

$$M_{\alpha,m}(\overrightarrow{f})(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha}{n}} \prod_{j=1}^m \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_i(y_i)| dy_i, \qquad 0 \le \alpha < nm.$$

In 1999, Kenig and Stein [33] studied the following multilinear fractional integral:

$$I_{\alpha,m}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{nm-\alpha}}\, dy_1 \dots\dots dy_m.$$

They showed that $I_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p_1 + \dots + 1/p_m - \beta/n > 0$ for each $p_i > 1 (i = 1, \dots, m)$. If some $p_i = 1$, then $I_{\alpha,m}$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$, where $WL_q(\mathbb{R}^n)$ denotes the weak L_p-space of measurable functions on $\mathbb{R}^n$. Obviously, the multilinear fractional integral operator $I_{\alpha,m}$ is a natural generalization of the classical fractional integral operator $I_\alpha \equiv I_{\alpha,1}$. For this operator, Moen [39] obtained two weighted L_p - L_q estimates and a Coifman type inequality, generalizing the results of [7] and [44] to multilinear context. As a consequence of the two weighted inequalities, he obtained a generalized version of the well known characterization proved in [42]. He also obtained weighted weak and strong inequalities for the multi-sublinear fractional maximal operator $M_{\alpha,m}$.

For the boundedness properties of multilinear fractional integrals on various function spaces, we refer the reader to [13, 30, 31, 32, 46, 48]. In 2009, Moen [39] considered the weighted norm inequalities for multilinear fractional integral operators and constructed the class of multiple $A_{\overrightarrow{P},q}$ weights (see also [2]).

Theorem C ([2, 39]) *Let $m \geq 2$, $0 < \alpha < mn$, $1/q = 1/p - \alpha/n$ and $I_{\alpha,m}$ be an m-linear fractional integral operator. If $p_1, \dots, p_m \in (1, \infty)$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q = 1/p - \alpha/n$ and $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies the $A_{\overrightarrow{P},q}$ condition, then there exists a constant $C > 0$ independent of $\overrightarrow{f} = (f_1, f_2, \dots, f_m)$ such that*

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{L_q((\nu_{\overrightarrow{w}})^q)} \leq C \prod_{i=1}^m \|f_i\|_{L_{p_i}(w_i^{p_i})},$$

where $\nu_{\overrightarrow{w}} = \prod_{i=1}^m w_i$.

Theorem D ([2, 39]) *Let $m \geq 2$, $0 < \alpha < mn$,$1/q = 1/p - \alpha/n$ and $I_{\alpha,m}$ be an m-linear fractional integral operator. If $p_1, \dots, p_m \in [1, \infty)$, $\min\{p_1, \dots, p_m\} = 1$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q = 1/p - \alpha/n$ and $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies the $A_{\overrightarrow{P},q}$ condition, then there exists a constant $C > 0$ independent of $\overrightarrow{f} = (f_1, f_2, \dots, f_m)$ such that*

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{WL_q((\nu_{\overrightarrow{w}})^q)} \leq C \prod_{i=1}^m \|f_i\|_{L_{p_i}(w_i^{p_i})},$$

where $\nu_{\overrightarrow{w}} = \prod_{i=1}^m w_i$.

Recently, Wang and Yi [49] established the boundedness properties of multilinear Calderón-Zygmund operators and multilinear fractional integrals on products of weighted Morrey spaces with multiple weights.

Theorem E ([49]) *Let $m \geq 2$ and T_m be an m-linear Calderón-Zygmund operator. If $p_1, \dots, p_m \in (1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, and $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies* *the* $A_{\vec{P}}$ *with* $w_1, \dots, w_m \in A_\infty$*, then for any* $0 < \kappa < 1$*, there exists a constant* $C > 0$ *independent of* $\vec{f} = (f_1, f_2, \dots, f_m)$ *such that*

$$\|T_m(\vec{f})\|_{L_{p,\kappa}(\nu_{\vec{w}})} \le C \prod_{i=1}^m \|f_i\|_{L_{p_i,\kappa}(w_i)},$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

Theorem F ([49]) *Let* $m \ge 2$ *and* T_m *be an* m*-linear Calderón-Zygmund operator. If* $p_1, \dots, p_m \in [1, \infty)$, $\min\{p_1, \dots, p_m\} = 1$ *and* $p \in (0, \infty)$ *with* $1/p = \sum_{k=1}^m 1/p_k$*, and* $\vec{w} = (w_1, \dots, w_m)$ *satisfies the* $A_{\vec{P}}$ *with* $w_1, \dots, w_m \in A_\infty$*, then for any* $0 < \kappa < 1$*, there exists a constant* $C > 0$ *independent of* $\vec{f} = (f_1, f_2, \dots, f_m)$ *such that*

$$\|T_m(\vec{f})\|_{WL_{p,\kappa}(\nu_{\vec{w}})} \le C \prod_{i=1}^m \|f_i\|_{L_{p_i,\kappa}(w_i)},$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

Theorem G ([49]) *Let* $m \ge 2$, $0 < \alpha < mn$, $1/q = 1/p - \alpha/n$ *and* $I_{\alpha,m}$ *be an* m*-linear fractional integral operator. If* $p_1, \dots, p_m \in (1, \infty)$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q = 1/p - \alpha/n$ *and* $\vec{w} = (w_1, \dots, w_m)$ *satisfies the* $A_{\vec{P},q}$ *with* $w_1^{q_1}, \dots, w_m^{q_m} \in A_\infty$*, then for any* $0 < \kappa < p/q$*, there exists a constant* $C > 0$ *independent of* $\vec{f} = (f_1, f_2, \dots, f_m)$ *such that*

$$\|I_{\alpha,m}(\vec{f})\|_{L_{q,\kappa}((\nu_{\vec{w}})^q)} \le C \prod_{i=1}^m \|f_i\|_{L_{p_i,\kappa}(w_i^{p_i})},$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i$.

Theorem H ([49]) *Let* $m \ge 2$, $0 < \alpha < mn$, $1/q = 1/p - \alpha/n$ *and* $I_{\alpha,m}$ *be an* m*-linear fractional integral operator. If* $p_1, \dots, p_m \in [1, \infty)$, $\min\{p_1, \dots, p_m\} = 1$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q = 1/p - \alpha/n$ *and* $\vec{w} = (w_1, \dots, w_m)$ *satisfies the* $A_{\vec{P},q}$ *with* $w_1^{q_1}, \dots, w_m^{q_m} \in A_\infty$*, then for any* $0 < \kappa < p/q$*, there exists a constant* $C > 0$ *independent of* $\vec{f} = (f_1, f_2, \dots, f_m)$ *such that*

$$\|I_{\alpha,m}(\vec{f})\|_{WL_{q,\kappa}((\nu_{\vec{w}})^q)} \le C \prod_{i=1}^m \|f_i\|_{L_{p_i,\kappa}(w_i^{p_i})},$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i$.

The main purpose of this paper is to establish the boundedness properties of multilinear Calderón-Zygmund operators, multilinear fractional integrals and their commutators on products of generalized weighted Morrey spaces with multiple weights.

We now formulate our main results.

Theorem 1.1. *Let $m \geq 2$ and T_m be an m-linear Calderón-Zygmund operator. Let also $p_1, \ldots, p_m \in (1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, $\overrightarrow{w} = (w_1, \ldots, w_m)$ satisfy the $A_{\overrightarrow{P}}$ with $w_1, \ldots, w_m \in A_\infty$, and $\vec{\varphi_k} = (\varphi_{k1}, \ldots, \varphi_{km})$, $k = 1, 2$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_{1i}(x,s) w_i(B(x,s))^{\frac{1}{p_i}}}{\prod_{i=1}^m w_i(B(x,t))^{\frac{1}{p_i}}} \frac{dt}{t} \lesssim \varphi_2(x,r), \tag{2}$$

where $\varphi_2 = \prod_{i=1}^m \varphi_{2i}$.

Then the operator T_m is bounded from product space $M_{p_1,\varphi_1}(w_1) \times \ldots \times M_{p_m,\varphi_m}(w_m)$ to $M_{p,\varphi_2}(\nu_{\overrightarrow{w}})$. Moreover, there exists a constant $C > 0$ independent of $\overrightarrow{f} = (f_1, f_2, \ldots, f_m)$ such that

$$\|T_m(\overrightarrow{f})\|_{M_{p,\varphi_2}(\nu_{\overrightarrow{w}})} \leq C \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_{1i}}(w_i)},$$

where $\nu_{\overrightarrow{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

Theorem 1.2. *Let $m \geq 2$ and T_m be an m-linear Calderón-Zygmund operator. Let also $p_1, \ldots, p_m \in [1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, and $\overrightarrow{w} = (w_1, \ldots, w_m)$ satisfy the $A_{\overrightarrow{P}}$ with $w_1, \ldots, w_m \in A_\infty$, $\vec{\varphi_k} = (\varphi_{k1}, \ldots, \varphi_{km})$, $k = 1, 2$ satisfy the condition (2).*

If at least one of the $p_i = 1$, then the operator T_m is bounded from product space $M_{p_1,\varphi_1}(w_1) \times \ldots \times M_{p_m,\varphi_m}(w_m)$ to $WM_{p,\varphi_2}(\nu_{\overrightarrow{w}})$. Moreover, there exists a constant $C > 0$ independent of $\overrightarrow{f} = (f_1, f_2, \ldots, f_m)$ such that

$$\|T_m(\overrightarrow{f})\|_{WM_{p,\varphi_2}(\nu_{\overrightarrow{w}})} \leq C \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_{1i}}(w_i)},$$

where $\nu_{\overrightarrow{w}} = \prod_{i=1}^m w_i^{p/p_i}$.

