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On The Partially Large Solutions For Semilinear Hyperbolic Systems With Damping

A. B.Aliev*, A. A.Kazymov

Abstract. In this paper we study existence of global solution for semilinear systems of hyperbolic equations with damping. Note that one component of the solution can be arbitrarily large, while another component is sufficiently small.

Key Words and Phrases: global solvability, semilinear hyperbolic system, Cauchy problem

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1. Introduction

We consider the Cauchy problem for the following system of semilinear hyperbolic equations

$$\left.\begin{array}{l} u_{1tt} + u_{1t} + (-1)^{l_1} \Delta^{l_1} u_1 = f_1(u_1, u_2) \\ u_{2tt} + u_{2t} + (-1)^{l_2} \Delta^{l_2} u_2 = f_2(u_1, u_2) \end{array}\right\}, \tag{1}$$

with initial data

$$\left.\begin{array}{l} u_1(0,x) = \varphi_1(x), \quad u_{1_t}(0,x) = \psi_1(x), \\ u_2(0,x) = \varepsilon\varphi_2(x), \quad u_{2_t}(0,x) = \varepsilon\psi_2(x) \end{array}\right\}, \ x \in R^N, \tag{2}$$

where $\varepsilon \in R$,

$$f_i(u_1, u_2) \sim u_1^{\rho_{i1}} u_2^{\rho_{i2}}. \tag{3}$$

The problem of the existence of global solutions of the Cauchy problem for semilinear hyperbolic equations with the condition (3) is the subject of numerous studies [1-9]. In these works, as well as in [10], initial data are quite small.

We assume that $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$ are continuous differentiable functions, and

$$|f_i(u,v)| \le b\,|u|^{\rho_{i1}}\,|v|^{\rho_{i2}}, \quad b > 0, \tag{4}$$

where

$$\rho_{ik} \ge 0, \qquad i,k = 1,2\ , \tag{5}$$

*Corresponding author.

$$\rho_{i1}+\rho_{i2}>\frac{2}{m_i},\ \ i=1,2, \tag{6}$$

$$\frac{\rho_{i1}}{l_1 m_1}+\frac{\rho_{i2}}{l_2 m_2}>\frac{2}{N}+\frac{r_i}{m_i};\ \ i=1,2\ ; \tag{7}$$

$$r_i=max\left\{r\left(l_1,l_2,\rho_{i1},\rho_{i2},m_i\right),r\left(l_2,l_1,\rho_{i2},\rho_{i1},m_i\right)\right\},$$

$$r(a,b,\alpha,\beta,m)=\begin{cases}\frac{1}{a}, & \alpha\geq\frac{2}{m}, \quad \beta\geq 0,\\ \frac{m\alpha}{2a}+\frac{2-m\alpha}{2b}, & 0\leq\ \alpha<\frac{2}{m},\ \beta>0;\end{cases}$$

$i=1,2$.

We also assume that

$$2\min(l_1,l_2)\geq N. \tag{8}$$

In [9] it is proved that under conditions (4)–(8) the problem (1), (2) has a global solution for "sufficiently small" initial data $(\varphi_i,\psi_i)\in(H^{l_i}(R^N)\cap L_1(R^N))\times(L_2(R^N)\cap L_1(R^N)),\ i=1,2$.

In this article we will investigate the global solvability of (1),(2) for arbitrary initial data $(\varphi_i,\psi_i)\in(H^{l_i}(R^N)\cap L_{m_i}(R^N))\times(L_2(R^N)\cap L_{m_i}(R^N))$, $1\leq m_i\leq 2$, $i=1,2$ and rather small $\varepsilon\in R$ under the assumptions (4)-(8) and

$$\rho_{12}>0,\quad \rho_{22}>1. \tag{9}$$

In other words, the initial data are arbitrary with respect to one variable and "sufficiently small" with respect to the other.

2. Preliminaries and statement of the results

In the sequel, by $\|\cdot\|_q$, we denote the usual $L_q(R^N)$-norm. For simplicity we write $\|\cdot\|$ instead of $\|\cdot\|_2$.

We denote by $C\left([0,T];H^l\left(R^N\right)\right)$ the space of all continuous functions $u:[0,T]\to H^l\left(R^N\right)$ with

$$\|u\|_{C\left([0,T];H^l(R^N)\right)}=max\|u\left(t,\cdot\right)\|_{H^l(R^N)},l=0,1,...,$$

and by $C^1\left([0,T];L_2\left(R^N\right)\right)$ the space of all differentiable functions $u:[0,T]\to L_2\left(R^N\right)$ with

$$\|u\|_{C^1([0,T];L_2(R^N))}=\|u\|_{C([0,T];L_2(R^N))}+\|u'\|_{C([0,T];L_2(R^N))}.$$

In addition, we denote $C\left([0,\infty);H^l\left(R^N\right)\right)=\underset{T>0}{\cup}C\left([0,T];H^l\left(R^N\right)\right)$, and $C^1\left([0,\infty);L_2\left(R^N\right)\right)=\underset{T>0}{\cup}C^1\left([0,T];L_2\left(R^N\right)\right)$. The purpose of this work is to prove the following result.

