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On Weighted Sobolev Type Inequalities in Spaces of Differentiable Functions

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Abstract. We prove two-weighted Sobolev type inequality that estimates $L_v^q(\Omega)$ weighted norm of differentiable function $u(x)$ vanishing on the boundary of domain Ω through $L_\omega^p(\Omega)$ weighted norm of its first derivatives for $1 < q < p < \infty$, some class of weights $v, \omega^{-\frac{1}{p-1}} \in L^{1,loc}$ and $\Omega \subset R^n$.

Key Words and Phrases: embedding theorems, weighted function spaces, Sobolev type inequality.
2010 Mathematics Subject Classifications: 26D10, 26D15, 46D35, 46B35

1. Introduction

This paper studies weighted Sobolev inequalities

$$\left(\int_\Omega |u(x)|^q v(x)dx\right)^{\frac{1}{q}} \leq \left(\int_\Omega |\nabla u(x)|^p \omega(x)\, dx\right)^{\frac{1}{p}} \tag{1}$$

for some class of domains $\Omega \subset R^n$ and differentiable functions $u(x)$ in Ω vanishing on the boundary $\partial\Omega$, where $1 \leq q < p < \infty$, $v(x)$ and $\omega(x)$ are a.e. positive functions in some neighborhood of Ω. Our approach is similar to that of [7] used for the case $\infty > q \geq p \geq 1$.

As $f(x) \leq I_1(|\nabla f|)(x)$, the inequalities like (1) are usually derived from two-weighted estimates for fractional integrals

$$I_\alpha f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

For example, it was shown in [1] that in case $1 < p < q$, for $\Omega = R^n$ and Lipshitz continuous functions $u(x)$ in R^n, these inequalities arise from the estimate for $I_\alpha f$ stated in [2] with $\alpha = 1$ and the weight functions $v(x)$ and $\omega(x)$ satisfying the condition

$$\left(\int_Q v dx\right)^{\frac{1}{q}} \left(\int_{R^n} \frac{\omega^{1-p'}}{\left(|Q|^{\frac{1}{n}} + |x_Q - x|\right)^{(n-1)p'}} dx\right)^{\frac{1}{p'}} \leq C \tag{2}$$

for all balls $Q \subset R^n$, where $p' = \frac{p}{p-1}$, x_Q denotes the center of the ball Q and $|Q|$ is its Lebesgue measure.

The condition (2) is equivalent to a simpler one

$$B_{pq} = \sup_Q |Q|^{\frac{1}{n}-1} \left(\int_Q v dx \right)^{\frac{1}{q}} \left(\int_Q \omega^{1-p'} dx \right)^{\frac{1}{p'}} < +\infty \qquad (A_{pq})$$

(see [1]), if $1 < p < q$ and $v(x)$ satisfies the reverse doubling condition, i.e. if there exist $\delta,\ \varepsilon \in (0,\,1)$ such that

$$\int_{\delta Q} v dx \le \varepsilon \int_Q v dx, \qquad (RD)$$

where δQ is a ball concentric to Q with a radius δ times as large as the radius of Q; or if $q = p$ and $v(x)$ satisfies the condition A_∞ (see, [7]), i.e. if there exist positive numbers β and δ such that

$$\frac{v(E)}{v(Q)} \le \beta \left(\frac{|E|}{|Q|} \right)^{\delta} (A_\infty) \qquad (A_\infty)$$

for every ball Q and every measurable subset $E \subset Q$ with $v(E) = \int_E v(x) dx$.