Remark 1.1. *Note that if $\varphi_{1i}(x,r) = \varphi_{2i}(x,r) = w_i(B(x,r))^{\frac{\kappa-1}{p}}$ and $\omega_i \in A_\infty$, $i = 1, \ldots, m$, $0 < k < 1$, then $(\vec{\varphi_1}, \vec{\varphi_2})$ satisfies condition (2), and from Theorems 1.1 and 1.2 we get the Theorems E and F, respectively. Also, in the unweighted case Theorems 1.1 and 1.2 were proved in [27].*

Remark 1.2. *Note that in the case $m = 1$ the Theorems 1.1 and 1.2 were proved in [22], while in the unweighted case they were proved in [19], see also [17, 18].*

Theorem 1.3. *Let $m \geq 2$, $0 < \alpha < mn$, $1/q = 1/p - \alpha/n$ and $I_{\alpha,m}$ be an m-linear fractional integral operator. Let also $p_1, \ldots, p_m \in [1, \infty)$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q = \sum_{i=1}^m 1/q_i = 1/p - \alpha/n$, $\overrightarrow{w} = (w_1, \ldots, w_m)$ satisfy the $A_{\overrightarrow{P},q}$ with $w_1^{q_1}, \ldots, w_m^{q_m} \in A_\infty$, and $\vec{\varphi_k} = (\varphi_{k1}, \ldots, \varphi_{km})$, $k = 1, 2$ satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_{1i}(x,s) \big(w_i^{p_i}(B(x,s))\big)^{\frac{1}{p_i}}}{\prod_{i=1}^m \big(w_i^{q_i}(B(x,t))\big)^{\frac{1}{q_i}}} \frac{dt}{t} \lesssim \varphi_2(x,r). \tag{3}$$

Then the operator $I_{\alpha,m}$ is bounded from product space $M_{p_1,\varphi_1}(w_1^{p_1})\times\ldots\times M_{p_m,\varphi_m}(w_m^{p_m})$ to $M_{p,\varphi_2}((\nu_{\vec{w}})^q)$. Moreover, there exists a constant $C>0$ independent of $\overrightarrow{f}=(f_1,f_2,\ldots,f_m)$ such that

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{M_{q,\varphi_2}((\nu_{\vec{w}})^q)}\leq C\prod_{i=1}^m\|f_i\|_{M_{p_i,\varphi_{1i}}(w_i^{p_i})},$$

where $\nu_{\vec{w}}=\prod_{i=1}^m w_i$.

Theorem 1.4. *Let $m\geq 2$, $0<\alpha<mn$, $1/q=1/p-\alpha/n$ and $I_{\alpha,m}$ be an m-linear fractional integral operator. If $p_1,\ldots,p_m\in[1,\infty)$, $\min\{p_1,\ldots,p_m\}=1$, $1/p=\sum_{k=1}^m 1/p_k$, $1/q=1/p-\alpha/n$, $\vec{w}=(w_1,\ldots,w_m)$ satisfies the $A_{\vec{P},q}$ with $w_1^{q_1},\ldots,w_m^{q_m}\in A_\infty$ and $(\varphi_1,\ldots,\varphi_m,\varphi)$ satisfies the condition (3), then the operator $I_{\alpha,m}$ is bounded from product space $M_{p_1,\varphi_1}(w_1^{p_1})\times\ldots\times M_{p_m,\varphi_m}(w_m^{p_m})$ to $WM_{p,\varphi_2}((\nu_{\vec{w}})^q)$. Moreover, there exists a constant $C>0$ independent of $\overrightarrow{f}=(f_1,f_2,\ldots,f_m)$ such that*

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{WM_{p,\varphi_2}((\nu_{\vec{w}})^q)}\leq C\prod_{i=1}^m\|f_i\|_{M_{p_i,\varphi_{1i}}(w_i^{p_i})},$$

where $\nu_{\vec{w}}=\prod_{i=1}^m w_i$.

Remark 1.3. *Note that if $\varphi_{1i}(x,r)=\varphi_{2i}(x,r)=w_i(B(x,r))^{\frac{\kappa-1}{p}}$ and $\omega_i^{q_i}\in A_\infty$ for $i=1,\ldots,m$, $0<k<1$, then $(\vec{\varphi_1},\vec{\varphi_2})$ satisfies condition (3), and from Theorems 1.3 and 1.4 we get the Theorems G and H, respectively. Also, in the unweighted case Theorems 1.3 and 1.4 were proved in [28].*

Remark 1.4. *Note that in the case $m=1$ the Theorems 1.3 and 1.4 were proved in [22], while in the unweighted case they were proved in [19], see also [17, 18].*

2. Notations and definitions

By a weight function (briefly weight), we mean a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E)=\int_E w(x)dx$, and denote the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E. Given a weight w, we say that w satisfies the doubling condition if there exists a constant $D>0$ such that for any ball B, we have $w(2B)\leq Dw(B)$. When w satisfies this condition, we write briefly $w\in\Delta_2$.

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L_p boundedness of Hardy-Littlewood maximal functions in [41]. A weight w is a nonnegative, locally integrable function on $\mathbb{R}^n$, $B=B(x_0,r_B)$ denotes the ball centered at x_0 with a radius r_B. For $1<p<\infty$, a weight function w is said to belong to the Muckenhoupt class A_p [41], if

$$[w]_{A_p}:=\sup_B[w]_{A_p(B)}$$

$$= \sup_B \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, by Hölder's inequality, for all balls B

$$[w]_{A_p(B)}^{1/p} = |B|^{-1} \|w\|_{L_1(B)}^{1/p} \|w^{-1/p}\|_{L_{p'}(B)} \geq 1. \tag{4}$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \leq Cw(x)$ with $[w]_{A_1} = \sup\limits_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$, $A_\infty = \bigcup_{1 \leq p < \infty} A_p$ and $[w]_{A_\infty} = \inf\limits_{1 \leq p < \infty} [w]_{A_p}$.

A weight function w belongs to the Muckenhoupt-Wheeden class $A_{p,q}$ [42] for $1 < p, q < \infty$ if

$$\begin{aligned} [w]_{A_{p,q}} &:= \sup_B [w]_{A_{p,q}(B)} \\ &= \sup_B \left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{1/q} \left(\frac{1}{|B|} \int_B w(x)^{-p'} dx \right)^{1/p'} < \infty, \end{aligned}$$

where the sup is taken with respect to all balls B. Note that, by Hölder's inequality, for all balls B

$$[w]_{A_{p,q}(B)} = |B|^{\frac{1}{p} - \frac{1}{q} - 1} \|w\|_{L_q(B)} \|w^{-1}\|_{L_{p'}(B)} \geq 1. \tag{5}$$

For $p = 1$, w is in $A_{1,q}$ with $1 < q < \infty$ if

$$\begin{aligned} [w]_{A_{1,q}} &:= \sup_B [w]_{A_{1,q}(B)} \\ &= \sup_B \left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{1/q} \left(\operatorname*{ess\,sup}_{x \in B} \frac{1}{w(x)} \right) < \infty, \end{aligned}$$

where the sup is taken with respect to all balls B.

Remark 2.5. *[9, 12] If $w \in A_{p,q}$ with $1 < p < q < \infty$, then the following statements are true:*

(a) $w^q \in A_t$ *with* $t = 1 + q/p'$.
(b) $w^{-p'} \in A_{t'}$ *with* $t' = 1 + p/q'$.
(c) $w \in A_{q,p}$.
(d) $w^p \in A_s$ *with* $s = 1 + p/q'$.
(e) $w^{-q'} \in A_{s'}$ *with* $s' = 1 + q'/p$.

Lemma 2.1. ([9, 12]) (1) *If $w \in A_p$ for some $1 \leq p < \infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$*

$$w(\lambda B) \leq \lambda^{np} [w]_{A_p} w(B).$$

(2) *If $w \in A_\infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$*

$$w(\lambda B) \leq 2^{\lambda^n} [w]_{A_\infty} w(B).$$

(3) *If $w \in A_p$ for some $1 \le p \le \infty$, then there exist $C > 0$ and $\delta > 0$ such that for any ball B and a measurable set $S \subset B$,*

$$\frac{w(S)}{w(B)} \le C\Big(\frac{|S|}{|B|}\Big)^{\delta}.$$

Lemma 2.2. ([8]) *Let $w \in A_\infty$. Then for all balls $B \subset \mathbb{R}^n$, the following reverse Jensen inequality holds:*

$$\int_B w(x)dx \le C\,|B| \cdot \exp\left(\frac{1}{|B|}\int_B w(x)dx\right).$$

Lemma 2.3. ([36]) *Let $p_1, \ldots, p_m \in [1,\infty)$ and $p \in (0,\infty)$ with $1/p = \sum_{k=1}^m 1/p_k$. Then $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$ if and only if*

$$\begin{cases} \nu_{\vec{w}} \in A_{mp}, \\ w_i^{1-p_i'} \in A_{mp_i'}, \quad i = 1, \ldots, m, \end{cases}$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$, and the condition $w_i^{1-p_i'} \in A_{mp_i'}$ in the case $p_i = 1$ is understood as $w_i^{1/m} \in A_1$.

