Azerbaijan Journal of Mathematics
V. 4, No 2, 2014, July
ISSN 2218-6816

Higher Order Commutators of Vector-Valued Intrinsic Square Functions on Vector-Valued Generalized Weighted Morrey Spaces

V. S. Guliyev*, M.N. Omarova

Abstract. In this paper, we will obtain the strong type and weak type estimates for vector-valued analogues of intrinsic square functions in the generalized weighted Morrey spaces $M_w^{p,\varphi}(R^n)$. We study the boundedness of intrinsic square functions including the Lusin area integral, Littlewood-Paley g-function and g_λ^* -function and their higher order commutators on vector-valued generalized weighted Morrey spaces $M_w^{p,\varphi}(l_2)$. In all the cases the conditions for the boundedness are given in terms of Zygmund-type integral inequalities on $\varphi(x,r)$ without assuming any monotonicity property of $\varphi(x,r)$ in r.

Key Words and Phrases: intrinsic square functions; vector-valued generalized weighted Morrey spaces; vector-valued inequalities; A_p weights; commutators; BMO

2010 Mathematics Subject Classifications: 42B25, 42B35

1. Introduction

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [2, 3] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(R^n)$. A well known result of Coifman, Rochberg and Weiss [9] states that the commutator operator $[b, K]f = K(bf) - b\,Kf$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [6]-[8], [5], [10], [11]).

The classical Morrey spaces were originally introduced by Morrey in [31] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [10, 11, 18, 31]. Recently, Komori and Shirai [28] first defined the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, Guliyev [21, 22]

*Corresponding author.

introduced the generalized weighted Morrey spaces $M_w^{p,\varphi}$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see, also [25, 27, 34]).

The intrinsic square functions were first introduced by Wilson in [39, 40]. They are defined as follows. For $0 < \alpha \le 1$, let C_α be the family of functions $\phi : R^n \to \mathbb{R}$ such that ϕ's support is contained in $\{x : |x| \le 1\}$, $\int_{R^n} \phi(x)dx = 0$, and for $x,\ x' \in R^n$,

$$|\phi(x) - \phi(x')| \le |x - x'|^\alpha.$$

For $(y,\ t) \in \mathbb{R}^{n+1}_+$ and $f \in L^{1,loc}(R^n)$, set

$$A_\alpha f(t,\ y) \equiv \sup_{\phi \in C_\alpha} |f * \phi_t(y)|,$$

where $\phi_t(y) = t^{-n}\phi(\frac{y}{t})$. Then we define the varying-aperture intrinsic square (intrinsic Lusin) function of f by the formula

$$G_{\alpha,\beta}(f)(x) = \left(\int\int_{\Gamma_\beta(x)} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

where $\Gamma_\beta(x) = \{(y,\ t) \in \mathbb{R}^{n+1}_+ : |x - y| < \beta t\}$. Denote $G_{\alpha,1}(f) = G_\alpha(f)$.

This function is independent of any particular kernel, such as Poisson kernel. It dominates pointwise the classical square function(Lusin area integral) and its real-variable generalizations. Although the function $G_{\alpha,\beta}(f)$ is dependent of kernels with uniform compact support, there is pointwise relation between $G_{\alpha,\beta}(f)$ with different β:

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x) .$$

Details can be found in [39].

The intrinsic Littlewood-Paley g-function and the intrinsic g^*_λ function are defined respectively by

$$\mathrm{g}_\alpha f(x) = \left(\int_0^\infty (A_\alpha f(y,t))^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

$$\mathrm{g}^*_{\lambda,\alpha} f(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x - y|} \right)^{n\lambda} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

When we say that f maps into l_2, we mean that $\vec{f}(x) = (f_j)_{j=1}^\infty$, where each f_j is Lebesgue measurable and, for almost every $x \in R^n$

$$\|\vec{f}(x)\|_{l_2} = \left(\sum_{j=1}^\infty |f_j(x)|^2 \right)^{1/2}.$$

Let $\vec{f} = (f_1,\ f_2,\ \ldots)$ be a sequence of locally integrable functions on R^n. For any $x \in R^n$, Wilson [40] also defined the vector-valued intrinsic square functions of $\vec{f}$ by $\|G_\alpha \vec{f}(x)\|_{l_2}$ and proved the following result.

Theorem A. *Let $1 \le p < \infty$, $0 < \alpha \le 1$ and $w \in A_p$. Then the operators G_α and $\mathrm{g}^*_{\lambda,\alpha}$ are bounded from $L^p_w(l_2)$ into itself for $p > 1$ and from $L^1_w(l_2)$ to $WL^1_w(l_2)$.*

Moreover, in [30], Lerner showed sharp L^p_w norm inequalities for the intrinsic square functions in terms of the A_p characteristic constant of w for all $1 < p < \infty$. Also Huang and Liu [12] studied the boundedness of intrinsic square functions on weighted Hardy spaces. Moreover, they characterized the weighted Hardy spaces by intrinsic square functions. In [37] and [38], Wang and Liu obtained some weak type estimates on weighted Hardy spaces. In [36], Wang considered intrinsic functions and the commutators generated by BMO functions on weighted Morrey spaces. Let b be a locally integrable function on R^n. Setting

$$A^k_{\alpha,b} f(t,y) \equiv \sup_{\phi\in C_\alpha} \left| \int_{R^n} [b(x) - b(z)]^k \phi_t(y-z) f(z)dz \right|,$$

the commutators are defined by

$$[b, G_\alpha]^k f(x) = \left(\int\int_{\Gamma(x)} (A^k_{\alpha,b} f(t,y))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

$$[b, \mathrm{g}_\alpha]^k f(x) = \left(\int_0^\infty (A^k_{\alpha,b} f(t,y))^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$[b, \mathrm{g}^*_{\lambda,\alpha}]^k f(x) = \left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} (A^k_{\alpha,b} f(t,y))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

A function $b \in L^{loc}_1(R^n)$ is said to be in $BMO(R^n)$ if

$$\|b\|_* = \sup_{x\in R^n,\, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy < \infty,$$

where $b_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} b(y)dy$.

In [36], Wang proved the following result.