Using the estimates for $I_\alpha f$ obtained in [1] and the results of [2] and [3], it is proven in [1] that in case $\Omega = Q_0$, where Q_0 is some ball in R^n, the inequality (1) is true for $q > p > 1$ if

$$\sup_{Q \subset 8Q_0} \left(\int_Q v dx \right)^{\frac{1}{q}} \left(\int_{8Q_0} \frac{\omega^{1-p'}}{\left(|Q|^{\frac{1}{n}} + |x_Q - x| \right)^{(n-1)p'}} dx \right)^{\frac{1}{p'}} < +\infty \qquad (3)$$

and for $q = p > 1$ if

$$\sup_{Q \subset 8Q_0} |Q|^{\frac{1}{n}-\frac{1}{r}} \left(\int_Q v^r dx \right)^{\frac{1}{pr}} \left(\int_Q \omega^{(1-p')r} dx \right)^{\frac{1}{p'r}} < +\infty \qquad (4)$$

with some $r > 1$. Besides, in case when $q > p$ and $v \in (RD)$ or when $q = p$ and $v \in A_\infty$ and $\omega^{1-p'} \in A_\infty$, (3) and (4) can be replaced by

$$\sup_{Q \subset 8Q_0} |Q|^{\frac{1}{n}-1} \left(\int_Q v dx \right)^{\frac{1}{p}} \left(\int_Q \omega^{1-p'} dx \right)^{\frac{1}{p'}} < +\infty.$$

The case $q < p$ is more complicated. For example, a necessary and sufficient condition on the measure μ in R^n which guarantees the validity of inequality

$$\|I_\alpha f\|_{L^q(d\mu)} \le C \, \|f\|_{L_p(R^n)}$$

is obtained in [4] in terms of the Wolf potential.

We finally note the works [5, 6, 8] which treat the inequality (1) for general weights. Also note the one-dimensional result for Hardy inequality with $1 \le q < p \le \infty$

$$\left(\int_0^\infty \left|\int_x^\infty f(t)\,dt\right|^q v(x)\,dx\right)^{\frac{1}{q}} \le c\left(\int_0^\infty |f(x)|^p \omega(x)\,dx\right)^{\frac{1}{p}} \tag{5}$$

which holds if and only if

$$\left(\int_0^\infty \omega^{1-p'}(z)\left[\int_0^z v(x)\,dx\left(\int_z^\infty \omega^{1-p'}(x)\,dx\right)^{q-1}\right]^{\frac{p}{p-q}} dz\right)^{\frac{p-q}{p}} < +\infty. \tag{6}$$

In the sequel, by C, C_0, C_1, etc. we will denote the positive constants with different meanings in different parts of the text which depend only on p, q, dimension n, and sometimes, on the constants β and δ included in condition (A_∞).

Definition 1. *The domain $\Omega \subset R^n$ is said to belong to the class H if there exist $\varepsilon > 0$ and $\rho_0 > 0$ such that*

$$\left|Q^x_{\rho_0} \backslash \Omega\right| \ge \varepsilon \left|Q^x_{\rho_0}\right| \tag{H}$$

for every point $x \in \Omega$, where $Q^x_{\rho_0}$ is a ball of radius ρ_0 centered at x.

As an example of such domains, we can mention any domain lying between two planes $x_n = a$ and $x_n = b$, $a \ne b$. In particular, any bounded domain belongs to the class H.

The following result was proved in [7].

Theorem 1. *Let $1 \le p \le q < \infty$, $\Omega \in H$ and $v(x) \in A_\infty$. If $v(x)$ and $\omega(x)$satisfy the condition (A_{pq}) with $p > 1$, or*

$$\left(\int_Q v dx\right)^{\frac{1}{q}} \le B_{1q}\, es \sin_{x \in Q} f\, \omega(x)$$

for every ball $Q \cap \Omega \ne \emptyset$ with $p = 1$, then the inequality (1) holds.

Now we state the main result of this paper.

Theorem 2. *Let $1 \le q < p < \infty$, $\Omega \in H$ and $v(x) \in A_\infty$. If*

$$\tilde{B}_{pq} = \left(\int_\Omega \omega^{1-p'}(z)\, M^{\frac{p}{p-q}}(z)\,dz\right)^{\frac{p-q}{pq}} < +\infty, (\tilde{A_{pq}})$$

where

$$M(z) = \sup_{Q^z} \frac{v(Q^z)}{|Q^z|^{\frac{n-1}{n}q}} \left(\int_{Q^z} \omega^{1-p'} dx\right)^{q-1}$$

with $q \ge 1$ and the supremum is taken over all balls Q^z centered at a point $z \in \Omega$, then the inequality (1) holds.