Lemma 2.4. ([2, 39]) *Let $0 < \alpha < mn$, $p_1, \ldots, p_m \in [1,\infty)$, $1/p = \sum_{k=1}^m 1/p_k$ and $1/q = 1/p - \alpha/n$. Then $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P},q}$ if and only if*

$$\begin{cases} \nu_{\vec{w}}^q \in A_{mq}, \\ w_i^{-p_i'} \in A_{mp_i'}, \quad i = 1, \ldots, m, \end{cases}$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i$.

Now let us recall the definitions of multiple weights. For m exponents $p_1, \ldots, p_m$, we will write $\vec{P}$ for the vector $\vec{P} = (p_1, \ldots, p_m)$. Let $p_1, \ldots, p_m \in [1,\infty)$ and $p \in (0,\infty)$ with $1/p = \sum_{k=1}^m 1/p_k$. Given $\vec{w} = (w_1, \ldots, w_m)$, set $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$. We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if it satisfies

$$\sup_B \left(\frac{1}{|B|}\int_B \nu_{\vec{w}} dx\right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|B|}\int_B w_i(x)^{1-p_i'}dx\right)^{1/p_i'} < \infty.$$

When $p_i = 1$, $\left(\frac{1}{|B|}\int_B w_i(x)^{1-p_i'}dx\right)^{1/p_i'}$ is understood as $(\inf_{x\in B} w_i(x))^{-1}$.

Let $p_1, \ldots, p_m \in [1,\infty)$, $1/p = \sum_{k=1}^m 1/p_k$ and $q > 0$. Given $\vec{w} = (w_1, \ldots, w_m)$, set $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$. We say that $\vec{w}$ satisfies the $A_{\vec{P},q}$ condition if it satisfies

$$\sup_B \left(\frac{1}{|B|}\int_B \nu_{\vec{w}}^q dx\right)^{1/q} \prod_{i=1}^m \left(\frac{1}{|B|}\int_B w_i(x)^{p_i'}dx\right)^{1/p_i'} < \infty.$$

When $p_i = 1$, $\left(\frac{1}{|B|}\int_B w_i(x)^{p_i'}dx\right)^{1/p_i'}$ is understood as $(\inf_{x\in B} w_i(x))$.

Lemma 2.5. ([49]) *Let $m \geq 2$, $p_1, \ldots, p_m \in [1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^{m} 1/p_k$. Assume that $w_1, \ldots, w_m \in A_\infty$ and $\nu_{\vec{w}} = \prod_{i=1}^{m} w_i^{p/p_i}$. Then for any ball B, there exists a constant $C > 0$ such that*

$$\prod_{i=1}^{m} \left(\int_B w_i(x) dx \right)^{p/p_i} \leq C \int_B \nu_{\vec{w}} dx.$$

Lemma 2.6. ([49]) *Let $m \geq 2$, $q_1, \ldots, q_m \in [1, \infty)$ and $q \in (0, \infty)$ with $1/q = \sum_{k=1}^{m} 1/q_k$. Assume that $w_1^{q_1}, \ldots, w_m^{q_m} \in A_\infty$ and $\nu_{\vec{w}} = \prod_{i=1}^{m} w_i$. Then for any ball B, there exists a constant $C > 0$ such that*

$$\prod_{i=1}^{m} \left(\int_B w_i^{q_i}(x) dx \right)^{q/q_i} \leq C \int_B \nu_{\vec{w}}^{q} dx.$$

3. Generalized weighted Morrey spaces

Given a weight function w on $\mathbb{R}^n$, the weighted Lebesgue space $L_p(w)$ for $0 < p < \infty$ is defined as the set of all functions f such that

$$\|f\|_{L_p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

We denote by $WL_p(w)$ the weighted weak space consisting of all measurable functions f such that

$$\|f\|_{WL_p(w)} = \sup_{t>0} t \cdot w(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p} < \infty.$$

In the study of local properties of solutions to partial differential equations, Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$, as well as weighted Lebesgue spaces, play an important role (see [10], [35]). Introduced by C. Morrey [40] in 1938, they are defined by the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \leq \lambda < n$, $1 \leq p < \infty$.

We also denote by $WM_{p,\lambda}$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p-space.

In 2009, Komori and Shirai [34] introduced the weighted Morrey spaces $L_{p,\kappa}(w)$ for $1 \leq p < \infty$. In order to deal with the multilinear case $m \geq 2$, we shall define $L_{p,\kappa}(w)$ for all $0 < p < \infty$.

Definition 3.1. *Let $0 < p < \infty$, $0 < \kappa < 1$ and w be a weight function on $\mathbb{R}^n$. Then the weighted Morrey space is defined by*

$$L_{p,\kappa}(w) = \Big\{ f \in L_p^{\rm loc}(w) : \|f\|_{L_{p,\kappa}(w)} < \infty \Big\},$$

where

$$\|f\|_{L_{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) dx \right)^{1/p}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

Definition 3.2. *Let $0 < p < \infty$, $0 < \kappa < 1$ and w be a weight function on $\mathbb{R}^n$. Then the weighted weak Morrey space is defined by*

$$WL_{p,\kappa}(w) = \Big\{ f \text{ measurable } : \|f\|_{WL_{p,\kappa}(w)} < \infty \Big\},$$

where

$$\|f\|_{WL_{p,\kappa}(w)} = \sup_B \sup_{t>0} \frac{1}{w(B)^{\kappa/p}} \; t \cdot w\left(\{x \in B : |f(x)| > t\}\right)^{1/p}.$$

Furthermore, in order to deal with the fractional order case, we need to consider the weighted Morrey spaces with two weights.

Definition 3.3. *Let $0 < p < \infty$ and $0 < \kappa < 1$. Then for two weights u and v, the weighted Morrey space is defined by*

$$L_{p,\kappa}(u,v) = \Big\{ f \in L_p^{\rm loc}(u) : \|f\|_{L_{p,\kappa}(u,v)} < \infty \Big\},$$

where

$$\|f\|_{L_{p,\kappa}(u,v)} = \sup_B \left(\frac{1}{v(B)^\kappa} \int_B |f(x)|^p u(x) dx \right)^{1/p}.$$

In 2011, Guliyev [22] introduced the generalized weighted Morrey spaces $M_{p,\varphi}(w)$. In order to deal with the multilinear case $m \geq 2$, we shall define $M_{p,\varphi}(w)$ for all $0 < p < \infty$.

Definition 3.4. *Let $1 \leq p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be a non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L_{p,w}^{\rm loc}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, w(B(x,r))^{-\frac{1}{p}} \, \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f \chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y) dy \right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ *we denote the weak generalized weighted Morrey space of all functions* $f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n)$ *for which*

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ *denotes the weak* $L_{p,w}$*-space of measurable functions* f *for which*

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t \cdot w\left(\{y \in B(x,r) : |f(y)| > t\}\right)^{1/p}.$$

Remark 3.6. (1) *If* $w \equiv 1$*, then* $M_{p,\varphi}(1) = M_{p,\varphi}$ *is the generalized Morrey space.*

(2) *If* $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$*, then* $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ *is the weighted Morrey space.*

(3) *If* $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$*, then* $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ *is the two weighted Morrey space.*

(4) *If* $w \equiv 1$ *and* $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ *with* $0 < \lambda < n$*, then* $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ *is the classical Morrey space and* $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ *is the weak Morrey space.*

(5) *If* $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$*, then* $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ *is the weighted Lebesgue space.*

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. Besides, we will denote the conjugate exponent of $p > 1$ by $p' = p/(p-1)$.

4. Multilinear Calderón-Zygmund operators in the product spaces $M_{p_1,\varphi_1}(w_1) \times \ldots \times M_{p_m,\varphi_m}(w_m)$

We are going to use the following result on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)d\mu(r),\ 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0,\infty)$.

Theorem 4.5. ([3]) *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all functions g *non-negative and non-increasing on* $(0,\infty)$ *if and only if*

$$A := \sup_{t>0} \frac{w(t)}{t}\int_0^t \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

In this section, we will prove the boundedness of multilinear Calderón-Zygmund operators on product generalized weighted Morrey space. First we prove the following theorem.

Theorem 4.6. *Let $m \geq 2$ and T_m be an m-linear Calderón-Zygmund operator. If $p_1, \dots, p_m \in (1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, and $\overrightarrow{w} = (w_1, \dots, w_m)$ satisfies the $A_{\overrightarrow{P}}$ with $w_1, \dots, w_m \in A_\infty$, then the inequality*

$$\|T_m(\overrightarrow{f})\|_{L_{p,\nu_{\overrightarrow{w}}}(B(x,r))} \lesssim \nu_{\overrightarrow{w}}(B(x,r))^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))} \, w_i(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \tag{6}$$

holds for any ball $B(x,r)$ and for all $\overrightarrow{f} \in L^{loc}_{p_1,w_1}(\mathbb{R}^n) \times \ldots \times L^{loc}_{p_m,w_m}(\mathbb{R}^n)$.