Theorem (Main results). *Suppose that the assumptions* (4)–(9) *are satisfied, and*

$$(\varphi_i,\psi_i)\in (H^{l_i}(R^N)\cap L_{m_i}(R^N))\times(L_2(R^N)\cap L_{m_i}(R^N)), i=1,2 \tag{10}$$

where $m_i\in[1,2]$. *Then there exists* $\varepsilon_0>0$ *such that, if* $0<\varepsilon\le\varepsilon_0$ *then the problem* (1), (2) *has a unique solution*

$$u=(u_1,u_2)\in C\left([0,\infty);H^{l_1}\left(R^N\right)\times H^{l_2}(R^N)\right)\cap C^1\left([0,\infty);L_2\left(R^N\right)\times L_2(R^N)\right).$$

Note that for every $M_1>M_{10}=b_1E_1(\varphi_1,\psi_1)$ *and* $M_2>M_{20}=b_1E_2(\varphi_2,\psi_2)$, *the number* $\varepsilon_0>0$ *is selected so that the following estimates are valid*

$$\sum_{|\alpha|=k}\|D^\alpha u_i(t,\cdot)\|\le M_i(\varepsilon_0)(1+t)^{-\frac{N\left(\frac{1}{m_i}-\frac{1}{2}\right)+k}{2l_i}},\quad k=0,1,...,l; \tag{11}$$

$$\|u_{it}(t,\cdot)\|\le M_i(\varepsilon_0)(1+t)^{-\gamma_i}. \tag{12}$$

Here

$$M_1(\varepsilon)=M_1,\quad M_2(\varepsilon)=\varepsilon M_2,$$

$$E_i(\varphi_i,\psi_i)=\left\|\nabla^{l_i}\varphi_i\right\|+\|\psi_i\|+\|\varphi_i\|+\|\varphi_i\|_{m_i}+\|\psi_i\|_{m_i},$$

$$\gamma_i=\min\left\{1+\frac{N(\frac{1}{m_i}-\frac{1}{2})}{2l_i},\frac{N}{2}\sum_{k=1}^{2}\frac{\rho_{ik}}{m_kl_k}-\frac{r_i}{m_i}\right\},\quad i=1,2.$$

The theorem will be proved by the method of successive approximations. First, let's state some facts about the asymptotics of solutions of the corresponding linear problem.

Let $\widehat{Z}_{ik}(t,\xi)$, $i,k=1,2$ be solutions of the following Cauchy problem

$$\begin{array}{ll} L_i\widehat{Z}_{i1}(t,\xi)=0, & \widehat{Z}_{i1}(0,\xi)=1,\quad \widehat{Z}_{i1_t}(t,\xi)=0,\quad i=1,2,\\ L_i\widehat{Z}_{i2}(t,\xi)=0, & \widehat{Z}_{i2}(0,\xi)=0,\quad \widehat{Z}_{i2_t}(t,\xi)=1,\quad i=1,2\end{array},$$

where

$$L_i\widehat{Z}(t,\xi)=\widehat{Z}_{tt}+\widehat{Z}_t+|\xi|^{2l_i}\widehat{Z}.$$

We denote $Z_{ik}(t,x)=F^{-1}(\widehat{Z}_{ik})(t,x)$, $i,k=1,2$, where F is the Fourier transform of the functions $\widehat{Z}_{ik}(t,\xi)$, $i,k=1,2$.

Lemma 1 (see [10]). *Let* $N\ge1$, $\varphi_i\in H^l(R^N)\cap L_{m_i}(R^N)$ *and* $\psi_i\in L_2(R^N)\cap L_{m_i}(R^N)$, *with* $1\le m_i\le2$. *Then the following estimates hold:*

$$\|Z_{i1}(t,x)*\varphi_i(x\|\le b_i(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})}(\|\varphi_i\|+\|\varphi_i\|_{m_i}), \tag{13}$$

$$\sum_{|\alpha|=l_i}\|D^\alpha Z_{i1}(t,x)*\varphi_i(x\|\le b_i(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})-\frac{1}{2}}(\left\|\nabla^{l_i}\varphi_i\right\|+\|\varphi_i\|_{m_i}), \tag{14}$$

$$\|D_t(Z_{i1}(t,x)*\varphi_i(x))\|\le b_i(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})-1}(\left\|\nabla^{l_i}\varphi_i\right\|+\|\varphi_i\|_{m_i}) \tag{15}$$

$$\|Z_{i2}(t,x) \ * \ \psi_i(x\| \le b_i\,(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})}\,(\|\psi_i\| + \|\psi_i\|_{m_i}), \tag{16}$$

$$\sum_{|\alpha|=l_i} \|D^\alpha Z_{i2}(t,x) \ * \ \psi_i(x)\| \le b_i\,(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})-\frac{1}{2}}\,(\|\psi_i\| + \|\psi_i\|_{m_i}), \tag{17}$$

$$\|D_t(Z_{i1}(t,x) \ * \ \psi_i(x))\| \le b_i\,(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})-1}\,(\|\psi_i\| + \|\psi_i\|_{m_i}), \tag{18}$$

where $b_i = const > 0, \ i = 1,2.$

3. Proof of main results

The weak solutions of problem (1), (2) can be represented in the form

$$\left.\begin{aligned} u_1(t,x) &= Z_{11}(t,x) \ * \ \varphi_1(x) + Z_{12}(t,x) \ * \ \psi_1(x) + \\ &\quad + \int_0^t Z_{12}(t-s,x) \ * \ f_1(u_1(s,x), u_2(s,x))ds, \\ u_2(t,x) &= \varepsilon Z_{21}(t,x) \ * \ \varphi_2(x) + \varepsilon Z_{22}(t,x) \ * \psi_2(x) + \\ &\quad + \int_0^t Z_{22}(t-s,x) \ * \ f_2(u_1(s,x), u_2(s,x))ds \end{aligned}\right\}. \tag{19}$$