2. Proof of main result

Let $\alpha > 0$,

$$e_\alpha = \{x \in \Omega : \; |u(x)| > \alpha\}$$

and let $\gamma \in (0,\, 1)$ be some number to be specified later. For every fixed $x \in e_{2\alpha}$ there exists a ball $Q^x_{\rho(x)}$ such that

$$\left| Q^x_{\rho(x)} \backslash e_\alpha \right| = \gamma \left| Q^x_{\rho(x)} \right|, \tag{7}$$

because the function $F(t) = |Q^x_t \backslash e_\alpha|$ is continuous on $(0, +\infty)$, $F(t) \to 0$ as $t \to 0$ and $F(t) \geq \gamma |Q^x_t|$ for sufficiently large values of t due to the conditions imposed on Ω and the function $u(x)$.

It is evident that all possible balls $Q^x_{\rho(x)}$, $x \in e_{2\alpha}$, form the cover of the set $e_{2\alpha}$ and their radii are uniformly bounded by the number ρ_0 due to the condition (H). By the Besicovitch covering lemma (see, e.g., [9]), one can find from the system $\left\{ Q^x_{\rho(x)} \right\}$ a subcover $\{Q_i\}$ of finite multiplicity for the set $e_{2\alpha}$.

Let Q_i be one of the balls of $\{Q_i\}$. There are two possibilities: either
a) $|Q_i \cap e_{2\alpha}| < \gamma |Q_i|$ or b) $|Q_i \cap e_{2\alpha}| \geq \gamma |Q_i|$.

Let us show that in case a) we have

$$v(e_{2\alpha} \cap Q_i) \leq \frac{\beta\gamma^\delta}{1 - \beta\gamma^\delta} v(Q_i \cap e_\alpha) \tag{8}$$

if

$$1 - \beta\gamma^\delta > 0, \tag{9}$$

and in case b) we have

$$v(Q_i \cap e_{2\alpha}) \leq c \left[\frac{1}{\gamma^2 |Q_i|^{1-\frac{1}{n}}} v^{\frac{1}{q}}(Q_i) \left(\int_{Q_i} \omega^{1-p'} dz \right)^{\frac{1}{p'}} \right]^q \cdot \frac{1}{\alpha^q} \left(\int_{Q_i \cap (e_\alpha \backslash e_{2\alpha})} \omega |\nabla u|^p \, dz \right)^{\frac{q}{p}}. \tag{10}$$

In case a), due to the condition (A_∞)

$$v(Q_i \cap e_{2\alpha}) \leq \beta \left(\frac{|Q_i \cap e_{2\alpha}|}{|Q_i|} \right)^\delta v(Q_i) \leq \beta\gamma^\delta v(Q_i). \tag{11}$$

On the other hand, again by virtue of (A_∞) and (7),

$$v(Q_i) = v(Q_i \cap e_\alpha) + v(Q_i \backslash e_\alpha) \leq v(Q_i \cap e_\alpha) + \beta \left(\frac{|Q_i \backslash e_\alpha|}{|Q_i|} \right)^\delta v(Q_i) \leq$$

$$\leq v(Q_i \cap e_\alpha) + \beta\gamma^\delta v(Q_i). \tag{12}$$

It is clear that γ defined by (9) guarantees the validity of condition (9). Therefore, from (13) we have

$$v(Q_i) \le \frac{1}{1-\beta\gamma^\delta} v(Q_i \cap e_\alpha).$$

Using the last inequality in (11), we get the estimate (8).