Proof. Let $1 < p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. For arbitrary $x \in \mathbb{R}^n$, set $B = B(x,r)$ for the ball centered at x with a radius r, $2B = B(x,2r)$. We represent $\overrightarrow{f} = (f_1, \dots, f_m)$ as

$$f_j = f_j^0 + f_j^\infty, \quad f_j^0 = f_j \chi_{2B}, \quad f_j^\infty = f_j \chi_{\complement_{(2B)}}, \quad j = 1, \dots, m. \tag{7}$$

Then we write

$$\begin{aligned}
\prod_{i=1}^m f_i(y_i) &= \prod_{i=1}^m \left(f_i^0(y_i) + f_i^\infty(y_i)\right) \\
&= \sum_{\beta_1,\dots,\beta_m \in \{0,\infty\}} f_1^{\beta_1}(y_1) \dots f_m^{\beta_m}(y_m) \\
&= \prod_{i=1}^m f_i^0(y_i) + \sum_{\beta_1,\dots,\beta_m}{}' f_1^{\beta_1}(y_1) \dots f_m^{\beta_m}(y_m),
\end{aligned}$$

where each term in $\sum'$ contains at least one $\beta_i \neq 0$.

Since T_m is an m-linear operator, we split $T_m(\overrightarrow{f})$ as follows:

$$\left|T_m(\overrightarrow{f})(y)\right| \leq c_0 \left|T_m(f_1^0, \dots, f_m^0)(y)\right| + \Big| \sum_{\beta_1,\dots,\beta_m}{}' T_m(f_1^{\beta_1}, \dots, f_m^{\beta_m})(y) \Big|,$$

where $\beta_1, \dots, \beta_m \in \{0, \infty\}$ and each term in $\sum'$ contains at least one $\beta_i \neq 0$. Then,

$$\begin{aligned}
\|T_m(\overrightarrow{f})\|_{L_{p,\nu_{\overrightarrow{w}}}(B(x,r))} &\leq \|T_m(f_1^0, \dots, f_m^0)\|_{L_{p,\nu_{\overrightarrow{w}}}(B(x,r))} \\
&\quad + \| \sum_{\beta_1,\dots,\beta_m}{}' T_m(f_1^{\beta_1}, \dots, f_m^{\beta_m})\|_{L_{p,\nu_{\overrightarrow{w}}}(B(x,r))}.
\end{aligned}$$

In view of Lemma 2.3, we have $\nu_{\vec{w}} \in A_{mp}$. Applying Theorem A, Lemma 2.5 and Lemma 2.1, we get

$$\begin{aligned}&\|T_m(\overrightarrow{f^0})\|_{L_{p,\nu_{\vec{w}}}(B(x,r))} \le \|T_m(\overrightarrow{f^0})\|_{L_{p,\nu_{\vec{w}}}(\mathbb{R}^n)}\\&\lesssim \prod_{i=1}^m \|f_i^0\|_{L_{p_i,w_i}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,2r))}.\end{aligned}$$

On the other hand, from Lemma 2.5 we have

$$\begin{aligned}&\prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(2B)} \approx |B|^m \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(2B)} \int_{2r}^\infty \frac{dt}{t^{nm+1}}\\&\le |B|^m \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))} \frac{dt}{t^{n+1}} \qquad (8)\\&\lesssim \prod_{i=1}^m w_i(B)^{\frac{1}{p_i}} \|w^{-1/p_i}\|_{L_{p_i'}(B)} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))} \frac{dt}{t^{n+1}}\\&\lesssim \prod_{i=1}^m w_i(B)^{\frac{1}{p_i}} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))} \|w^{-1/p_i}\|_{L_{p_i'}(B(x,t))} \frac{dt}{t^{n+1}}\\&\lesssim \prod_{i=1}^m [w_i]_{A_{p_i}}\, w_i(B)^{\frac{1}{p_i}} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))}\, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t}\\&\approx \nu_{\vec{w}}(B)^{\frac{1}{p}} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))}\, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t},\end{aligned}$$

where $\nu_{\vec{w}}(B)^{1/p} = \prod_{i=1}^m w_i(B)^{1/p_i}$. Thus

$$\|T_m(\overrightarrow{f^0})\|_{L_{p,\nu_{\vec{w}}}(B(x,r))} \lesssim \nu_{\vec{w}}(B(x,r))^{\frac{1}{p}} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))}\, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t}. \qquad (9)$$

For the other terms, let us first deal with the case when $\beta_1 = \cdots = \beta_m = \infty$.

When $|x-y_i| \le r$, $|z-y_i| \ge 2r$, we have $\frac{1}{2}|z-y_i| \le |x-y_i| \le \frac{3}{2}|z-y_i|$, and therefore,

$$\begin{aligned}|T_m(f_1^\infty,\dots,f_m^\infty)(z)| &\lesssim \int_{\left(\complement_{B(x,2r)}\right)^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\dots,x-y_m)|^{mn}} d\vec{y}\\&\lesssim \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i\end{aligned}$$

and

$$\|T_m(f_1^\infty,\dots,f_m^\infty)\|_{L_{p,\nu_{\vec{w}}}(B(x,r))} \le \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \ \|\chi_{B(x,r)}\|_{L_{p,\nu_{\vec{w}}}(\mathbb{R}^n)}$$

$$\lesssim \nu_{\vec{w}}(B(x,r))^{1/p} \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i.$$

By Fubini's theorem we have

$$\begin{aligned}
\int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i &\approx \int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m |f_i(y_i)| \int_{|x-y_i|}^{\infty} \frac{dt}{t^{n+1}} dy_i \\
&\approx \int_{2r}^{\infty} \prod_{i=1}^m \int_{2r\le|x-y_i|<t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \prod_{i=1}^m \int_{B(x,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}}.
\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}
&\int_{\left(^{\complement}B(x,2r)\right)^m} \prod_{i=1}^m \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \lesssim \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))} \|w_i^{-1/p_i}\|_{L_{p_i'}(B(x,t))} \frac{dt}{t^{n+1}} \\
&\le \prod_{i=1}^m [w_i]_{A_{p_i}}^{1/p_i} \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))} \, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t}. \qquad (10)
\end{aligned}$$

Moreover, for all $p_i \in [1,\infty)$, $i = 1,\dots,m$ the inequality

$$\begin{aligned}
&\|T_m(f_1^\infty,\dots,f_m^\infty)\|_{L_{p,\nu_{\vec{w}}}(B(x,r))} \\
&\lesssim \nu_{\vec{w}}(B(x,r))^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))} \, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t} \qquad (11)
\end{aligned}$$

is valid.

We now consider the cases when exactly l of the β_l's are ∞ for some $1 \le l < m$. We only give the arguments for one of these cases. The rest are similar and can easily be obtained from the arguments below by permuting the indices. To this end we may assume that $\beta_1 = \ldots = \beta_l = \infty$ and $\beta_{l+1} = \ldots = \beta_m = 0$. Recall the fact that $|x-y_i| \approx |z-y_i|$ for $z \in B(x,r)$, $y_i \in {}^{\complement}B(x,2r)$ and $1 \le i \le l$. We have

$$\begin{aligned}
&\left|T_m(f_1^\infty,\dots,f_l^\infty,f_{l+1}^0,\dots,f_m^0)(z)\right| \\
&\lesssim \int_{\left(^{\complement}B(x,2r)\right)^l} \int_{\left(B(x,2r)\right)^{m-l}} \frac{|f_1(y_1)\cdots f_m(y_m)| \, d\vec{y}}{(|x-y_1|+\ldots+|x-y_l|)^{mn}} \\
&\lesssim r^{-(m-l)n} \int_{\left(^{\complement}B(x,2r)\right)^l} \frac{|f_1(y_1)\cdots f_l(y_m)| \, dy_1\ldots dy_l}{(|x-y_1|+\ldots+|x-y_l|)^{ln}} \\
&\times \int_{\left(B(x,2r)\right)^{m-l}} |f_1(y_{l+1})\cdots f_m(y_m)| \, dy_{l+1}\ldots dy_m
\end{aligned}$$

$$\lesssim \int_{\left(\complement_{B(x,2r)}\right)^l} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \; r^{-n(m-l)} \int_{\left(B(x,2r)\right)^{m-l}} \prod_{i=l+1}^{m} |f_i(y_i)| dy_i.$$

Applying Hölder's inequality, we get

$$\int_{\left(\complement_{B(x,2r)}\right)^l} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \lesssim \int_{2r}^{\infty} \prod_{i=1}^{l} \|f_i\|_{L_{p_i,w_i}(B(x,t))} \, \|w_i^{-1/p_i}\|_{L_{p_i'}(B(x,t))} \frac{dt}{t^{n+1}}$$

$$\leq \prod_{i=1}^{l} [w_i]_{A_{p_i}}^{1/p_i} \int_{2r}^{\infty} \prod_{i=1}^{l} \|f_i\|_{L_{p_i,w_i}(B(x,t))} \, w_i(B(x,t))^{-1/p_i} \frac{dt}{t} \tag{12}$$

and

$$\begin{aligned} &r^{-n(m-l)} \int_{\left(B(x,2r)\right)^{m-l}} \prod_{i=l+1}^{m} |f_i(y_i)| dy_i \\ &\lesssim \prod_{i=l+1}^{m} \|f_i\|_{L_{p_i,w_i}(B(x,2r))} \, \|w_i^{-1/p_i}\|_{L_{p_i'}(B(x,2r))} \int_{2r}^{\infty} \frac{dt}{t^{n(m-l)+1}} \\ &\leq \int_{2r}^{\infty} \prod_{i=l+1}^{m} \|f\|_{L_{p_i,w_i}(B(x,t))} \, \|w_i^{-1/p_i}\|_{L_{p_i'}(B(x,t))} \frac{dt}{t^{n(m-l)+1}} \\ &\leq \prod_{i=l+1}^{m} [w_i]_{A_{p_i}}^{1/p_i} \int_{2r}^{\infty} \prod_{i=l+1}^{m} \|f\|_{L_{p_i,w_i}(B(x,t))} \, w_i(B(x,t))^{-1/p_i} \frac{dt}{t}. \end{aligned} \tag{13}$$