Theorem B. *Let $1 < p < \infty$, $0 < \alpha \le 1$, $w \in A_p$ and $b \in BMO(R^n)$. Then the commutator operators $[b, G_\alpha]$ and $[b, \mathrm{g}^*_{\lambda,\alpha}]$ are bounded from $L^p_w(l_2)$ into itself.*

By the similar argument as in [13] and [36], we can get

Theorem B'. *Let $1 < p < \infty$, $0 < \alpha \le 1$, $w \in A_p$ and $b \in BMO(R^n)$. Then the kth-order commutator operators $[b, G_\alpha]^k$ and $[b, \mathrm{g}^*_{\lambda,\alpha}]^k$ are bounded from $L^p_w(l_2)$ into itself.*

In this paper, we will consider the boundedness of the operators G_α, g_α, $\mathrm{g}^*_{\lambda,\alpha}$ and their commutators on vector-valued generalized weighted Morrey spaces. Let $\varphi(x,\ r)$ be a positive measurable function on $R^n \times \mathbb{R}_+$ and w be non-negative measurable function on

R^n. For any $\vec{f} \in L_w^{p,loc}(l_2)$, we denote by $M_w^{p,\varphi}(l_2)$ the vector-valued generalized weighted Morrey spaces, if

$$\|\vec{f}\|_{M_w^{p,\varphi}(l_2)} = \sup_{x\in R^n, r>0} \varphi(x,\ r)^{-1}\, w(B(x,r))^{-\frac{1}{p}} \, \|\|\vec{f}(\cdot)\|_{l_2}\|_{L_w^p(B(x,r))} < \infty.$$

If $w \equiv 1$, then $M_w^{p,\varphi}(l_2)$ coincide with the vector-valued generalized Morrey spaces $M^{p,\varphi}(l_2)$. There are many papers which discussed the conditions on $\varphi(x, r)$ to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [17] (see also [18]), Guliyev imposed the following condition on the pair (φ_1, φ_2) :

$$\int_r^\infty \varphi_1(x,t)\frac{dt}{t} \le C\varphi_2(x,r), \tag{1}$$

where $C > 0$ does not depend on x and r. Under the above condition, Guliyev obtained the boundedness of Calderón-Zygmund singular integral operators from $M^{p,\varphi_1}(R^n)$ to $M^{p,\varphi_2}(R^n)$. Also, in [1] and [20], Guliyev et al. introduced a weaker condition: If $1 \le p < \infty$, then there exits a constant $C > 0$ such that for any $x \in R^n$ and $r > 0$,

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\,\varphi_2(x,r). \tag{2}$$

If the pair (φ_1, φ_2) satisfies condition (1), then (φ_1, φ_2) satisfies condition (2). But the opposite is not true. See Remark 4.7 in [20] for details.

Recently, in [21, 22] (see also [25, 27, 34]), Guliyev introduced a weighted condition: If $1 \le p < \infty$, then there exits a constant $C > 0$ such that for any $x \in R^n$ and $t > 0$,

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \le C\,\varphi_2(x,r). \tag{3}$$

In this paper, we will obtain the boundedness of the vector-valued intrinsic function, the intrinsic Littlewood-Paley g function, the intrinsic g_λ^* function and their commutators on vector-valued generalized weighted Morrey spaces when $w \in A_p$ and the pair (φ_1, φ_2) satisfies condition (3) or the following inequalities

$$\int_r^\infty \ln^k\left(e+\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \le C\,\varphi_2(x,r), \tag{4}$$

where C does not depend on x and r. Our main results in this paper are stated as follows.

Theorem 1. *Let $1 \le p < \infty$, $0 < \alpha \le 1$, $w \in A_p$ and (φ_1, φ_2) satisfy the condition (3). Then the operator G_α is bounded from $M_w^{p,\varphi_1}(l_2)$ to $M_w^{p,\varphi_2}(l_2)$ for $p > 1$ and from $M_w^{1,\varphi_1}(l_2)$ to $WM_w^{1,\varphi_2}(l_2)$.*

Theorem 2. *Let* $1 \le p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, $\lambda > 3 + \dfrac{\alpha}{n}$ *and* (φ_1, φ_2) *satisfy the condition* (3)*. Then the operator* $g^*_{\lambda,\alpha}$ *is bounded from* $M^{p,\varphi_1}_w(l_2)$ *to* $M^{p,\varphi_2}_w(l_2)$ *for* $p > 1$ *and from* $M^{1,\varphi_1}_w(l_2)$ *to* $WM^{1,\varphi_2}_w(l_2)$*.*

Theorem 3. *Let* $1 < p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, $b \in BMO$ *and* (φ_1, φ_2) *satisfy the condition* (4)*. Then* $[b, G_\alpha]^k$ *is bounded from* $M^{p,\varphi_1}_w(l_2)$ *to* $M^{p,\varphi_2}_w(l_2)$ *.*

Theorem 4. *Let* $1 < p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, $b \in BMO$ *and* (φ_1, φ_2) *satisfy the condition* (4)*. Then for* $\lambda > 3 + \frac{\alpha}{n}$, $[b, g^*_{\lambda,\alpha}]^k$ *is bounded from* $M^{p,\varphi_1}_w(l_2)$ *to* $M^{p,\varphi_2}_w(l_2)$*.*

In [39], the author proved that the functions $G_\alpha f$ and $g_\alpha f$ are pointwise comparable. Thus, as a consequence of Theorem 1 and Theorem 3, we have the following results.

Corollary 1. *Let* $1 \le p < \infty$, $0 < \alpha \le 1$, $w \in A_p$ *and* (φ_1, φ_2) *satisfy the condition* (3)*. Then* g_α *is bounded from* $M^{p,\varphi_1}_w(l_2)$ *to* $M^{p,\varphi_2}_w(l_2)$ *for* $p > 1$ *and from* $M^{1,\varphi_1}_w(l_2)$ *to* $WM^{1,\varphi_2}_w(l_2)$*.*

Corollary 2. *Let* $1 < p < \infty$, $0 < \alpha \le 1$, $w \in A_p$, $b \in BMO$ *and* (φ_1, φ_2) *satisfy the condition* (4)*. Then* $[b, g_\alpha]$ *is bounded from* $M^{p,\varphi_1}_w(l_2)$ *to* $M^{p,\varphi_2}_w(l_2)$*.*