Define a function space $U = U_{l_1}(M_1) \times U_{l_2}(\varepsilon M_2)$, where $M_i > M_{i0}$, $i = 1,2$,

$$U_{l_i}(M_i(\varepsilon)) = \Big\{ u \ : \cdot u \in C\left([0,\infty); W_2^{l_i}\left(R^N\right)\right),$$

$$(1+t)^{\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})}\,\|u(t,\cdot)\| + (1+t)^{\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})+\frac{1}{2}}\left\|\nabla^{l_i} u(t,\cdot)\right\| \le M_i(\varepsilon)\Big\}, \ \ i = 1,2.$$

We define the mapping

$$(v_1(t,x), v_2(t,x)\,) = \Psi(u_1(t,x), u_2(t,x)\,)$$

by the following equalities

$$v_1(t,x) \ = \ Z_{11}(t,x) \ * \ \varphi_1(x) \ + Z_{12}(t,x) \ * \ \psi_1(x) +$$

$$+ \int_0^t Z_{12}(t-s,x) \ * \ f_1(u_1(s,x), u_2(s,x))ds, \tag{20}$$

$$v_2(t,x) = \varepsilon Z_{21}(t,x) \ * \ \varphi_2(x) \ + \varepsilon Z_{22}(t,x) \ * \psi_2(x) +$$

$$+ \int_0^t Z_{22}(t-s,x) \ * \ f_2(u_1(s,x), u_2(s,x))ds. \tag{21}$$

Lemma 2. *Let the conditions* (4)–(9) *be satisfied. Then we can choose* $\varepsilon_0 > 0$ *such that the operator* Ψ *maps* $U = U_{l_1}(M_1) \times U_{l_2}(\varepsilon M_2)$ *into* $U = U_{l_1}(M_1) \times U_{l_2}(\varepsilon M_2)$, *where* $0 < \varepsilon \le \varepsilon_0$.

Proof. Let

$$(u_1,u_2)\in U\ =U_{l_1}(M_1)\ \times U_{l_2}(\varepsilon M_2). \tag{22}$$

Then, from (18), (11) and (12),(21) we have

$$\begin{gathered}\|v_i\,(t,\cdot)\|\leq(1+t)^{-\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)}\,M_{i0}(\varepsilon)+\\ +b_1\int\limits_0^t(1+t-\tau)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})}\times\\ \times[\|f_i(u_1(\tau,.),\,u_2(\tau,.))\|+\|f_i(u_1(\tau,.),\,u_2(\tau,.))\|_{m_i}]d\tau\end{gathered} \tag{23}$$

and

$$\begin{gathered}\sum_{|\alpha|=l_i}\|D^\alpha v_i\,(t,\cdot)\|\leq(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})-\frac{1}{2}}\,M_{i0}(\varepsilon)+\\ +b_1\int\limits_0^t(1+t-\tau)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})-\frac{1}{2}}\,[\|f_i(u_1(\tau,.),\,u_2(\tau,.))\|+\\ +\|f_i(u_1(\tau,.),\,u_2(\tau,.))\|_{m_i}]d\tau,\end{gathered} \tag{24}$$

where $M_{10}(\varepsilon)=M_{10}=b_1E_1(\varphi_1,\psi_1)$, $M_{20}(\varepsilon)=\varepsilon M_{20}=\varepsilon b_2E_2(\varphi_2,\psi_2)$.

According to (5)-(7) there exist $q_i>1$, $p_i>1$ such that $\dfrac{1}{p_i}+\dfrac{1}{q_i}=1$ and $q_im_i\rho_{i1}>2$, $p_im_i\rho_i>2$, $i=1,2$ (see [13], Lemma 3). Then using the Hölder inequality, we obtain

$$\begin{gathered}\|f_i(u_1(t,\cdot),u_2(t,\cdot))\|_{m_i}\leq b\left\{\int\limits_{R^n}|u_1(t,\cdot)|^{\rho_{i1}m_i}\,|u_2(t,\cdot)|^{\rho_{i2}m_i}\,d\tau\right\}^{1/m_i}\leq\\ \leq b\,\|u_1(t,\cdot)\|^{\rho_{i1}}_{q_im_i\rho_{i1}}\,\|u_2(t,\cdot)\|^{\rho_{i2}}_{p_im_i\rho_{i2}}.\end{gathered} \tag{25}$$