From (12) and (13) we get

$$\begin{aligned} &\|T_m(f_1^{\infty}, \dots, f_l^{\infty}, f_{l+1}^0, \dots, f_m^0)\|_{L_{p,\nu_{\vec{w}}}(B(x,r))} \\ &\lesssim \nu_{\vec{w}}(B(x,r))^{1/p} \int_{\left(\complement_{B(x,2r)}\right)^l} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \; r^{-n(m-l)} \int_{\left(B(x,2r)\right)^{m-l}} |f_i(y_i)| dy_i \\ &\lesssim \nu_{\vec{w}}(B(x,r))^{1/p} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f\|_{L_{p_i,w_i}(B(x,t))} \, w_i(B(x,t))^{-1/p_i} \frac{dt}{t}. \end{aligned}$$

Thus we get the following inequality:

$$\begin{aligned} &\Big\| \sum_{\beta_1,\dots,\beta_m}{}' T_m(f_1^{\beta_1}, \dots, f_m^{\beta_m}) \Big\|_{L_{p,\nu_{\vec{w}}}(B(x,r))} \\ &\lesssim \nu_{\vec{w}}(B(x,r))^{1/p} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f\|_{L_{p_i,w_i}(B(x,t))} \, w_i(B(x,t))^{-1/p_i} \frac{dt}{t}. \end{aligned}$$

Consequently, the inequality (15) is valid.◀

Theorem 4.7. *Let $m \geq 2$ and T_m be an m-linear Calderón-Zygmund operator. If $p_1, \ldots, p_m \in [1,\infty)$, $\min\{p_1, \ldots, p_m\} = 1$ and $p \in (0,\infty)$ with $1/p = \sum_{k=1}^m 1/p_k$, and $\vec{w} = (w_1, \ldots, w_m)$ satisfies the $A_{\vec{P}}$ with $w_1, \ldots, w_m \in A_\infty$, then the inequality*

$$\|T_m(\vec{f})\|_{WL_{p,\nu_{\vec{w}}}(B(x,r))} \lesssim \nu_{\vec{w}}(B(x,r))^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))}\, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t} \quad (14)$$

holds for any ball $B(x,r)$ and for all $\vec{f} \in L^{loc}_{p_1,w_1}(\mathbb{R}^n) \times \ldots \times L^{loc}_{p_m,w_m}(\mathbb{R}^n)$.

Proof. For any ball $B = B(x,r) \subset \mathbb{R}^n$, decompose $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i \chi_{2B}$, $2B = B(x,2r)$, $i = 1, \ldots, m$. Then for any given $\lambda > 0$, we can write

$$\begin{aligned}
&\nu_{\vec{w}}\big(\{y \in B(x,r) : |T_m(\vec{f})(y)| > \lambda\}\big)^{\frac{1}{p}} \\
&\nu_{\vec{w}}\big(\{y \in B(x,r) : |T_m(f_1^0, \ldots, f_m^0)(y)| > \lambda/2^m\}\big)^{\frac{1}{p}} \\
&+ {\sum}' \nu_{\vec{w}}\big(\{y \in B(x,r) : |T_m(f_1^{\beta_1}, \ldots, f_m^{\beta_m})(y)| > \lambda/2^m\}\big)^{\frac{1}{p}} \\
&= I_*^0 + {\sum}' I_*^{\beta_1,\ldots,\beta_m},
\end{aligned}$$

where each term in $\sum'$ contains at least one $\beta_i \neq 0$. By Lemma 2.4 again, we know that $\nu_{\vec{w}} \in A_{mp}$. Applying Theorem B, Lemma 3.1 and Lemma 2.1, we have

$$\begin{aligned}
I_*^0 &= \|T_m(\vec{f^0})\|_{WL_{p,\nu_{\vec{w}}}(B(x,r))} \leq \|T_m(\vec{f^0})\|_{WL_{p,\nu_{\vec{w}}}(\mathbb{R}^n)} \\
&\lesssim \prod_{i=1}^m \|f_i^0\|_{L_{p_i,w_i}(\mathbb{R}^n)} = \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,2r))}.
\end{aligned}$$

In the proof of Theorem 1.1, we have already shown the validity of the following estimate (see (3.1) and (3.3)):

$$\begin{aligned}
&\|T_m(f_1^{\beta_1}, \ldots, f_m^{\beta_m})\|_{L_{p,\nu_{\vec{w}}}(B(x,r))} \\
&\lesssim \nu_{\vec{w}}(B(x,r))^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{i=1}^m \|f\|_{L_{p_i,w_i}(B(x,t))}\, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t}.
\end{aligned}$$

Then

$$\begin{aligned}
&{\sum}' I_*^{\beta_1,\ldots,\beta_m} = {\sum}' \|T_m(f_1^{\beta_1}, \ldots, f_m^{\beta_m})\|_{WL_{p,\nu_{\vec{w}}}(B(x,r))} \\
&\leq {\sum}' \|T_m(f_1^{\beta_1}, \ldots, f_m^{\beta_m})\|_{L_{p,\nu_{\vec{w}}}(B(x,r))} \\
&\lesssim \nu_{\vec{w}}(B(x,r))^{\frac{1}{p}} \int_{2r}^{\infty} \prod_{i=1}^m \|f\|_{L_{p_i,w_i}(B(x,t))}\, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t}. \blacktriangleleft
\end{aligned}$$

Remark 4.7. *Note that in the case $m = 1$ the Theorems 4.6 and 4.7 were proved in [22], while in the unweighted case they were proved in [19], see also [17, 18].*

Now we prove the boundedness of multilinear Calderón-Zygmund operators on product generalized weighted Morrey space.

Proof of Theorem 1.1. Let $1 < p_1, \ldots, p_m < \infty$ and $\overrightarrow{f} \in M_{p_1,\varphi_1}(w_1) \times \ldots \times M_{p_m,\varphi_m}(w_m)$. By Theorems 4.5 and 4.6 we have

$$\|T_m(\overrightarrow{f})\|_{M_{p,\varphi}(\nu_{\overrightarrow{w}})} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1}\nu_{\overrightarrow{w}}(B(x,r))^{-\frac{1}{p_i}}\, \|T_m(\overrightarrow{f})\|_{L_{p,w}(B(x,r))}$$
$$\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \int_r^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))}\, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t}$$
$$= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \int_0^{r^{-1}} \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t^{-1}))}\, w_i(B(x,t^{-1}))^{-\frac{1}{p_i}} \frac{dt}{t}$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \prod_{i=1}^m \varphi_{1i}(x,r^{-1})^{-1} w_i(B(x,r^{-1}))^{-\frac{1}{p_i}}\, \|f\|_{L_{p_i,w_i}(B(x,r^{-1}))}$$
$$= \sup_{x\in\mathbb{R}^n, r>0} \prod_{i=1}^m \varphi_{1i}(x,r)^{-1} w_i(B(x,r)^{-\frac{1}{p_i}}\, \|f\|_{L_{p_i,w_i}(B(x,r))}$$
$$= \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_{1i}}(w_i)}. \blacktriangleleft$$

Now we turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $p_1, \ldots, p_m \in [1,\infty)$, $\min\{p_1, \ldots, p_m\} = 1$, $p \in (0,\infty)$ with $1/p = \sum_{k=1}^m 1/p_k$ and $\overrightarrow{f} \in M_{p_1,\varphi_1}(w_1) \times \ldots \times M_{p_m,\varphi_m}(w_m)$. By Theorems 4.5 and 4.7 we have

$$\|T_m(\overrightarrow{f})\|_{WM_{p,\varphi}(\nu_{\overrightarrow{w}})} = \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1}\nu_{\overrightarrow{w}}(B(x,r))^{-\frac{1}{p}}\, \|T_m(\overrightarrow{f})\|_{WL_{p,w}(B(x,r))}$$
$$\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \int_r^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t))}\, w_i(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t}$$
$$= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \int_0^{r^{-1}} \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i}(B(x,t^{-1}))}\, w_i(B(x,t^{-1}))^{-\frac{1}{p_i}} \frac{dt}{t}$$
$$\lesssim \sup_{x\in\mathbb{R}^n, r>0} \prod_{i=1}^m \varphi_{1i}(x,r^{-1})^{-1} w_i(B(x,r^{-1}))^{-\frac{1}{p_i}}\, \|f\|_{L_{p_i,w_i}(B(x,r^{-1}))}$$
$$= \sup_{x\in\mathbb{R}^n, r>0} \prod_{i=1}^m \varphi_{1i}(x,r)^{-1} w_i(B(x,r)^{-\frac{1}{p_i}}\, \|f\|_{L_{p_i,w_i}(B(x,r))}$$
$$= \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_{1i}}(w_i)}.$$