Remark 1. *Note that, in the scalar valued case with* $w \equiv 1$ *the Theorems 1 - 4 and Corollaries 1 - 2 were proved in [26]. Also, in the scalar valued case with* $w \equiv A_p$ *and* $\varphi_1(x, r) = \varphi_2(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$, $0 < \kappa < 1$ *Theorems 1-4 and Corollaries 1-2 were proved by Wang in [36, 35]. How as, if* $\varphi(x, r) \equiv w(B(x, r))^{\frac{\kappa-1}{p}}$*, then the vector-valued generalized weighted Morrey space* $M^{p,\varphi}_w(l_2)$ *coincides with the vector-valued weighted Morrey space* $L^{p,\kappa}_w(l_2)$ *and the pair* $(w(B(x, r))^{\frac{\kappa-1}{p}}, w(B(x, r))^{\frac{\kappa-1}{p}})$ *satisfies both conditions* (3) *and* (4)*. Indeed, by Lemma 1 there exist* $C > 0$ *and* $\delta > 0$ *such that for all* $x \in R^n$ *and* $t > r$*:*

$$w(B(x, t)) \ge C\Big(\frac{t}{r}\Big)^{n\delta} w(B(x, r)).$$

Then

$$\begin{aligned}
\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} w(B(x, s))^{\frac{\kappa}{p}}}{w(B(x, t))^{1/p}} \frac{dt}{t} &\le \int_r^\infty \ln^k\Big(e + \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} w(B(x, s))^{\frac{\kappa}{p}}}{w(B(x, t))^{1/p}} \frac{dt}{t} \\
&= \int_r^\infty \ln^k\Big(e + \frac{t}{r}\Big) w(B(x, t))^{\frac{\kappa-1}{p}} \frac{dt}{t} \\
&\lesssim \int_r^\infty \ln^k\Big(e + \frac{t}{r}\Big) \Big(\Big(\frac{t}{r}\Big)^{n\delta} w(B(x, r))\Big)^{\frac{\kappa-1}{p}} \frac{dt}{t} \\
&= w(B(x, r))^{\frac{\kappa-1}{p}} \int_r^\infty \ln^k\Big(e + \frac{t}{r}\Big) \Big(\frac{t}{r}\Big)^{n\delta\frac{\kappa-1}{p}} \frac{dt}{t} \\
&= w(B(x, r))^{\frac{\kappa-1}{p}} \int_1^\infty \ln^k\Big(e + \tau\Big) \tau^{n\delta\frac{\kappa-1}{p}} \frac{d\tau}{\tau} \\
&\approx w(B(x, r))^{\frac{\kappa-1}{p}}.
\end{aligned}$$

Throughout this paper, we use the notation $A \lesssim B$ to mean that there is a positive constant C independent of all essential variables such that $A \le CB$. Moreover, C may be different from place to place.

2. Vector-valued generalized weighted Morrey spaces

The classical Morrey spaces $M^{p,\lambda}$ were originally introduced by Morrey in [31] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [15, 29].

We denote by $M^{p,\lambda}(l_2) \equiv M^{p,\lambda}(R^n, l_2)$ the vector-valued Morrey space, the space of all vector-valued functions $\vec{f} \in L^{p,\mathrm{loc}}(l_2)$ with finite quasinorm

$$\left\|\vec{f}\right\|_{M^{p,\lambda}(l_2)} = \sup_{x\in R^n,\, r>0} r^{-\frac{\lambda}{p}} \|\vec{f}\|_{L^p(B(x,r),l_2)},$$

where $1 \le p < \infty$ and $0 \le \lambda \le n$.

Note that $M^{p,0}(l_2) = L^p(l_2)$ and $M^{p,n}(l_2) = L^\infty(l_2)$. If $\lambda < 0$ or $\lambda > n$, then $M^{p,\lambda}(l_2) = \Theta$, where Θ is the set of all vector-valued functions equivalent to 0 on R^n.

We define the vector-valued generalized weighed Morrey spaces as follows.

Definition 1. *Let $1 \le p < \infty$, φ be a positive measurable vector-valued function on $R^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_w^{p,\varphi}(l_2)$ the vector-valued generalized weighted Morrey space, the space of all vector-valued functions $\vec{f} \in L_w^{p,\mathrm{loc}}(l_2)$ with finite norm*

$$\|\vec{f}\|_{M_w^{p,\varphi}(l_2)} = \sup_{x\in R^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_w^p(B(x,r),l_2)},$$

where $L_w^p(B(x,r),l_2)$ denotes the vector-valued weighted L^p-space of measurable functions f for which

$$\|\vec{f}\|_{L_w^p(B(x,r))} \equiv \|\vec{f}\chi_{B(x,r)}\|_{L_w^p(\mathbb{R}^n)} = \left(\int_{B(x,r)} \|\vec{f}(y)\|_{l_2}^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_w^{p,\varphi}(l_2)$ we denote the vector-valued weak generalized weighted Morrey space of all functions $f \in WL_w^{p,\mathrm{loc}}(l_2)$ for which

$$\|\vec{f}\|_{WM_w^{p,\varphi}(l_2)} = \sup_{x\in R^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|\vec{f}\|_{WL_w^p(B(x,r),l_2)} < \infty,$$

where $WL_w^p(B(x,r),l_2)$ denotes the weak L_w^p-space of measurable functions f for which

$$\|\vec{f}\|_{WL_w^p(B(x,r),l_2)} \equiv \|\vec{f}\chi_{B(x,r)}\|_{WL_w^p(l_2)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\, \|\vec{f}(y)\|_{l_2}>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

Remark 2. (1) *If* $w \equiv 1$, *then* $M_1^{p,\varphi}(l_2) = M^{p,\varphi}(l_2)$ *is the vector-valued generalized Morrey space.*

(2) *If* $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, *then* $M_w^{p,\varphi}(l_2) = L_w^{p,\kappa}(l_2)$ *is the vector-valued weighted Morrey space.*

(3) *If* $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, *then* $M_w^{p,\varphi}(l_2) = L_{v,w}^{p,\kappa}(l_2)$ *is the vector-valued two weighted Morrey space.*

(4) *If* $w \equiv 1$ *and* $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ *with* $0 < \lambda < n$, *then* $M_w^{p,\varphi}(l_2) = L^{p,\lambda}(l_2)$ *is the vector-valued Morrey space and* $WM_w^{p,\varphi}(l_2) = WL^{p,\lambda}(l_2)$ *is the vector-valued weak Morrey space.*

(5) *If* $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, *then* $M_w^{p,\varphi}(l_2) = L_w^p(l_2)$ *is the vector-valued weighted Lebesgue space.*

3. Preliminaries and some lemmas

By a weight function, briefly weight, we mean a locally integrable function on R^n which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)dx$, and denote the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E. Given a weight w, we say that w satisfies the doubling condition if there exists a constant $D > 0$ such that for any ball B, we have $w(2B) \le Dw(B)$. When w satisfies this condition, we write briefly $w \in \Delta_2$.

If w is a weight function, we denote by $L_w^p(l_2) \equiv L_w^p(R^n, l_2)$ the vector-valued weighted Lebesgue space defined by finiteness of the norm

$$\|\vec{f}\|_{L_w^p(l_2)} = \left(\int_{R^n} \|\vec{f}(x)\|_{l_2}^p w(x)dx \right)^{\frac{1}{p}} < \infty, \quad \text{if } 1 \le p < \infty$$

and by $\|\vec{f}\|_{L_w^\infty(l_2)} = \operatorname*{ess\,sup}_{x \in R^n} \|\vec{f}(x)\|_{l_2} w(x)$ if $p = \infty$.