Further, using the multiplicative inequality of the Gagliardo-Nirenberg type (see[11]), we have

$$\|u_1(t,\cdot)\|^{\rho_{i1}}_{q_im_i\rho_{i1}}\leq\|u_1(t,\cdot)\|^{\rho_{i1}(1-\theta_{i,1,m_i})}\left(\sum_{|\alpha|=l_i}\|D^\alpha u_1(t,\cdot)\|\right)^{\rho_{i1}\theta_{i,1,m_i}}, \tag{26}$$

$$\|u_2(t,\cdot)\|^{\rho_{i2}}_{p_im_i\rho_{i2}}\leq\|u_2(t,\cdot)\|^{\rho_{i1}(1-\theta_{i,2,m_i})}\left(\sum_{|\alpha|=l_i}\|D^\alpha u_2(t,\cdot)\|\right)^{\rho_{i1}\theta_{i,2,m_i}}, \tag{27}$$

where

$$\theta_{i,1,m_i}=\frac{N}{l_1}\left(\frac{1}{2}-\frac{1}{m_iq_i\rho_{i1}}\right),\quad i=1,2, \tag{28}$$

$$\theta_{i,2,m_i}=\frac{N}{l_2}\left(\frac{1}{2}-\frac{1}{m_ip_i\rho_{i2}}\right),\quad i=1,2. \tag{29}$$

From our assumption it follows that $0<\theta_{i,1,m_i}\leq1$ and $0<\theta_{i,2,m_i}\leq1$, $i=1,2$.

Therefore, from (25)-(27) it follows that

$$\begin{gathered} \|f_i(u_1(t,\cdot),u_2(t,\cdot))\|_{m\,i} \leq \\ \leq b\,\|u_1\,(t,\cdot)\ \|^{\rho_{i1}(1-\theta_{i,1,m_i})}\left(\sum_{|\alpha|=l_i}\|D^\alpha u_1(t,\cdot)\ \|\right)^{\rho_{i1}\theta_{i,1,m_i}}\times \\ \times\,\|u_2(t,\cdot)\ \|^{\rho_{i1}(1-\theta_{i,2,m_i})}\left(\sum_{|\alpha|=l_2}\|D^\alpha u_2(t,\cdot)\ \|\right)^{\rho_{i1}\theta_{i,2,m_i}}. \end{gathered} \tag{30}$$

Similarly, we have

$$\begin{gathered} \|f_i(u_1(t,\cdot),u_2(t,\cdot))\,\| \leq \\ \leq b\,\|u_1(t,\cdot)\,\|^{\rho_{i1}(1-\theta_{i,1,2})}\left(\sum_{|\alpha|=l_i}\|D^\alpha u_1(t,\cdot)\ \|\right)^{\rho_{i1}\theta_{i,1,2}}\times \\ \times\,\|u_2(t,\cdot)\ \|^{\rho_{i1}(1-\theta_{i,2,2})}\left(\sum_{|\alpha|=l_2}\|D^\alpha u_2(t,\cdot)\ \|\right)^{\rho_{i1}\theta_{i,2,2}}, \end{gathered} \tag{31}$$

where

$$\theta_{i,1,2} = \frac{N}{2l_1}\left(1-\frac{1}{q_i\rho_{i1}}\right),\quad i=1,2, \tag{32}$$

$$\theta_{i,2,2} = \frac{N}{2l_2}\left(1-\frac{1}{p_i\rho_{i2}}\right),\quad i=1,2. \tag{33}$$

Using (22) in (23)-(27) and (30),(31) we obtain that

$$\begin{gathered} (1+t)^{\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})}\,\|v_i\,(t,\cdot)\| \leq \\ \leq M_{i0}(\varepsilon)+b_1 b\,(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})}\int\limits_0^t (1+t-\tau)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})}\,[(1+\tau)^{-\gamma_{m_i}}+ \\ +(1+\tau)^{-\gamma'_{m_i}}]\,(M_1)^{\rho_{11}}\,(\varepsilon M_2)^{\rho_{12}}\,d\tau \end{gathered} \tag{34}$$

$$\begin{gathered} (1+t)^{\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})+\frac{1}{2}}\sum_{|\alpha|=l_i}\|D^\alpha v_i\,(t,\cdot)\| \leq M_{i0}(\varepsilon)+ \\ +b_1 b\,(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})+\frac{1}{2}}\int\limits_0^t (1+t-\tau)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})-\frac{1}{2}}\,[(1+\tau)^{-\gamma_{m_i}}+ \\ +(1+\tau)^{-\gamma'_{m_i}}]\,(M_1)^{\rho_{11}}\,(\varepsilon M_2)^{\rho_{12}}\,d\tau, \end{gathered} \tag{35}$$

where

$$\begin{gathered} \gamma_{i,m_i} = \rho_{i1}(1-\theta_{i,1,m_i})\tfrac{N}{2l_i}(\tfrac{1}{m_i}-\tfrac{1}{2})+\rho_{i1}\theta_{i,1,m}[\tfrac{N}{2l_i}(\tfrac{1}{m_i}-\tfrac{1}{2})+\tfrac{1}{2}]+ \\ +\rho_{i2}(1-\theta_{i,2,m})\tfrac{N}{2l_i}(\tfrac{1}{m_i}-\tfrac{1}{2})+\rho_{i2}\theta_{i,2,m}[\tfrac{N}{2l_i}(\tfrac{1}{m_i}-\tfrac{1}{2})+\tfrac{1}{2}], \end{gathered} \tag{36}$$