Consider case b). For simplicity, we denote $A = \{Q_i \backslash e_\alpha\} \times \{Q_i \cap e_{2\alpha}\}$. Due to (7), we have

$$\iint_A dxdy \ge \gamma^2 |Q_i|^2. \tag{13}$$

For fixed $x \in Q_i \backslash e_\alpha$ and $y \in Q_i \cap e_{2\alpha}$ we can find $t_1 = t_1(x, y)$ and $t_2 = t_2(x, y)$, $t_2 \ge t_1$, such that

$$|u(x + t_1(y - x))| = \alpha \text{ and } |u(x + t_2(y - x))| = 2\alpha.$$

Put $x_k = x + t_k(y - x)$, $k = 1, 2$. Then from (13) we get

$$1 \le \frac{1}{\left(\gamma^2 |Q_i|^2\right)^q} \frac{1}{\alpha^q} \left(\iint_A |u(x_2) - u(x_1)| \, dxdy\right)^q,$$

or

$$v(Q_i \cap e_{2\alpha}) \le \frac{v(Q_i)}{\left(\gamma^2 |Q_i|^2\right)^q} \frac{1}{\alpha^q} \left(\iint_A |x - y| \int_{t_1}^{t_2} |\nabla u(x + t(y - x))| \, dtdxdy\right)^q.$$

As $|x - y| \le c_0 |Q_i|^{\frac{1}{n}}$, by the Hölder inequality we have

$$v(Q_i \cap e_{2\alpha}) \le c_0 \left(\frac{v^{\frac{1}{q}}(Q_i) |Q_i|^{\frac{1}{n}}}{\gamma^2 |Q_i|^2}\right)^q \frac{1}{\alpha^q} \left(I_1^{\frac{1}{p'}} \cdot I_2^{\frac{1}{p}}\right)^q, \tag{14}$$

where

$$I_1 = \iint_A \int_{t_1}^{t_2} \omega^{1-p'}(x + t(y - x)) \, dtdxdy,$$

$$I_2 = \iint_A \int_{t_1}^{t_2} \omega(x + t(y - x)) |\nabla u(x + t(y - x))|^p \, dtdxdy.$$

According to Fubini theorem,

$$I_1 \le \int_0^1 \iint_{\substack{A, \\ x + t(y - x) \in Q_i \cap (e_\alpha \backslash e_{2\alpha})}} \omega^{1-p'}(x + t(y - x)) \, dxdydt \le$$

$$\leq \int_0^1 \int_{Q_i \backslash e_\alpha} \left(\int_{\substack{Q_i \cap e_\alpha, \\ x+t(y-x) \in Q_i \cap (e_\alpha \backslash e_{2\alpha})}} \omega^{1-p'} (x + t(y-x))\, dy \right) dxdt. \qquad (15)$$

Change of variables $z = x + t(y-x)$ reshapes the right-hand side as follows

$$c_1 \int_0^1 \left(\int_{Q_i \backslash e_\alpha} \left(\int_{\substack{z \in Q_i \cap (e_\alpha \backslash e_{2\alpha}), \\ \frac{z-x}{t} + x \in Q_i \cap e_{2\alpha}}} \omega^{1-p'}(z)\, dz \right) dx \right) t^{-n} dt =$$

$$= c_1 \int_0^1 \left(\int_{z \in Q_i \cap (e_\alpha \backslash e_{2\alpha})} \omega^{1-p'}(z) \left(\int_{\substack{Q_i \backslash e_\alpha, \\ \frac{z-x}{t} + x \in Q_i \cap e_{2\alpha}}} dx \right) dz \right) t^{-n} dt. \qquad (16)$$

Noting that for fixed $z \in Q_i \cap (e_\alpha \backslash e_{2\alpha})$ the set of points $x \in Q_i \backslash e_\alpha$ with $\frac{z-x}{t} + x \in Q_i \cap e_{2\alpha}$ is contained in the ball $Q^z_{c_0 t |Q^i|^{1/n}}$, we change the domain of integration from the set $Q_i \cap (e_\alpha \backslash e_{2\alpha})$ to the entire ball Q_i. Then the right-hand side of last equality will be majorized by

$$c_2 \int_0^1 \int_{Q_i} \omega^{1-p'}(z) \left(\int_{|x-z| \leq c_0 t |Q_i|^{\frac{1}{n}}} dx \right) dz t^{-n} dt \leq c_3 |Q_i| \int_{Q_i} \omega^{1-p'} dz. \qquad (17)$$

Thus, succession of estimates (17), (16), (15) leads to the following inequality

$$I_1 \leq c_4 |Q_i| \int_{Q_i} \omega^{1-p'} dz.$$

Performing the similar procedures for I_2, we get

$$I_2 \leq \int_0^1 \int_{Q_i \cap (e_\alpha \backslash e_2 \alpha)} \omega(z) |\nabla u(z)|^p \left(\int_{|x-z| \leq c_0 t |Q_i|^{\frac{1}{n}}} dx \right) dz t^{-n} dt \leq$$

$$c_5 |Q_i| \int_{Q_i \cap (e_\alpha \backslash e_2 \alpha)} \omega(z) |\nabla u(z)|^p dz.$$

Substituting the above inequalities for I_1 and I_2 into (14), we finally obtain the required estimate (10).