We recall that a weight function w is in the Muckenhoupt's class A_p [32], $1 < p < \infty$, if

$$\begin{aligned} [w]_{A_p} : &= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} < \infty, \end{aligned}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B by Hölder's inequality

$$[w]_{A_p(B)}^{1/p} = |B|^{-1} \|w\|_{L^1(B)}^{1/p} \, \|w^{-1/p}\|_{L^{p'}(B)} \ge 1.$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \le Cw(x)$ with $[w]_{A_1} = \sup\limits_{x \in \mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$ $A_\infty = \bigcup_{1 \le p < \infty} A_p$ and $[w]_{A_\infty} = \inf\limits_{1 \le p < \infty} [w]_{A_p}$.

Lemma 1. ([16]) (1) *If $w \in A_p$ for some $1 \leq p < \infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$*

$$w(\lambda B) \leq \lambda^{np}[w]_{A_p} w(B).$$

(2) *If $w \in A_\infty$, then $w \in \Delta_2$. Moreover, for all $\lambda > 1$*

$$w(\lambda B) \leq 2^{\lambda^n}[w]_{A_\infty} w(B).$$

(3) *If $w \in A_p$ for some $1 \leq p \leq \infty$, then there exist $C > 0$ and $\delta > 0$ such that for any ball B and a measurable set $S \subset B$,*

$$\frac{w(S)}{w(B)} \leq C\Big(\frac{|S|}{|B|}\Big)^{\delta}.$$

We are going to use the following result on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)d\mu(r),\ 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0, \infty)$.

Theorem 5. ([4]) *The inequality*

$$\operatorname*{ess\,sup}_{t>0} \omega(t)Hg(t) \leq c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all functions g non-negative and non-increasing on $(0, \infty)$ if and only if

$$A := \sup_{t>0} \frac{\omega(t)}{t}\int_0^t \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

We also need the following statement on the boundedness of the Hardy type operator

$$(H_1 g)(t) := \frac{1}{t}\int_0^t \ln^k\Big(e + \frac{t}{r}\Big) g(r)d\mu(r),\ 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0, \infty)$.

Theorem 6. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} \omega(t)H_1 g(t) \leq c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all functions g non-negative and non-increasing on $(0, \infty)$ if and only if

$$A_1 := \sup_{t>0} \frac{\omega(t)}{t}\int_0^t \ln^k\Big(e + \frac{t}{r}\Big) \frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A_1$.

Note that, Theorem 6 can be proved analogously to Theorem 4.3 in [19].

Definition 2. *$BMO(R^n)$ is the Banach space modulo constants with the norm $\|\cdot\|_*$ defined by*

$$\|b\|_* = \sup_{x\in R^n, r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y)-b_{B(x,r)}|dy < \infty,$$

where $b \in L_1^{\rm loc}(R^n)$ and

$$b_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} b(y)dy.$$

Lemma 2. ([33], Theorem 5, p. 236) *Let $w \in A_\infty$. Then the norm $\|\cdot\|_*$ is equivalent to the norm*

$$\|b\|_{*,w} = \sup_{x\in R^n, r>0} \frac{1}{w(B(x,r))}\int_{B(x,r)} |b(y)-b_{B(x,r),w}|w(y)dy,$$

where

$$b_{B(x,r),w} = \frac{1}{w(B(x,r))}\int_{B(x,r)} b(y)w(y)dy.$$

Remark 3. (1) *The John-Nirenberg inequality : there are constants C_1, $C_2 > 0$, such that for all $b \in BMO(R^n)$ and $\beta > 0$*

$$|\{x \in B : |b(x)-b_B| > \beta\}| \le C_1|B|e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset R^n.$$

(2) *For $1 \le p < \infty$ the John-Nirenberg inequality implies that*

$$\|b\|_* \approx \sup_B \left(\frac{1}{|B|}\int_B |b(y)-b_B|^p dy\right)^{\frac{1}{p}} \tag{5}$$

and for $1 \le p < \infty$ and $w \in A_\infty$

$$\|b\|_* \approx \sup_B \left(\frac{1}{w(B)}\int_B |b(y)-b_B|^p w(y)dy\right)^{\frac{1}{p}}. \tag{6}$$

Note that, by the John-Nirenberg inequality and Lemma 1 (part 3) it follows that

$$w(\{x \in B : |b(x)-b_B| > \beta\}) \le C_1^\delta w(B)e^{-C_2\beta\delta/\|b\|_*}$$

for some $\delta > 0$. Hence

$$\int_B |b(y)-b_B|^p w(y)dy = p\int_0^\infty \beta^{p-1}\, w(\{x \in B : |b(x)-b_B| > \beta\})d\beta$$
$$\le pC_1^\delta\, w(B)\int_0^\infty \beta^{p-1}\, e^{-C_2\beta\delta/\|b\|_*}\, d\beta = C_3 w(B)\|b\|_*^p,$$

where $C_3 > 0$ depends only on C_1^δ, C_2, p, and δ, which implies (6).

Also (5) is a particular case of (6) with $w \equiv 1$.

The following lemma was proved in [22].

Lemma 3. *i) Let $w \in A_\infty$ and $b \in BMO(\omega)$. Let also $1 \le p < \infty$, $x \in R^n$, $k > 0$ and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{w(B(x,r_1))}\int\limits_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{kp} w(y)dy\Big)^{\frac{1}{p}} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)^k \|b\|_*^k,$$

where $C > 0$ is independent of f, w, x, r_1, and r_2.

ii) Let $w \in A_p$ and $b \in BMO(\omega)$. Let also $1 < p < \infty$, $x \in R^n$, $k > 0$ and $r_1, r_2 > 0$. Then

$$\Big(\frac{1}{w^{1-p'}(B(x,r_1))}\int\limits_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{kp'} w(y)^{1-p'}dy\Big)^{\frac{1}{p'}} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)^k \|b\|_*^k,$$

where $C > 0$ is independent of f, w, x, r_1, and r_2.

4. Proofs of main theorems

Before proving the main theorems, we need the following lemmas.