$$\begin{gathered} \gamma_{i,2} = \rho_{i1}(1-\theta_{i,1,2})\tfrac{N}{2l_i}(\tfrac{1}{m_i}-\tfrac{1}{2})+\rho_{i1}\theta_{i,1,2}[\tfrac{N}{2l_i}(\tfrac{1}{m_i}-\tfrac{1}{2})+\tfrac{1}{2}]+ \\ +\rho_{i2}(1-\theta_{i,1,2})\tfrac{N}{2l_i}(\tfrac{1}{m_i}-\tfrac{1}{2})+\rho_{i2}\theta_{i,1,2}[\tfrac{N}{2l_i}(\tfrac{1}{m_i}-\tfrac{1}{2})+\tfrac{1}{2}]. \end{gathered} \tag{37}$$

Combining (28)-(29), (32), (33) and (36),(37), we get

$$\gamma_{i,m_i} = \frac{N}{2}\left[\left(\frac{\rho_{i1}}{l_1 m_1} + \frac{\rho_{i2}}{l_2 m_2}\right) - \left(\frac{1}{p_i l_1 m_i} + \frac{1}{q_i l_2 m_i}\right)\right]. \tag{38}$$

$$\gamma_{i,2} = \frac{N}{2}\left[\left(\frac{\rho_{i1}}{l_1 m_1} + \frac{\rho_{i2}}{l_2 m_2}\right) - \frac{1}{2}\left(\frac{1}{p_i l_1 m_i} + \frac{1}{q_i l_2 m_i}\right)\right]. \tag{39}$$

It follows from (7), (8) and (38),(39) that

$$\gamma_{i,2} > \gamma_{i,m_i} > 1, \quad i = 1,2.$$

It is known that (see [12]) in this case the following inequalities are true

$$(1+t)^{\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)} \int_0^t (1+t-\tau)^{-\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)} \left[(1+\tau)^{\gamma_{i,m_i}} + (1+\tau)^{\gamma_{i,2}}\right] d\tau \le c, \tag{40}$$

$$(1+t)^{\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)+\frac{1}{2}} \int_0^t (1+t-\tau)^{-\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)-\frac{1}{2}} \left[(1+\tau)^{\gamma_{i,m_i}} + (1+\tau)^{\gamma_{i,2}}\right] d\tau \le c. \tag{41}$$

By virtue of (34),(35) and (40),(41) we have

$$(1+t)^{\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)} \|v_i(t,\cdot)\| \le M_{i0}(\varepsilon) + bb_1 c\, (M_1)^{\rho_{11}} (\varepsilon M_2)^{\rho_{12}},$$

$$(1+t)^{\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)+\frac{1}{2}} \sum_{|\alpha|=l_i} \|D^\alpha v_i(t,\cdot)\| \le M_{i0}(\varepsilon) + bb_1 c\, (M_1)^{\rho_{11}} (\varepsilon M_2)^{\rho_{12}}.$$

Therefore, we deduce

$$(1+t)^{\frac{N}{2l_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \|v_1(t,\cdot)\| \le M_{10} + b_1 bc\, M_1^{\rho_{11}} (\varepsilon M_2))^{\rho_{12}}, \tag{42}$$

$$(1+t)^{\frac{N}{2l_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)} \|v_2(t,\cdot)\| \le \varepsilon M_{20} + bb_2 c\, M_1^{\rho_{21}} (\varepsilon M_2)^{\rho_{22}}, \tag{43}$$

$$(1+t)^{\frac{N}{2l_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\frac{1}{2}} \sum_{|\alpha|=l_1} \|D^\alpha v_1(t,\cdot)\| \le M_{10} + b_1 bc\, M_1^{\rho_{11}} (\varepsilon M_2))^{\rho_{12}}, \tag{44}$$

$$(1+t)^{\frac{N}{2l_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)+\frac{1}{2}} \sum_{|\alpha|=l_2} \|D^\alpha v_2(t,\cdot)\| \le \varepsilon M_{20} + bb_2 c\, M_1^{\rho_{21}} (\varepsilon M_2)^{\rho_{22}}, \tag{45}$$

We choose $\varepsilon_0 > 0$ as follows

$$\varepsilon_0 = \min\left\{\frac{1}{M_2}\left(\frac{M_1 - M_{10}}{bb_1 c M_1^{\rho_{11}}}\right)^{\frac{1}{\rho_{12}}}, \left(\frac{M_2 - M_{20}}{bb_2 c M_1^{\rho_{21}} M_2^{\rho_{22}}}\right)^{\frac{1}{\rho_{22}-1}}\right\}.$$

Using (9) in (44), (45), we obtain that

$$(1+t)^{\frac{N}{2l_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \|v_1(t,\cdot)\| \le M_1, \tag{46}$$

$$(1+t)^{\frac{N}{2l_2}(\frac{1}{m_2}-\frac{1}{2})}\left\|v_2(t,\cdot)\right\| \le \varepsilon M_2, \tag{47}$$

$$(1+t)^{\frac{N}{2l_1}(\frac{1}{m_1}-\frac{1}{2})+\frac{1}{2}}\sum_{|\alpha|=l_1}\left\|D^\alpha v_1(t,\cdot)\right\| \le M_1, \tag{48}$$

$$(1+t)^{\frac{N}{2l_2}(\frac{1}{m_2}-\frac{1}{2})+\frac{1}{2}}\sum_{|\alpha|=l_2}\left\|D^\alpha v_2(t,\cdot)\right\| \le \varepsilon M_2. \tag{49}$$

Then (40),(41) yield that

$$(v_1, v_2) \in U = U_{l_1}(M_1) \times U_{l_2}(\varepsilon M_2),$$

i.e., Ψ maps U into itself.