By using Hölder's inequality, it is easy to verify that if each w_i is in A_{p_i}, then

$$\prod_{i=1}^{m} A_{p_i} \subset A_{\overrightarrow{P}}$$

and this inclusion is strict (see [36]). Thus, as direct consequences of Theorems 1.1 and 1.2, we immediately obtain the following

Corollary 4.1. *Let $m \geq 2$ and T_m be an m-linear Calderón-Zygmund operator. If $p_1, \ldots, p_m \in (1, \infty)$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^{m} 1/p_k$, $\overrightarrow{w} = (w_1, \ldots, w_m) \in \prod_{i=1}^{m} A_{p_i}$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition (2), then the operator T_m is bounded from product space $M_{p_1,\varphi_1}(w_1) \times \ldots \times M_{p_m,\varphi_m}(w_m)$ to $M_{p,\varphi}(\nu_{\overrightarrow{w}})$. Moreover, there exists a constant $C > 0$ independent of $\overrightarrow{f} = (f_1, f_2, \ldots, f_m)$ such that*

$$\|T_m(\overrightarrow{f})\|_{M_{p,\varphi}(\nu_{\overrightarrow{w}})} \leq C \prod_{i=1}^{m} \|f_i\|_{M_{p_i,\varphi_i}(w_i)},$$

where $\nu_{\overrightarrow{w}} = \prod_{i=1}^{m} w_i^{p/p_i}$.

Corollary 4.2. *Let $m \geq 2$ and T_m be an m-linear Calderón-Zygmund operator. If $p_1, \ldots, p_m \in [1, \infty)$, $\min\{p_1, \ldots, p_m\} = 1$ and $p \in (0, \infty)$ with $1/p = \sum_{k=1}^{m} 1/p_k$, $\overrightarrow{w} = (w_1, \ldots, w_m) \in \prod_{i=1}^{m} A_{p_i}$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition (2), then the operator T_m is bounded from product space $M_{p_1,\varphi_1}(w_1) \times \ldots \times M_{p_m,\varphi_m}(w_m)$ to $WM_{p,\varphi}(\nu_{\overrightarrow{w}})$. Moreover, there exists a constant $C > 0$ independent of $\overrightarrow{f} = (f_1, f_2, \ldots, f_m)$ such that*

$$\|T_m(\overrightarrow{f})\|_{WM_{p,\varphi}(\nu_{\overrightarrow{w}})} \leq C \prod_{i=1}^{m} \|f_i\|_{M_{p_i,\varphi_i}(w_i)},$$

where $\nu_{\overrightarrow{w}} = \prod_{i=1}^{m} w_i^{p/p_i}$.

5. Multilinear Riesz potential operators in the product spaces $M_{p_1,\varphi_1}(w_1) \times \ldots \times M_{p_m,\varphi_m}(w_m)$

In this section, we will prove the boundedness of multilinear Riesz potential operators on product generalized weighted Morrey space. First we prove the following theorem.

Theorem 5.8. *Let $m \geq 2$, $0 < \alpha < mn$, $1/q = 1/p - \alpha/n$ and $I_{\alpha,m}$ be an m-linear fractional integral operator. If $p_1, \ldots, p_m \in (1, \infty)$, $1/p = \sum_{k=1}^{m} 1/p_k$, $1/q_i = 1/p_i -$*

$\alpha/(nm)$, $1/q = 1/p - \alpha/n$ and $\overrightarrow{w} = (w_1, \ldots, w_m)$ satisfies the $A_{\overrightarrow{P},q}$ with $w_1^{q_1}, \ldots, w_m^{q_m} \in A_\infty$, then the inequality

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{L_{q,(\nu_{\overrightarrow{w}})^q}(B(x,r))} \lesssim \nu_{\overrightarrow{w}}^q(B(x,r))^{\frac{1}{q}} \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))} w_i^{q_i}(B(x,t))^{-\frac{1}{q_i}} \frac{dt}{t} \tag{15}$$

holds for any ball $B(x,r)$ and for all $\overrightarrow{f} \in L_{p_1,w_1^{p_1}}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m,w_m^{p_m}}^{loc}(\mathbb{R}^n)$.

Proof. Let $p_1, \ldots, p_m \in (1,\infty)$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q = 1/p - \alpha/n$. Arguing as in the proof of Theorem 4.6, fix a ball $B = B(x,r) \subset \mathbb{R}^n$ and decompose $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{2B}$, $2B = B(x,2r)$, $i = 1, \ldots, m$. Since $I_{\alpha,m}$ is an m-linear operator, then we split $I_{\alpha,m}(\overrightarrow{f})$ as follows:

$$\left|I_{\alpha,m}(\overrightarrow{f})(y)\right| \le \left|I_{\alpha,m}(f_1^0, \ldots, f_m^0)(y)\right| + \left|\sum_{\beta_1,\ldots,\beta_m}{}' I_{\alpha,m}(f_1^{\beta_1}, \ldots, f_m^{\beta_m})(y)\right|,$$

where $\beta_1, \ldots, \beta_m \in \{0,\infty\}$ and each term in $\sum'$ contains at least one $\beta_i \neq 0$. Then,

$$\begin{aligned}\|I_{\alpha,m}(\overrightarrow{f})\|_{L_{q,\nu_{\overrightarrow{w}}^q}(B(x,r))} &\le \|I_{\alpha,m}(f_1^0, \ldots, f_m^0)\|_{L_{q,\nu_{\overrightarrow{w}}^q}(B(x,r))} \\ &\quad + \|\sum_{\beta_1,\ldots,\beta_m}{}' I_{\alpha,m}(f_1^{\beta_1}, \ldots, f_m^{\beta_m})\|_{L_{q,\nu_{\overrightarrow{w}}^q}(B(x,r))} \\ &\le J^0 + \sum{}' J^{\beta_1,\ldots,\beta_m}.\end{aligned}$$

In view of Lemma 2.3, we have $(\nu_{\overrightarrow{w}})^q \in A_{mq}$. Applying Theorem B, Lemma 2.6 and Lemma 2.1, we get

$$\begin{aligned}J^0 = \|I_{\alpha,m}(\overrightarrow{f^0})\|_{L_{q,\nu_{\overrightarrow{w}}^q}(B(x,r))} &\le \|I_{\alpha,m}(\overrightarrow{f^0})\|_{L_{q,\nu_{\overrightarrow{w}}^q}(\mathbb{R}^n)} \\ \lesssim \prod_{i=1}^m \|f_i^0\|_{L_{p_i,w_i^{p_i}}(\mathbb{R}^n)} &\lesssim \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,2r))}.\end{aligned}$$

On the other hand, from Lemma 2.6 we have

$$\begin{aligned}\prod_{i=1}^m \|f_i\|_{L_{p_i,w_i^{p_i}}(2B)} &\approx |B|^{m\left(1-\frac{\alpha}{n}\right)} \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i^{p_i}}(2B)} \int_{2r}^\infty \frac{dt}{t^{nm+1-\alpha}} \\ &\le |B|^{m\left(1-\frac{\alpha}{n}\right)} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))} \frac{dt}{t^{nm+1-\alpha}} \\ &\lesssim \prod_{i=1}^m w_i^{q_i}(B)^{\frac{1}{q_i}} \|w_i^{-1}\|_{L_{p_i'}(B)} \int_{2r}^\infty \prod_{i=1}^m \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))} \frac{dt}{t^{nm+1-\alpha}}\end{aligned} \tag{16}$$

$$\lesssim \prod_{i=1}^{m} w_i^{q_i}(B)^{\frac{1}{q_i}} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i,w_i}(B(x,t))} \|w^{-1/p_i}\|_{L_{p_i'}(B(x,t))} \frac{dt}{t^{nm+1-\alpha}}$$

$$\lesssim \prod_{i=1}^{m} [w_i]_{A_{p_i,q_i}} w_i^{q_i}(B)^{\frac{1}{q_i}} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i,w_i^{p_i}(B(x,t))}} w_i^{q_i}(B(x,t))^{-\frac{1}{q_i}} \frac{dt}{t}$$

$$\approx \nu_{\vec{w}}^{q}(B)^{1/q} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))} w_i^{q_i}(B(x,t))^{-\frac{1}{q_i}} \frac{dt}{t},$$

where $\nu_{\vec{w}}(B) = \prod_{i=1}^{m} w_i(B)$. Thus

$$J^0 \lesssim \nu_{\vec{w}}^{q}(B(x,r))^{1/q} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))} w_i^{p_i}(B(x,t))^{-\frac{1}{p_i}} \frac{dt}{t}. \tag{17}$$

For the other terms, let us first deal with the case when $\beta_1 = \cdots = \beta_m = \infty$.