Lemma 4. *[36] For $j \in \mathbb{Z}_+$, denote*

$$G_{\alpha,2^j}(f)(x) = \left(\int_0^\infty \int_{|x-y|\le 2^j t} (A_\alpha f(y,t))^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}.$$

Let $0 < \alpha \le 1$, $1 < p < \infty$ and $w \in A_p$. Then for any $j \in \mathbb{Z}_+$, we have

$$\|G_{\alpha,2^j}(f)\|_{L^p_w} \lesssim 2^{j\left(\frac{3n}{2}+\alpha\right)} \|G_\alpha(f)\|_{L^p_w}.$$

This lemma is easily derived from the following inequality which is proved in [39].

$$G_{\alpha,\beta}(f)(x) \le \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x).$$

By the similar argument as in [3], we can get the following lemma.

Lemma 5. *Let $1 < p < \infty$, $0 < \alpha \le 1$ and $w \in A_p$. Then the commutator $[b, G_\alpha]$ is bounded from $L^p_w(l_2)$ to itself whenever $b \in BMO$.*

Now we are in a position to prove theorems.

Lemma 6. *Let $1 \le p < \infty$, $0 < \alpha \le 1$ and $w \in A_p$.*

Then, for $p > 1$ the inequality

$$\|G_\alpha \vec{f}\|_{L^p_w(B,l_2)} \lesssim \big(w(B)\big)^{\frac{1}{p}} \int_{2r}^\infty \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \big(w(B(x_0,t))\big)^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ *and for all* $\vec{f} \in L^{1,loc}_{\omega}(l_2)$.

Moreover, for $p = 1$ *the inequality*

$$\|G_\alpha \vec{f}\|_{WL^1_w(B,l_2)} \lesssim w(B) \int_{2r}^{\infty} \|\vec{f}\|_{L^1_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t))\right)^{-1} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ *and for all* $\vec{f} \in L^{1,\mathrm{loc}}_{w}(l_2)$.

Proof. The main ideas of these proofs come from [22]. For arbitrary $x \in R^n$, set $B = B(x_0, r)$, $2B \equiv B(x_0, 2r)$. We decompose $\vec{f} = \vec{f_0} + \vec{f_\infty}$, where $\vec{f_0}(y) = \vec{f}(y)\chi_{2B}(y)$, $\vec{f_\infty}(y) = \vec{f}(y) - \vec{f_0}(y)$. Then,

$$\|G_\alpha \vec{f}\|_{L^p_w\left(B(x_0,r),l_2\right)} \le \|G_\alpha \vec{f_0}\|_{L^p_w\left(B(x_0,r),l_2\right)} + \|G_\alpha \vec{f_\infty}\|_{L^p\left(B(x_0,r),l_2\right)} := I + II.$$

First, let us estimate I. By Theorem A, we can obtain that

$$I \le \|G_\alpha \vec{f_0}\|_{L^p_w(l_2)} \lesssim \|\vec{f_0}\|_{L^p_w(l_2)} = \|\vec{f}\|_{L^p_w(2B,l_2)}. \tag{7}$$

On the other hand,

$$\begin{aligned}
\|\vec{f}\|_{L^p_w(2B,l_2)} &\approx |B| \|\vec{f}\|_{L^p_w(2B,l_2)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}} \\
&\le |B| \int_{2r}^{\infty} \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \frac{dt}{t^{n+1}} \\
&\lesssim w(B)^{\frac{1}{p}} \|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \frac{dt}{t^{n+1}} \\
&\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}} \\
&\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t))\right)^{-\frac{1}{p}} \frac{dt}{t}.
\end{aligned} \tag{8}$$

Therefore from (7) and (8) we get

$$I \lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t))\right)^{-\frac{1}{p}} \frac{dt}{t}. \tag{9}$$

Now let us estimate II.

$$\|\vec{f} * \phi_t(y)\|_{l_2} = \left\| t^{-n} \int_{|y-z|\le t} \phi\left(\frac{y-z}{t}\right) \vec{f_\infty}(z) dz \right\|_{l_2} \le t^{-n} \int_{|y-z|\le t} \|\vec{f_\infty}(z)\|_{l_2} dz.$$

Since $x \in B(x_0, r)$, $(y, t) \in \Gamma(x)$, we have $|z - x| \le |z - y| + |y - x| \le 2t$, and

$$r \le |z - x_0| - |x_0 - x| \le |x - z| \le |x - y| + |y - z| \le 2t.$$

So, we obtain

$$\begin{aligned}\left\|G_\alpha \vec{f}_\infty(x)\right\|_{l_2} &\le \left(\int\int_{\Gamma(x)}\left(t^{-n}\int_{|y-z|\le t}\|\vec{f}_\infty(z)\|_{l_2}dz\right)^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\ &\le \left(\int_{t>r/2}\int_{|x-y|<t}\left(\int_{|x-z|\le 2t}\|\vec{f}_\infty(z)\|_{l_2}dz\right)^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\ &\lesssim \left(\int_{t>r/2}\left(\int_{|z-x|\le 2t}\|\vec{f}_\infty(z)\|_{l_2}dz\right)^2\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}.\end{aligned}$$

By Minkowski and Hölder's inequalities and $|z-x|\ge|z-x_0|-|x_0-x|\ge\frac{1}{2}|z-x_0|$, we have

$$\begin{aligned}\left\|G_\alpha \vec{f}_\infty(x)\right\|_{l_2} &\lesssim \int_{R^n}\left(\int_{t>\frac{|z-x|}{2}}\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}}\|\vec{f}_\infty(z)\|_{l_2}dz\\ &\lesssim \int_{|z-x_0|>2r}\frac{\|\vec{f}(z)\|_{l_2}}{|z-x|^n}dz \lesssim \int_{|z-x_0|>2r}\frac{\|\vec{f}(z)\|_{l_2}}{|z-x_0|^n}dz\\ &= \int_{|z-x_0|>2r}\|\vec{f}(z)\|_{l_2}\int_{|z-x_0|}^{+\infty}\frac{dt}{t^{n+1}}dz\\ &= \int_{2r}^{+\infty}\int_{2r<|z-x_0|<t}\|\vec{f}(z)\|_{l_2}dz\frac{dt}{t^{n+1}}\\ &\lesssim \int_{2r}^{\infty}\|\|\vec{f}(z)\|_{l_2}\|_{L_w^p(B(x_0,t))}\|w^{-1}\|_{L_{p'}(B(x_0,t))}\frac{dt}{t^{n+1}}\\ &\lesssim \int_{2r}^{\infty}\|\vec{f}\|_{L_w^p\left(B(x_0,t),l_2\right)}\left(w(B(x_0,t))\right)^{-\frac{1}{p}}\frac{dt}{t}. \end{aligned}\tag{10}$$