Now, combining the estimates (8) and (10) and summing over i we get

$$v(e_{2\alpha}) \leq \frac{c_2 \beta \gamma^\delta}{1 - \beta \gamma^\delta} v(e_\alpha) + \frac{c_1}{\gamma^{2q}} \frac{1}{\alpha^q} \sum_i \left[\frac{v(Q_i \cap e_{2\alpha})}{|Q_i|^{\frac{n-1}{n} q}} \left(\int_{Q_i \cap (e_\alpha \backslash e_{2\alpha})} \omega^{1-p'} dz \right)^{\frac{q(p-1)}{p}} \times \right.$$

$$\times\left(\int_{Q_i\cap(e_\alpha\setminus e_{2\alpha})}\omega\,|\nabla u|^p\,dz\right)^{\frac{q}{p}}\Bigg].$$

By virtue of the Hölder inequality with the exponent $p/\,(p-q)$, we have

$$v\,(e_{2\alpha})\le\frac{c_2\beta\gamma^\delta}{1-\beta\gamma^\delta}v\,(e_\alpha)+$$

$$+\frac{c_1}{\gamma^{2q}}\frac{1}{\alpha^q}\left[\sum_i\left(\frac{v\,(Q_i)}{|Q_i|^{\frac{n-1}{n}q}}\left(\int_{Q_i\cap(e_\alpha\setminus e_{2\alpha})}\omega^{1-p'}dz\right)^{\frac{p-1}{p}q}\right)^{\frac{p}{p-q}}\right]^{\frac{p-q}{p}}\times$$

$$\times\left[\sum_i\int_{Q_i\cap(e_\alpha\setminus e_{2\alpha})}\omega\,|\nabla u|^p\,dz\right]^{\frac{p}{q}}.$$

Choose γ by (9) and integrate both sides of the last inequality over α. Then we get

$$\int_0^\infty v\,(e_\alpha)\,d\alpha^q\le c_3\int\frac{d\alpha}{\alpha^{\frac{p-q}{p}}}\left[\sum_i\left(\frac{v\,(Q_i)}{|Q_i|^{\frac{n-1}{n}q}}\left(\int_{Q_i(e_\alpha\setminus e_{2\alpha})}\omega^{1-p'}dz\right)^{\frac{p-1}{p}q}\right)^{\frac{p}{p-q}}\right]^{\frac{p-q}{p}}\times$$

$$\times\left[\frac{1}{\alpha^{\frac{q}{p}}}\left(\int_{Q_i\cap(e_\alpha\setminus e_{2\alpha})}\omega\,|\nabla u|^p\,dz\right)^{\frac{q}{p}}\right].\tag{18}$$

By virtue of the Hölder inequality, again with the index $p/\,(p-q)$, the right-hand side of (18) is bounded by

$$c_3\cdot\left\{\int_0^\infty\frac{d\alpha}{\alpha}\sum_i\left[\left(\frac{v\,(Q_i)}{|Q_i|^{\frac{n-1}{n}q}}\right)^{\frac{p}{p-q}}\left(\int_{Q_i\cap(e_\alpha\setminus e_{2\alpha})}\omega^{1-p'}dz\right)^{\frac{p-1}{p-q}q}\right]\right\}^{\frac{p-q}{q}}\cdot$$

$$\cdot\left\{\int_0^\infty\frac{d\alpha}{\alpha}\int_{e_\alpha\setminus e_{2\alpha}}\omega\,|\nabla u|^p\,dz\right\}^{\frac{q}{p}}\le$$