When $|x-y_i| \le r$, $|z-y_i| \ge 2r$, we have $\frac{1}{2}|z-y_i| \le |x-y_i| \le \frac{3}{2}|z-y_i|$, and therefore,

$$|I_{\alpha,m}(f_1^{\infty},\dots,f_m^{\infty})(z)| \lesssim \int_{\left(\complement_{B(x,2r)}\right)^m} \frac{|f_1(y_1)\cdots f_m(y_m)|}{(|(x-y_1|+\ldots+|x-y_m|)^{mn-\alpha}} d\vec{y}$$
$$\lesssim \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^{m} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i$$

and

$$\|I_{\alpha,m}(f_1^{\infty},\dots,f_m^{\infty})\|_{L_{q,(\nu_{\vec{w}})^q}(B(x,r))} \le \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^{m} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i \, \|\chi_{B(x,r)}\|_{L_{q,\nu_{\vec{w}}}(\mathbb{R}^n)}$$
$$\lesssim \nu_{\vec{w}}^{q}(B(x,r))^{1/q} \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^{m} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i.$$

By Fubini's theorem we have

$$\int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^{m} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i \approx \int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^{m} |f_i(y_i)| dy_i \int_{|x-y_i|}^{\infty} \frac{dt}{t^{n+1-\alpha/m}}$$
$$\approx \int_{2r}^{\infty} \prod_{i=1}^{m} \int_{2r \le |x-y_i| < t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1-\alpha/m}}$$
$$\lesssim \int_{2r}^{\infty} \prod_{i=1}^{m} \int_{B(x,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1-\alpha/m}}.$$

Applying Hölder's inequality, we get

$$\int_{\left(\complement_{B(x,2r)}\right)^m} \prod_{i=1}^{m} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha/m}} dy_i \lesssim \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))} \|w_i^{-1}\|_{L_{p_i'}(B(x,t))} \frac{dt}{t^{n+1-\alpha/m}}$$

$$\lesssim \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))}\, w_i^{q_i}(B(x,t))^{-1/q_i}\, \frac{dt}{t}, \tag{18}$$

where $1/p_i - 1/q_i = \alpha/m$, $i = 1, \ldots, m$.

Moreover, for all $p_i \in [1,\infty)$, $1/p_i - 1/q_i = \alpha/(nm)$, $i = 1, \ldots, m$ the inequality

$$\begin{aligned} &\|I_{\alpha,m}(f_1^{\infty}, \ldots, f_m^{\infty})\|_{L_{q,(\nu_{\vec{w}})^q}(B(x,r))} \\ &\lesssim \nu_{\vec{w}}^{q}(B(x,r))^{1/q} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))}\, w_i^{q_i}(B(x,t))^{-1/q_i}\, \frac{dt}{t} \end{aligned} \tag{19}$$

is valid.

We now consider the cases when exactly l of the β_l's are ∞ for some $1 \le l < m$. We only give the arguments for one of these cases. The rest are similar and can easily be obtained from the arguments below by permuting the indices. To this end we may assume that $\beta_1 = \ldots = \beta_l = \infty$ and $\beta_{l+1} = \ldots = \beta_m = 0$. Recall the fact that $|x - y_i| \approx |z - y_i|$ for $z \in B(x,r)$, $y_i \in {}^{\complement}B(x,2r)$ and $1 \le i \le l$. We have

$$\begin{aligned} &\big|I_{\alpha,m}(f_1^{\infty}, \ldots, f_l^{\infty}, f_{l+1}^{0}, \ldots, f_m^{0})(z)\big| \\ &\lesssim \int_{\left({}^{\complement}B(x,2r)\right)^l} \int_{\left(B(x,2r)\right)^{m-l}} \frac{|f_1(y_1)\cdots f_m(y_m)|\, d\vec{y}}{(|x-y_1| + \ldots + |x-y_l|)^{mn-\alpha}} \\ &\lesssim r^{(m-l)(n-\alpha)} \int_{\left({}^{\complement}B(x,2r)\right)^l} \frac{|f_1(y_1)\cdots f_l(y_m)|\, dy_1 \ldots dy_l}{(|x-y_1| + \ldots + |x-y_l|)^{l(n-\alpha)}} \\ &\times \int_{\left(B(x,2r)\right)^{m-l}} |f_1(y_{l+1})\cdots f_m(y_m)|\, dy_{l+1} \ldots dy_m \\ &\lesssim \prod_{i=1}^{l} \int_{{}^{\complement}B(x,2r)} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha}} dy_i \prod_{i=l+1}^{m} r^{\alpha-n} \int_{B(x,2r)} |f_i(y_i)| dy_i. \end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned} &\int_{{}^{\complement}(2B)} \prod_{i=1}^{l} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha}} dy_i \lesssim \int_{2r}^{\infty} \prod_{i=1}^{l} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))}\, \|w_i^{-1}\|_{L_{p_i'}(B(x,t))}\, \frac{dt}{t^{n+1-\alpha}} \\ &\le \prod_{i=1}^{l} [w_i]_{A_{p_i,q_i}}^{1/p_i} \int_{2r}^{\infty} \prod_{i=1}^{l} \|f\|_{L_{p_i,w_i^{p_i}}(B(x,t))}\, w_i^{q_i}(B(x,t))^{-1/q_i}\, \frac{dt}{t} \end{aligned} \tag{20}$$

and

$$\begin{aligned} &r^{\alpha-n(m-l)} \int_{\left(B(x,2r)\right)^{m-l}} \prod_{i=l+1}^{m} |f_i(y_i)| dy_i \\ &\lesssim \prod_{i=l+1}^{m} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,2r))}\, \|w_i^{-1}\|_{L_{p_i'}(B(x,2r))} \int_{2r}^{\infty} \frac{dt}{t^{n(m-l)+1-\alpha}} \end{aligned}$$

$$\leq \int_{2r}^{\infty} \prod_{i=l+1}^{m} \|f\|_{L_{p_i,w_i^{p_i}}(B(x,t))} \|w_i^{-1}\|_{L_{p_i'}(B(x,t))} \frac{dt}{t^{n(m-l)+1-\alpha}} \tag{21}$$

$$\leq \prod_{i=l+1}^{m} [w_i]_{A_{p_i,q_i}}^{1/p_i} \int_{2r}^{\infty} \prod_{i=l+1}^{m} \|f\|_{L_{p_i,w_i^{p_i}}(B(x,t))} w_i^{q_i}(B(x,t))^{-1/q_i} \frac{dt}{t}.$$

From (20) and (21) we get

$$\|I_{\alpha,m}(f_1^\infty,\ldots,f_l^\infty,f_{l+1}^0,\ldots,f_m^0)\|_{L_{p,(\nu_{\vec{w}})^q}(B(x,r))}$$

$$\lesssim \nu_{\vec{w}}^q(B(x,r))^{1/q} \prod_{i=1}^{l} \int_{\complement_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^{n-\alpha}} dy_i \prod_{i=l+1}^{m} r^{\alpha-n} \int\limits_{B(x,2r)} |f_i(y_i)| dy_i$$

$$\lesssim \nu_{\vec{w}}^q(B(x,r))^{1/q} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f\|_{L_{p_i,w_i^{p_i}}(B(x,t))} w_i^{q_i}(B(x,t))^{-1/q_i} \frac{dt}{t}. \blacktriangleleft$$

Theorem 5.9. *Let $m \geq 2$, $0 < \alpha < mn$, $1/q = 1/p - \alpha/n$ and $I_{\alpha,m}$ be an m-linear fractional integral operator. If $p_1,\ldots,p_m \in [1,\infty)$, $\min\{p_1,\ldots,p_m\} = 1$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q_i = 1/p_i - \alpha/(nm)$, $1/q = 1/p - \alpha/n$ and $\vec{w} = (w_1,\ldots,w_m)$ satisfies the $A_{\vec{P},q}$ with $w_1^{q_1},\ldots,w_m^{q_m} \in A_\infty$, then the inequality*

$$\|I_{\alpha,m}(\vec{f})\|_{WL_{q,(\nu_{\vec{w}})^q}(B(x,r))} \lesssim \nu_{\vec{w}}^q(B(x,r))^{\frac{1}{q}} \int_{2r}^{\infty} \prod_{i=1}^{m} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))} w_i^{q_i}(B(x,t))^{-\frac{1}{q_i}} \frac{dt}{t} \tag{22}$$

holds for any ball $B(x,r)$ and for all $\vec{f} \in L_{p_1,w_1^{p_1}}^{loc}(\mathbb{R}^n) \times \ldots \times L_{p_m,w_m^{p_m}}^{loc}(\mathbb{R}^n)$.