Thus,

$$\|G_\alpha \vec{f}_\infty\|_{L_w^p(B,l_2)} \lesssim w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|\vec{f}\|_{L_w^p\left(B(x_0,t),l_2\right)}\left(w(B(x_0,t))\right)^{-\frac{1}{p}}\frac{dt}{t}.\tag{11}$$

By combining (9) and (11), we have

$$\|G_\alpha \vec{f}\|_{L_w^p(B,l_2)} \lesssim w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|\vec{f}\|_{L_w^p\left(B(x_0,t),l_2\right)}\left(w(B(x_0,t))\right)^{-\frac{1}{p}}\frac{dt}{t}.$$

Proof of Theorem 1

By Lemma 6 and Theorem 5 we have for $p>1$

$$\|G_\alpha \vec{f}\|_{M_w^{p,\varphi_2}(l_2)} \lesssim \sup_{x_0\in R^n, r>0}\varphi_2(x_0,\ r)^{-1}\int_{r}^{\infty}\|\vec{f}\|_{L_w^p\left(B(x_0,t),l_2\right)}\left(w(B(x_0,t))\right)^{-\frac{1}{p}}\frac{dt}{t}$$

$$= \sup_{x_0\in R^n, r>0} \varphi_2(x_0,\ r)^{-1} \int_0^{r^{-1}} \|\vec{f}\|_{L^p_w\left(B(x_0,t^{-1}),l_2\right)} \left(w(B(x_0,t^{-1}))\right)^{-\frac{1}{p}} \frac{dt}{t}$$
$$= \sup_{x_0\in R^n, r>0} \varphi_2(x_0,r^{-1})^{-1}\, r\, \frac{1}{r} \int_0^{r} \|\vec{f}\|_{L^p_w\left(B(x_0,t^{-1}),l_2\right)} \left(w(B(x_0,t^{-1}))\right)^{-\frac{1}{p}} \frac{dt}{t}$$
$$\lesssim \sup_{x_0\in R^n, r>0} \varphi_1(x_0,r^{-1})^{-1} \left(w(B(x_0,r^{-1}))\right)^{-\frac{1}{p}} \|\vec{f}\|_{L^p_w\left(B(x_0,r^{-1}),l_2\right)}$$
$$= \sup_{x_0\in R^n, r>0} \varphi_1(x_0,r)^{-1} \left(w(B(x_0,r))\right)^{-\frac{1}{p}} \|\vec{f}\|_{L^p_w\left(B(x_0,r),l_2\right)} = \|\vec{f}\|_{M^{p,\varphi_1}_w(l_2)}$$

and for $p = 1$

$$\|G_\alpha \vec{f}\|_{WM^{1,\varphi_2}_w(l_2)} \lesssim \sup_{x_0\in R^n, r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|\vec{f}\|_{L^1_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t))\right)^{-1} \frac{dt}{t}$$
$$= \sup_{x_0\in R^n, r>0} \varphi_2(x_0,\ r)^{-1} \int_0^{r^{-1}} \|\vec{f}\|_{L^1_w\left(B(x_0,t^{-1}),l_2\right)} \left(w(B(x_0,t^{-1}))\right)^{-1} \frac{dt}{t}$$
$$= \sup_{x_0\in R^n, r>0} \varphi_2(x_0,r^{-1})^{-1}\, r\, \frac{1}{r} \int_0^{r} \|\vec{f}\|_{L^1_w\left(B(x_0,t^{-1}),l_2\right)} \left(w(B(x_0,t^{-1}))\right)^{-1} \frac{dt}{t}$$
$$\lesssim \sup_{x_0\in R^n, r>0} \varphi_1(x_0,r^{-1})^{-1} \left(w(B(x_0,r^{-1}))\right)^{-1} \|\vec{f}\|_{L^1_w\left(B(x_0,r^{-1}),l_2\right)}$$
$$= \sup_{x_0\in R^n, r>0} \varphi_1(x_0,r)^{-1} \left(w(B(x_0,r))\right)^{-1} \|\vec{f}\|_{L^1_w\left(B(x_0,r),l_2\right)} = \|\vec{f}\|_{M^{1,\varphi_1}_w(l_2)}.$$

Lemma 7. *Let $1 \le p < \infty$, $0 < \alpha \le 1$, $\lambda > 3 + \dfrac{\alpha}{n}$ and $w \in A_p$. Then, for $p > 1$ the inequality*

$$\left\|g^*_{\lambda,\alpha}(\vec{f})\right\|_{L^p_w\left(B,l_2\right)} \lesssim \left(w(B)\right)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t))\right)^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B = B(x_0,r)$ and for all $\vec{f} \in L^{1,loc}_\omega()l_2$.

Moreover, for $p = 1$ the inequality

$$\left\|g^*_{\lambda,\alpha}(\vec{f})\right\|_{WL^1_w\left(B,l_2\right)} \lesssim w(B) \int_{2r}^{\infty} \|\vec{f}\|_{L^1_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t))\right)^{-1} \frac{dt}{t}$$

holds for any ball $B = B(x_0,r)$ and for all $\vec{f} \in L^{1,\mathrm{loc}}_w(l_2)$.

Proof. From the definition of $g^*_{\lambda,\alpha}(f)$, we readily see that

$$\left\|g^*_{\lambda,\alpha}(\vec{f})(x)\right\|_{l_2} = \Big\|\Big(\int_0^\infty \int_{R^n} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left(A_\alpha \vec{f}(y,t)\right)^2 \frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l_2}$$
$$\le \Big\|\Big(\int_0^\infty \int_{|x-y|<t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda} \left(A_\alpha \vec{f}(y,t)\right)^2 \frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l_2}$$

$$+\Big\|\Big(\int_0^\infty\int_{|x-y|\geq t}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}\Big(A_\alpha\vec{f}(y,t)\Big)^2\frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l_2}$$
$$:=III+IV.$$

First, let us estimate III.

$$III\leq\Big\|\Big(\int_0^\infty\int_{|x-y|<t}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}\Big(A_\alpha\vec{f}(y,t)\Big)^2\frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l_2}\leq\|G_\alpha\vec{f}(x)\|_{l_2}.$$

Now, let us estimate IV.

$$\begin{aligned}IV&\leq\Big\|\Big(\sum_{j=1}^\infty\int_0^\infty\int_{2^{j-1}t\leq|x-y|\leq 2^jt}\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}\Big(A_\alpha\vec{f}(y,t)\Big)^2\frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l_2}\\&\lesssim\Big\|\Big(\sum_{j=1}^\infty\int_0^\infty\int_{2^{j-1}t\leq|x-y|\leq 2^jt}2^{-jn\lambda}\Big(A_\alpha\vec{f}(y,t)\Big)^2\frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l_2}\\&\lesssim\sum_{j=1}^\infty 2^{-jn\lambda}\Big\|\Big(\int_0^\infty\int_{|x-y|\leq 2^jt}\Big(A_\alpha\vec{f}(y,t)\Big)^2\frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l_2}\\&:=\sum_{j=1}^\infty 2^{-jn\lambda}\Big\|G_{\alpha,2^j}(\vec{f})(x)\Big\|_{l_2}.\end{aligned}$$