Therefore, we can construct the sequence$\left\{ (u_1^{(n)}, u_2^{(n)}) \right\} \subset U$ as follows

$$(u_1^{(n+1)}, u_2^{(n+1)}) = \Psi(u_1^{(n)}, u_2^{(n)}), n = 0, 1, 2, ..., \tag{50}$$

where

$$\left.\begin{array}{l} u_1^{(0)} = Z_{11}(t,x) * \varphi_1(x) + Z_{12}(t,x) * \psi_1(x) \\ u_2^{(0)} = \varepsilon Z_{21}(t,x) * \varphi_2(x) + \varepsilon Z_{22}(t,x) * \psi_2(x) \end{array}\right\}. \tag{51}$$

Now we state some a priori estimates for the sequence $\left\{ (u_1^{(n)}, u_2^{(n)}) \right\}$ which follow directly from the construction of this sequence.

Lemma 3. *Let the conditions* (4)–(9) *be satisfied. Then for any* $0 < \varepsilon < \varepsilon_0$ *and the sequence* $\left\{(u_1^{(n)}, u_2^{(n)})\right\}$ *the following estimates hold*

$$\left\|u_1^{(n)}(t,\cdot)\right\| \le M_1 (1+t)^{-\frac{N}{2l_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)}, \tag{52}$$

$$\left\|u_2^{(n)}(t,\cdot)\right\| \le \varepsilon M_2 (1+t)^{-\frac{N}{2l_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)}, \tag{53}$$

$$\sum_{|\alpha|=l_1}\left\|D^\alpha u_1^{(n)}(t,\cdot)\right\| \le M_1 (1+t)^{-\frac{N}{2l_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}}, \tag{54}$$

$$\sum_{|\alpha|=l_2}\left\|D^\alpha u_2^{(n)}(t,\cdot)\right\| \le \varepsilon M_2 (1+t)^{-\frac{N}{2l_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{1}{2}}. \tag{55}$$

It follows from (20), (21) and (43) that

$$\left\|D_t u_i^{(n)}(t,\cdot)\right\| \le (1+t)^{-\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)-1} M_{i0}(\varepsilon) +$$

$$+b_1\int_0^t (1+t-\tau)^{-\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)-1}\left[(1+\tau)^{-\gamma_{m_i}} + (1+\tau)^{-\gamma'_{m_i}}\right](M_1)^{\rho_{i1}}(\varepsilon M_2)^{\rho_{i2}}\,d\tau.$$

Finally, by using the well known estimate

$$(1+t)^{\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)+1}\int\limits_0^t(1+t-\tau)^{-\frac{N}{2l_i}\left(\frac{1}{m_i}-\frac{1}{2}\right)-1}\left[(1+\tau)^{\gamma_{i,m_i}}+(1+\tau)^{\gamma_{i,2}}\right]d\tau\le c,$$

we have the following Lemma.

Lemma 4. *Let the conditions* (4)-(9) *be satisfied. Then for any* $0<\varepsilon<\varepsilon_0$ *the sequence* $\left\{(u_1^{(n)},u_2^{(n)})\right\}$ *satisfies the following estimates*

$$\left\|D_t u_1^{(n)}(t,\cdot)\right\|\le M_1(1+t)^{-\gamma_1}, \tag{56}$$

$$\left\|D_t u_2^{(n)}(t,\cdot)\right\|\le\varepsilon M_2(1+t)^{-\gamma_2}. \tag{57}$$

Next, using the embedding theorems, from (52), (57), we obtain the following result.

Lemma 5. *Let the conditions* (4)–(9) *be satisfied. Then for any* $0<\varepsilon<\varepsilon_0$ *the sequence* $\left\{(u_1^{(n)},u_2^{(n)})\right\}$ *satisfies the following estimates*

$$\left|u_1^{(n)}(t,x)\right|\le 2c_1\,bE_1(\varphi_1,\psi_1)=K_1\,,\ \ (t,x)\in[0,\infty)\times R^N \tag{58}$$

$$\left|u_2^{(n)}(t,x)\right|\le 2c_2\,b\varepsilon E_2(\varphi_2,\psi_2)=K_2\,,\ \ (t,x)\in[0,\infty)\times R^N, \tag{59}$$

where c_i *is the norm of the embedding* $H^{l_i}(R^N)\subset C(R^N)$.

Now we shall prove that $\left\{(u_1^{(n)},u_2^{(n)})\right\}$ is a fundamental sequence.