Proof. For any ball $B = B(x,r) \subset \mathbb{R}^n$, decompose $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{2B}$, $2B = B(x,2r)$, $i = 1,\ldots,m$. Then for any given $\lambda > 0$, we can write

$$\begin{aligned}
&\nu_{\vec{w}}^q\big(\{y \in B(x,r) : \big|I_{\alpha,m}(\vec{f})(y)\big| > \lambda\}\big)^{1/q} \\
&\leq \nu_{\vec{w}}^q\big(\{y \in B(x,r) : \big|I_{\alpha,m}(f_1^0,\ldots,f_m^0)(y)\big| > \lambda/2^m\}\big)^{1/q} \\
&+ \sum{}' \nu_{\vec{w}}^q\big(\{y \in B(x,r) : \big|I_{\alpha,m}(f_1^{\beta_1},\ldots,f_m^{\beta_m})(y)\big| > \lambda/2^m\}\big)^{1/q} \\
&= J_*^0 + \sum{}' J_*^{\beta_1,\ldots,\beta_m},
\end{aligned}$$

where each term in $\sum'$ contains at least one $\beta_i \neq 0$. By Lemma 2.4 again, we know that $(\nu_{\vec{w}})^q \in A_{mq}$. Applying Theorem B, Lemma 3.1 and Lemma 2.1, we have

$$\begin{aligned}
J_*^0 &= \|I_{\alpha,m}(\vec{f^0})\|_{WL_{q,(\nu_{\vec{w}})^q}(B(x,r))} \leq \|I_{\alpha,m}(\vec{f^0})\|_{WL_{q,(\nu_{\vec{w}})^q}(\mathbb{R}^n)} \\
&\lesssim \prod_{i=1}^{m} \|f_i^0\|_{L_{p_i,w_i^{p_i}}(\mathbb{R}^n)} = \prod_{i=1}^{m} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,2r))}.
\end{aligned}$$

In the proof of Theorem 1.1, we have already shown the validity of the following estimate (see (3.1) and (3.3)):

$$\|I_{\alpha,m}(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{L_{q,(\nu_{\overrightarrow{w}})^q}(B(x,r))}$$
$$\lesssim (\nu_{\overrightarrow{w}})^q(B(x,r))^{1/q}\int_{2r}^{\infty}\prod_{i=1}^m \|f\|_{L_{p_i,w_i^{p_i}}(B(x,t))}\, w_i^{q_i}(B(x,t))^{-1/q_i}\,\frac{dt}{t}.$$

Then

$$\sum{}' J_*^{\beta_1,\dots,\beta_m} = \sum{}' \|I_{\alpha,m}(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{WL_{q,(\nu_{\overrightarrow{w}})^q}(B(x,r))}$$
$$\le \sum{}' \|I_{\alpha,m}(f_1^{\beta_1},\dots,f_m^{\beta_m})\|_{L_{q,(\nu_{\overrightarrow{w}})^q}(B(x,r))}$$
$$\lesssim (\nu_{\overrightarrow{w}})^q(B(x,r))^{1/q}\int_{2r}^{\infty}\prod_{i=1}^m \|f\|_{L_{p_i,w_i^{p_i}}(B(x,t))}\, w_i^{q_i}(B(x,t))^{-1/q_i}\,\frac{dt}{t}. \blacktriangleleft$$

Now we give the boundedness of multilinear fractional integral operator on product generalized weighted Morrey space.

Proof of Theorem 1.3. Let $p_1,\dots,p_m \in (1,\infty)$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q_i = 1/p_i - \alpha/(nm)$, $1/q = 1/p - \alpha/n$ and $\overrightarrow{f} \in M_{p_1,\varphi_1}(w_1)\times\ldots\times M_{p_m,\varphi_m}(w_m)$. By Theorems 4.5 and 5.8 we have

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{M_{q,\varphi}((\nu_{\overrightarrow{w}})^q)} = \sup_{x\in\mathbb{R}^n,\, r>0}\varphi(x,r)^{-1}\nu_{\overrightarrow{w}}^q(B(x,r))^{-\frac{1}{q}}\,\|I_{\alpha,m}(\overrightarrow{f})\|_{L_{q,w^q}(B(x,r))}$$
$$\lesssim \sup_{x\in\mathbb{R}^n,\, r>0}\varphi(x,r)^{-1}\prod_{i=1}^m\int_r^{\infty}\|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))}\, w_i^{q_i}(B(x,t))^{-\frac{1}{q_i}}\,\frac{dt}{t}$$
$$= \sup_{x\in\mathbb{R}^n,\, r>0}\varphi(x,r)^{-1}\prod_{i=1}^m\int_0^{r^{-1}}\|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t^{-1}))}\, w_i^{q_i}(B(x,t^{-1}))^{-\frac{1}{q_i}}\,\frac{dt}{t}$$
$$\lesssim \prod_{i=1}^m\sup_{x\in\mathbb{R}^n, r>0}\varphi_{1i}(x,r^{-1})^{-1}w_i^{p_i}(B(x,r^{-1}))^{-\frac{1}{p_i}}\,\|f\|_{L_{p_i,w_i^{p_i}}(B(x,r^{-1}))}$$
$$= \prod_{i=1}^m\sup_{x\in\mathbb{R}^n, r>0}\varphi_{1i}(x,r)^{-1}w_i^{p_i}(B(x,r)^{-\frac{1}{p_i}}\,\|f\|_{L_{p_i,w_i^{p_i}}(B(x,r))}$$
$$= \prod_{i=1}^m\|f_i\|_{M_{p_i,\varphi_{1i}}(w_i^{p_i})}. \quad\blacktriangleleft$$

Now we turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $p_1,\dots,p_m \in [1,\infty)$, $\min\{p_1,\dots,p_m\}=1$, $p\in(0,\infty)$ with $1/p=\sum_{k=1}^m 1/p_k$ and $\overrightarrow{f} \in M_{p_1,\varphi_1}(w_1)\times\ldots\times M_{p_m,\varphi_m}(w_m)$. By Theorems 4.5 and 5.9 we have

$$\|I_{\alpha,m}(\overrightarrow{f})\|_{WM_{q,\varphi}((\nu_{\overrightarrow{w}})^q)} = \sup_{x\in\mathbb{R}^n,\, r>0}\varphi(x,r)^{-1}\nu_{\overrightarrow{w}}^q(B(x,r))^{-\frac{1}{q}}\,\|I_{\alpha,m}(\overrightarrow{f})\|_{WL_{q,w^q}(B(x,r))}$$

$$
\begin{aligned}
&\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \prod_{i=1}^m \int_r^\infty \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t))}\, w_i^{q_i}(B(x,t))^{-\frac{1}{q_i}} \frac{dt}{t} \\
&= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi(x,r)^{-1} \prod_{i=1}^m \int_0^{r^{-1}} \|f_i\|_{L_{p_i,w_i^{p_i}}(B(x,t^{-1}))}\, w_i^{q_i}(B(x,t^{-1}))^{-\frac{1}{q_i}} \frac{dt}{t} \\
&\lesssim \prod_{i=1}^m \sup_{x\in\mathbb{R}^n, r>0} \varphi_{1i}(x,r^{-1})^{-1} w_i^{p_i}(B(x,r^{-1}))^{-\frac{1}{p_i}} \|f\|_{L_{p_i,w_i^{p_i}}(B(x,r^{-1}))} \\
&= \prod_{i=1}^m \sup_{x\in\mathbb{R}^n, r>0} \varphi_{1i}(x,r)^{-1} w_i^{p_i}(B(x,r))^{-\frac{1}{p_i}} \|f\|_{L_{p_i,w_i^{p_i}}(B(x,r))} \\
&= \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_{1i}}(w_i^{p_i})}. \quad \blacktriangleleft
\end{aligned}
$$

By using Hölder's inequality, it is easy to verify that if each w_i is in A_{p_i,q_i}, then

$$\prod_{i=1}^m A_{p_i,q_i} \subset A_{\vec{P},q}$$

and this inclusion is strict (see [36]). Thus, as direct consequences of Theorems 1.3 and 1.4, we immediately obtain the following

Corollary 5.3. *Let $m \geq 2$, $0 < \alpha < mn$ and $I_{\alpha,m}$ be an m-linear fractional integral operator. If $p_1,\ldots,p_m \in (1,\infty)$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q_i = 1/p_i - \alpha/(nm)$, $1/q = 1/p - \alpha/n$, $\vec{w} = (w_1,\ldots,w_m) \in \prod_{i=1}^m A_{p_i,q_i}$ and $(\varphi_1,\ldots,\varphi_m,\varphi)$ satisfies the condition (3), then the operator $I_{\alpha,m}$ is bounded from product space $M_{p_1,\varphi_1}(w_1^{p_1}) \times \ldots \times M_{p_m,\varphi_m}(w_m^{p_m})$ to $M_{q,((\nu_{\vec{w}})^q)}$. Moreover, there exists a constant $C > 0$ independent of $\vec{f} = (f_1, f_2, \ldots, f_m)$ such that*

$$\|I_{\alpha,m}(\vec{f})\|_{M_{q,\varphi}((\nu_{\vec{w}})^q)} \leq C \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_{1i}}(w_i^{p_i})},$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i$.

Corollary 5.4. *Let $m \geq 2$, $0 < \alpha < mn$ and $I_{\alpha,m}$ be an m-linear fractional integral operator. If $p_1,\ldots,p_m \in [1,\infty)$, $\min\{p_1,\ldots,p_m\} = 1$, $1/p = \sum_{k=1}^m 1/p_k$, $1/q_i = 1/p_i - \alpha/(nm)$, $1/q = 1/p - \alpha/n$, $\vec{w} = (w_1,\ldots,w_m) \in \prod_{i=1}^m A_{p_i,q_i}$ and $(\varphi_1,\ldots,\varphi_m,\varphi)$ satisfies the condition (3), then the operator $I_{\alpha,m}$ is bounded from product space $M_{p_1,\varphi_1}(w_1^{p_1}) \times \ldots \times M_{p_m,\varphi_m}(w_m^{p_m})$ to $WM_{q,\varphi}((\nu_{\vec{w}})^q)$. Moreover, there exists a constant $C > 0$ independent of $\vec{f} = (f_1, f_2, \ldots, f_m)$ such that*

$$\|I_{\alpha,m}(\vec{f})\|_{M_{q,\varphi}((\nu_{\vec{w}})^q)} \leq C \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_{1i}}(w_i^{p_i})},$$

where $\nu_{\vec{w}} = \prod_{i=1}^m w_i$.

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