Thus,

$$\|g^*_{\lambda,\alpha}(\vec{f})\|_{L^p_w(B,l_2)}\leq\|G_\alpha\vec{f}\|_{L^p_w(B,l_2)}+\sum_{j=1}^\infty 2^{-\frac{jn\lambda}{2}}\|G_{\alpha,2^j}(\vec{f})\|_{L^p_w(B,l_2)}.\tag{12}$$

By Lemma 6, we have

$$\|G_\alpha\vec{f}\|_{L^p_w(B,l_2)}\lesssim\big(w(B)\big)^{\frac{1}{p}}\int_{2r}^\infty\|\vec{f}\|_{L^p_w(B(x_0,t))}\big(w(B(x_0,t))\big)^{-\frac{1}{p}}\frac{dt}{t}.\tag{13}$$

Now we proceed to estimate $\|G_{\alpha,2^j}(\vec{f})\|_{L^p_w(B,l_2)}$. We divide $\|G_{\alpha,2^j}(\vec{f})\|_{L^p_w(B,l_2)}$ into two parts.

$$\|G_{\alpha,2^j}(\vec{f})\|_{L^p_w(B,l_2)}\leq\|G_{\alpha,2^j}(\vec{f_0})\|_{L^p_w(B,l_2)}+\|G_{\alpha,2^j}(\vec{f_\infty})\|_{L^p_w(B,l_2)},\tag{14}$$

where $\vec{f_0}(y)=\vec{f}(y)\chi_{2B}(y)$, $\vec{f_\infty}(y)=\vec{f}(y)-\vec{f_\infty}(y)$. For the first part, by Lemma 4,

$$\begin{aligned}\|G_{\alpha,2^j}(\vec{f_0})\|_{L^p_w(B,l_2)}&\lesssim 2^{j(\frac{3n}{2}+\alpha)}\|G_\alpha(\vec{f_0})\|_{L^p_w(l_2)}\lesssim 2^{j(\frac{3n}{2}+\alpha)}\|f\|_{L^p_w(B,l_2)}\\&\lesssim 2^{j(\frac{3n}{2}+\alpha)}w(B)^{\frac{1}{p}}\int_{2r}^\infty\|\vec{f}\|_{L^p_w(B(x_0,t),l_2)}\big(w(B(x_0,t))\big)^{-\frac{1}{p}}\frac{dt}{t}.\end{aligned}\tag{15}$$

For the second part.

$$\Big\|G_{\alpha,2^j}(\vec{f_\infty})(x)\Big\|_{l_2}=\Big\|\Big(\int_0^\infty\int_{|x-y|\leq 2^jt}\Big(A_\alpha\vec{f}(y,t)\Big)^2\frac{dydt}{t^{n+1}}\Big)^{l/2}\Big\|_{l_2}$$

$$= \Big\| \left(\int_0^\infty \int_{|x-y|\le 2^j t} \left(\sup_{\phi\in C_\alpha} |\vec{f} * \phi_t(y)| \right)^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \Big\|_{l_2}$$

$$\le \left(\int_0^\infty \int_{|x-y|\le 2^j t} \left(\int_{|z-y|\le t} \|\vec{f}_\infty(z)\|_{l_2} dz \right)^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}}.$$

Since $|x-z| \le |y-z| + |x-y| \le 2^{j+1}t$, we get

$$\left\| G_{\alpha,2^j}(\vec{f}_\infty)(x) \right\|_{l_2} \le \left(\int_0^\infty \int_{|x-y|\le 2^j t} \left(\int_{|x-z|\le 2^{j+1}t} \|\vec{f}_\infty(z)\|_{l_2} dz \right)^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}}$$

$$\le \left(\int_0^\infty \left(\int_{|z-x|\le 2^{j+1}t} \|\vec{f}_\infty(z)\|_{l_2} dz \right)^2 \frac{2^{jn}dt}{t^{2n+1}} \right)^{\frac{1}{2}}$$

$$\le 2^{\frac{jn}{2}} \int_{R^n} \left(\int_{t\ge \frac{|x-z|}{2^{j+1}}} \|\vec{f}_\infty(z)\|_{l_2}^2 \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} dz$$

$$\le 2^{\frac{3jn}{2}} \int_{|x_0-z|>2r} \frac{\|\vec{f}(z)\|_{l_2}}{|x-z|^n} dz.$$

As $|z-x| \ge |x_0-z| - |x-x_0| \ge |x_0-z| - \frac{1}{2}|x_0-z| = \frac{1}{2}|x_0-z|$, by Fubini's theorem and Hölder's inequality, we obtain

$$\left\| G_{\alpha,2^j}(\vec{f}_\infty)(x) \right\|_{l_2} \le 2^{\frac{3jn}{2}} \int_{|x_0-z|>2r} \frac{\|\vec{f}(z)\|_{l_2}}{|x_0-z|^n} dz$$

$$= 2^{\frac{3jn}{2}} \int_{|x_0-z|>2r} \|\vec{f}(z)\|_{l_2} \int_{|x_0-z|}^\infty \frac{dt}{t^{n+1}} dz$$

$$\le 2^{\frac{3jn}{2}} \int_{2r}^\infty \int_{|x_0-z|<t} \|\vec{f}(z)\|_{l_2} dz \frac{dt}{t^{n+1}}$$

$$\le 2^{\frac{3jn}{2}} \int_{2r}^\infty \| \|\vec{f}(\cdot)\|_{l_2} \|_{L^1(B(x_0,t))} \frac{dt}{t^{n+1}}.$$

$$\le 2^{\frac{3jn}{2}} \int_{2r}^\infty \|\vec{f}(\cdot)\|_{l_2} \|_{L^p_w(B(x_0,t))} \|w^{-1}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}$$

$$\le 2^{\frac{3jn}{2}} \int_{2r}^\infty \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t)) \right)^{-\frac{1}{p}} \frac{dt}{t}.$$

So,

$$\|G_{\alpha,2^j}(\vec{f}_\infty)\|_{L^p_w\left(B,l_2\right)} \le 2^{\frac{3jn}{2}} w(B)^{\frac{1}{p}} \int_{2r}^\infty \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t)) \right)^{-\frac{1}{p}} \frac{dt}{t}. \qquad (16)$$

Combining (14), (15) and (16), we have

$$\|G_{\alpha,2^j}(\vec{f})\|_{L^p_w\left(B,l_2\right)} \lesssim 2^{j(\frac{3n}{2}+\alpha)}\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t))\right)^{-\frac{1}{p}} \frac{dt}{t}. \tag{17}$$

Thus,

$$\|g^*_{\lambda,\alpha}(\vec{f})\|_{L^p_w\left(B,l_2\right)} \leq \|G_\alpha \vec{f}\|_{L^p_w\left(B,l_2\right)} + \sum_{j=1}^{\infty} 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(\vec{f})\|_{L^p_w\left(B,l_2\right)}. \tag{18}$$

Since $\lambda > 3 + \dfrac{\alpha}{n}$, by (13), (17) and (18), we have the desired lemma.