Indeed, from (20)–(21) and (50) we have

$$\left\|u_i^{(n+1)}(t,\cdot)-u_i^{(n)}(t,\cdot)\right\|\le$$

$$\le\int\limits_o^t\left\|Z_{i2}(t-\tau,x)*[f_i(u_1^{(n)}(\tau,x),u_2^{(n)}(\tau,x)-f_i(u_1^{(n-1)}(\tau,x),u_2^{(n-1)}(\tau,x)\right\|d\tau\le$$

$$\le\int\limits_0^t\int\limits_0^1\left\|f'_{iu_1}(u_1^{(n-1)}(\tau,x)+\lambda(u_1^{(n)}(\tau,x)-u_1^{(n-1)}(\tau,x),u_2^{(n-1)}(\tau,x)+\lambda(u_2^{(n)}(\tau,x)\right.$$

$$-u_2^{(n-1)}(\tau,x))d\lambda\left(u_1^{(n)}(\tau,x)-u_1^{(n-1)}(\tau,x)\right)\|\,d\tau+$$

$$+\int\limits_0^t\int\limits_0^1\left\|f'_{iu_2}(u_1^{(n-1)}(\tau,x)+\lambda(u_1^{(n)}(\tau,x)-u_1^{(n-1)}(\tau,x),\right.$$

$$u_2^{(n-1)}(\tau,x)+\lambda(u_2^{(n)}(\tau,x)-u_2^{(n-1)}(\tau,x))d\lambda\left(u_2^{(n)}(\tau,x)-u_2^{(n-1)}(\tau,x)\right)\|\,d\tau.$$

Taking into account (58), (59),we get

$$\left\|u_i^{(n+1)}(t,\cdot)-u_i^{(n)}(t,\cdot)\right\|\le$$

$$\le\int\limits_0^t C_i\left[\left\|\left(u_1^{(n)}(\tau,x)-u_1^{(n-1)}(\tau,x)\right)\right\|+\left\|\left(u_2^{(n)}(\tau,x)-u_2^{(n-1)}(\tau,x)\right)\right\|\right]d\tau,$$

where $C_i = \max\limits_{|\xi|\leq 2K_1, |\eta|\leq 2K_1} [|f'_{iu}(\xi,\eta)| + |f'_{iux}(\xi,\eta)|]$, $i = 1,2$.

Letting

$$w_n(t) = \max_{0\leq\tau\leq t} \sum_{i=1}^{2} \left\| u_i^{(n+1)}(t,\cdot) - u_i^{(n)}(t,\cdot) \right\|,$$

we obtain that

$$w_{n+1}(t) \leq C\, t\, w_n(t),$$

where $C = C_1 + C_2$.

Applying this inequality repeatedly, we have

$$w_{n+1}(t) \leq \frac{(Ct)^n}{n!}.$$

Therefore $\left\{ (u_1^{(n)}, u_2^{(n)}) \right\}$ is a fundamental sequence in space $C\left([0,T]; L_2\left(R^N\right)\right)$, for every $T > 0$.

Then, there exist the functions $u_1, u_2 \in C\left([0,T]; L_2\left(R^N\right)\right)$ such that, $u_i^{(n)} \to u_i$ in $C\left([0,T]; L_2\left(R^N\right)\right)$, $i = 1,2$, for every $T > 0$.

In view of the $*-$weak compactness of bounded sets in $L_\infty\left((0,\infty); L_2\left(R^N\right)\right)$, from (52)-(57) it follows that from the sequence $\{(u_1^n, u_2^n)\}$ we can choose a subsequence $\Big\{(u_1^{(n_k)}, u_2^{(n_k)})\Big\}$ with the following properties

$$u_i^{(n_k)} \to u_i \quad *-\text{weakly in } L_\infty\Big((0,\infty); H^{l_i}\left(R^N\right)\Big), \quad i = 1,2, \tag{60}$$

$$D_t u_i^{(n_k)} \to D_t u_i \quad *-\text{weakly in } L_\infty\left((0,\infty); L_2\left(R^N\right)\right) \quad i = 1,2. \tag{61}$$

From (52) –(57) and (60), (61) it follows that

$$\|u_i(t,\cdot)\| \leq c\, M_i(\varepsilon)\,(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})}, i = 1,2,, \tag{62}$$

$$\sum_{|\alpha|=l_i} \|D^\alpha u_i(t,\cdot)\| \leq c\, M_i(\varepsilon)\,(1+t)^{-\frac{N}{2l_i}(\frac{1}{m_i}-\frac{1}{2})-\frac{1}{2}}, \quad i = 1,2. \tag{63}$$

$$\|D_t u_i(t,\cdot)\| \leq c\, M_i(\varepsilon)\,(1+t)^{-\gamma_i}, \quad i = 1,2, \tag{64}$$

where $\gamma_i = min\left\{ 1 + \dfrac{N(\frac{1}{m_i} - \frac{1}{2})}{2l_i}, \dfrac{N}{2} \sum\limits_{k=1}^{2} \dfrac{\rho_{ik}}{m_k l_k} - \dfrac{r_i}{m_i} \right\}$.

Now we prove that the couple of functions (u_1, u_2) is a solution of problem (1), (2).