$$c_4\cdot\left\{\int_0^\infty\frac{d\alpha}{\alpha}\sum_i\left[\int_{Q_i\cap(e_\alpha\setminus e_{2\alpha})}\omega^{1-p'}dz\left(\frac{v\,(Q_i)}{|Q_i|^{\frac{n-1}{n}q}}\right)^{\frac{p}{p-q}}\left(\int_{Q_i\cap(e_\alpha\setminus e_{2\alpha})}\omega^{1-p'}dx\right)^{\frac{p-1}{p-q}q-1}\right]\right\}^{\frac{p-q}{p}}$$

$$\left(\int_\Omega\omega\,|\nabla u|^p\,dz\right)^{\frac{q}{p}}.\tag{19}$$

Let Q_i^z be a ball centered at a point $z \in Q_i \cap (e_\alpha \backslash e_{2\alpha})$ with a radius twice as large as the radius of Q_i, $Q_i \subset Q_i^z$. Taking into account that $\frac{p-1}{p-q}q - 1 \geq 0$ with $q \geq 1$ implies

$$\left(\int_{Q_i \cap (e_\alpha \backslash e_{2\alpha})} \omega^{1-p'} dx\right)^{\frac{p-1}{p-q}q-1} \leq \left(\int_{Q_i^z} \omega^{1-p'} dx\right)^{\frac{p-1}{p-q}q-1},$$

and the comparableness of balls Q_i and Q_i^z implies $\frac{v(Q_i)}{|Q_i|^{\frac{n-1}{n}q}} \leq c_5 \frac{v(Q_i^z)}{|Q_i^z|^{\frac{n-1}{n}q}}$, we come to a conclusion that the right-hand side of (19) is majorized by

$$c_6 \left(\int_\Omega \omega |\nabla u|^p\right)^{\frac{q}{p}} \left\{\int_0^\infty \frac{d\alpha}{\alpha} \sum_i \int_{Q_i \cap (e_\alpha \backslash e_{2\alpha})} \omega^{1-p'}(z) \left(\frac{v(Q_i^z)}{|Q_i^z|^{\frac{n-1}{n}q}}\right)^{\frac{p}{p-q}} \times\right.$$

$$\left. \times \left(\int_{Q_i^z} \omega^{1-p'}(x)\, dx\right)^{\frac{p-1}{p-q}q-1}\right\}^{\frac{p-q}{p}} \leq$$

$$c_7 \left(\int_\Omega \omega |\nabla u|^p\right)^{\frac{q}{p}} \left\{\int_0^\infty \frac{d\alpha}{\alpha} \int_{e_\alpha \backslash e_{2\alpha}} \omega^{1-p'}(z)\, M^{\frac{p}{p-q}}(z)\, dz\right\}^{\frac{p-q}{p}} \leq$$

$$\leq c_8 \left(\int_\Omega \omega |\nabla u|^p\right)^{\frac{q}{p}} \left(\int_\Omega \omega^{1-p'} M^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}. \qquad (20)$$

Thus, the succession of inequalities (20)-(19)-(18) and the condition $\left(\tilde{A}_{pq}\right)$lead to the inequality

$$\left(\int_\Omega |u|^q v\right)^{\frac{1}{q}} \leq c\tilde{B}_{pq} \left(\int_\Omega \omega |\nabla u|^p\right)^{\frac{1}{p}}.$$

Theorem is proved. ◄

Note that Theorem 1 in [7] was proved for more general class of unbounded domains satisfying the following condition:

there exists $\varepsilon \in (0, 1)$ such that for every point $x \in \Omega$ one can find the largest ball $Q_{R(x)}^x$ with

$$\left|Q_{Q(x)}^x \backslash \Omega\right| \geq \varepsilon \left|Q_{R(x)}^x\right|.$$

Theorem 2 is true for such class of domains too, we have considered the domains $\Omega \in H$ only for the sake of simplicity. It follows from the proof that both theorems remain valid for any open domains Ω if the condition $\lim_{x \in \infty} u(x) = 0$ as $x \in \infty$ is satisfied.

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