From (52)–(57) and (62)–(64) it follows that

$$\left\| f_i(u_1^{(n)}, u_2^{(n)}) - f_i(u_1, u_2) \right\| \leq$$

$$\leq \sum_{k=1}^{2} \left\| \int_0^1 f_{iu_k}(u_1 + \lambda(u_1^{(n)} - u_1), u_2 + \lambda(u_2^{(n)} - u_2)) \left(u_k^{(n)}(\lambda,x) - u_k(\lambda,x) \right) d\lambda \right\| \leq .$$

$$\leq K \sum_{k=1}^{2} \left\| u_k^{(n)} - u_k \right\|$$

Therefore, for every $T > 0$

$$f_i(u_1^{(n)}, u_2^{(n)}) \to f_i(u_1, u_2) \text{ in } C\left([0,T]; L_2\left(R^N\right)\right).$$

Passing to the limit in (50) we see that the pair of functions (u_1, u_2) satisfies the equation

$$(u_1, u_2) = \Psi(u_1, u_2). \tag{65}$$

By (60), (61) and (65), we have

$$(u_1, u_2) \in C\left([0,\infty); H^{l_1}\left(R^N\right) \times H^{l_2}(R^N)\right) \cap C^1\left([0,\infty); L_2\left(R^N\right) \times L_2(R^N)\right).$$

4. Analysis of conditions (4)–(9). Examples.

As stated above, the conditions (4)–(8) provide existence of global solutions with small initial data. The assertion of Theorem becomes invalid without condition $\rho_{12} > 0$. Indeed, if $\rho_{12} = 0$, then the function $f_1(u_1, u_2) = |u_1|^{\rho_{11}-1} u_1$, where $1 < \rho_{11} \leq \min\left\{\frac{2l_1 m_1}{N} + 1, \frac{2}{m_1}\right\}$, satisfies the conditions ((4) –(8). In that case the Cauchy problem

$$\left.\begin{array}{l} u_{1tt} + u_{1t} + (-1)^{l_1} \Delta^{l_1} u_1 = |u_1|^{\rho_{11}-1} u_1 \\ u_1(0,x) = \varphi_1(x), u_{1t}(0,x) = \psi_1(x) \end{array}\right\} \tag{66}$$

has a global solution only for "small" $E_1(\varphi_1, \psi_1)$ (see, for example [2]), and in general, for "large" $E_1(\varphi_1, \psi_1)$ the problem (66) has no global solution (see, for example [4, 5, 13, 14]).

We give some examples satisfying the conditions (4)–(9).

a) Consider the Cauchy problem

$$\left.\begin{array}{l} u_{1tt} + u_{1t} + (-1)^l \Delta^l u_1 = u_1^m u_2^r \\ u_{2tt} + u_{2t} + (-1)^l \Delta^l u_2 = u_1^k u_2^s \end{array}\right\}, \tag{67}$$

$$\left.\begin{array}{l} u_1(0,x) = \varphi_1(x), \ u_{1t}(0,x) = \psi_1(x), \\ u_2(0,x) = \epsilon\varphi_2(x), \ u_{2t}(0,x) = \epsilon\psi_2(x) \end{array}\right\}, x \in R^N \tag{68}$$

where $m, k \in \{0, 1, ...\}, l, r \in \{1, 2, ...\}, s \in \{2, 3...\}$, $m + r > \frac{2l}{N} + 1$, $k + s > \frac{2l}{N} + 1$.

The assertion of Theorem is true for problem (67)-(68), where $m_1 = m_2 = 1$.

b) Consider the function

$$f_i(u_1, u_2) = \varphi_{i1}(u_1)\varphi_{i2}(u_2), \quad u_1, u_2 \in R,$$

where

$$\varphi_{ij}(\eta) = \begin{cases} |\eta|^{p_{ij}-1}\eta, & |\eta| \le 1 \\ \left(\frac{p_{ij}}{q_{ij}}\right)^{q_{ij}} \left|\eta - \frac{p_{ij}-q_{ij}}{p_{ij}}\right|^{q_{ij}-1} \left(\eta - \frac{p_{ij}-q_{ij}}{p_{ij}}\right), & |\eta| > 1 \end{cases}$$

$$p_{ij} \ge 1 or p_{ij} = 0 \;\; q_{ij} \ge 0, i, j = 1, 2.$$

It is obvious that,

$$|f_i(u,v)| \le b\,|u|^{\rho_{i1}}\,|u|^{\rho_{i2}}, \;\; b > 0, \quad \text{where} \quad \rho_{ij} = \min\{p_{ij}, q_{ij}\}.$$

Under conditions (4) –(9) the assertion of Theorem is true.

For example, if

$$l_1 = l_2 = 1,\; m_1 = m_2 = 1, p_{11} = q_{11} = 0, p_{12} = q_{12} = p > \frac{2}{N} + 1,$$

$$p_{21} = q_{21} = 0, p_{22} = q_{22} > \frac{2}{N} + 1,$$

then the assertion of Theorem is valid.

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Akbar B.Aliev
Azerbaijan Technical University, Institute of Mathematics and Mechanics of NAS of Azerbaijan. 9, B. Vahabzadeh str., AZ1141, Baku, Azerbaijan.
E-mail: aliyevagil@yahoo.com

Anar A.Kazymov
Nakhchivan State University
*E-mail:*anarkazimov1979@gmail.com

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