Proof of Theorem 2

From inequality (19) we have

$$\|g^*_{\lambda,\alpha}(\vec{f})\|_{M^{p,\varphi_2}_w(l_2)} \leq \|G_\alpha \vec{f}\|_{M^{p,\varphi_2}_w(l_2)} + \sum_{j=1}^{\infty} 2^{-\frac{jn\lambda}{2}} \|G_{\alpha,2^j}(\vec{f})\|_{M^{p,\varphi_2}_w(l_2)}. \tag{19}$$

By Theorem 1, we have

$$\|G_\alpha \vec{f}\|_{M^{p,\varphi_2}_w(l_2)} \lesssim \|\vec{f}\|_{M^{p,\varphi_1}_w(l_2)}. \tag{20}$$

Now we proceed to estimate $\|G_{\alpha,2^j}(\vec{f})\|_{M^{p,\varphi_2}_w(l_2)}$. By change of variables and Theorem 5, we get

$$\begin{aligned}
&\|G_{\alpha,2^j}(\vec{f})\|_{M^{p,\varphi_2}_w(l_2)} \\
&\lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in R^n, r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t))\right)^{-\frac{1}{p}} \frac{dt}{t} \\
&= 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in R^n, r>0} \varphi_2(x_0,r^{-1})^{-1}\, r\, \frac{1}{r} \int_0^{r} \|\vec{f}\|_{L^p_w\left(B(x_0,t^{-1}),l_2\right)} \left(w(B(x_0,t^{-1}))\right)^{-\frac{1}{p}} \frac{dt}{t} \\
&\lesssim 2^{j(\frac{3n}{2}+\alpha)} \sup_{x_0\in R^n, r>0} \varphi_1(x_0,r^{-1})^{-1} \left(w(B(x_0,r^{-1}))\right)^{-\frac{1}{p}} \|\vec{f}\|_{L^p_w\left(B(x_0,r^{-1}),l_2\right)} \\
&= 2^{j(\frac{3n}{2}+\alpha)} \|\vec{f}\|_{M^{p,\varphi_1}_w(l_2)}.
\end{aligned} \tag{21}$$

Since $\lambda > 3 + \dfrac{\alpha}{n}$, by (19), (20) and (21), we have the desired theorem.

Lemma 8. *Let $1 < p < \infty$, $0 < \alpha \leq 1$, $w \in A_p$ and $b \in BMO$.*
Then the inequality

$$\|[b, G_\alpha]^k \vec{f}\|_{L^p_w\left(B,l_2\right)} \lesssim \left(w(B)\right)^{\frac{1}{p}} \int_{2r}^{\infty} \ln^k\left(e + \frac{t}{r}\right) \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \left(w(B(x_0,t))\right)^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B = B(x_0, r)$ and for all $\vec{f} \in L^{1,loc}_\omega(l_2)$.

Proof. We decompose $\vec{f}=\vec{f_0}+\vec{f_\infty}$, where $\vec{f_0}=\vec{f}\chi_{2B}$ and $\vec{f_\infty}=\vec{f}-\vec{f_0}$. Then

$$\|[b,G_\alpha]^k\vec{f}\|_{L^p_w\left(B,l_2\right)} \le \|[b,G_\alpha]^k\vec{f_0}\|_{L^p_w\left(B,l_2\right)}+\|[b,G_\alpha]^k\vec{f_\infty}\|_{L^p_w\left(B,l_2\right)}.$$

By Lemma 5, we have

$$\begin{aligned}\|[b,\ G_\alpha]^k\vec{f_0}\|_{L^p_w\left(B,l_2\right)} &\lesssim \|b\|_*^k\,\|\vec{f_0}\|_{L^p_w(l_2)}=\|b\|_*^k\,\|\vec{f}\|_{L^p_w\left(2B,l_2\right)}\\ &\lesssim \|b\|_*^k\,w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)}\left(w(B(x_0,t))\right)^{-\frac{1}{p}}\frac{dt}{t}.\end{aligned}$$

As for the second part, we divide it into two parts:

$$\begin{aligned}\Big\|[b,G_\alpha]^k\vec{f_\infty}(x)\Big\|_{l_2} &= \Big\|\Big(\int\int_{\Gamma(x)}\sup_{\phi\in C_\alpha}\Big|\int_{R^n}[b(x)-b(z)]^k\phi_t(y-z)\vec{f_\infty}(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac{1}{2}}\Big\|_{l_2}\\ &\le \Big\|\Big(\int\int_{\Gamma(x)}\sup_{\phi\in C_\alpha}\Big|\int_{R^n}[b(x)-b_{B,w}]^k\phi_t(y-z)\vec{f_\infty}(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac{1}{2}}\Big\|_{l_2}\\ &+\Big\|\Big(\int\int_{\Gamma(x)}\sup_{\phi\in C_\alpha}\Big|\int_{R^n}[b(z)-b_{B,w}]^k\phi_t(y-z)\vec{f_\infty}(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac{1}{2}}\Big\|_{l_2}\\ &:= A(x)+B(x).\end{aligned}$$

Therefore

$$\|[b,G_\alpha]^k\vec{f_\infty}\|_{L^p_w\left(B,l_2\right)} \le \|A(\cdot)\|_{L^p_w(B)}+\|B(\cdot)\|_{L^p_w(B)}.$$

First, for $A(x)$, we find that

$$\begin{aligned}A(x) &= |b(x)-b_{B,w}|^k\,\Big\|\Big(\iint_{\Gamma(x)}\sup_{\phi\in C_\alpha}\Big|\int_{R^n}\phi_t(y-z)\vec{f_\infty}(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac{1}{2}}\Big\|_{l_2}\\ &= \big|b(x)-b_{B,w}\big|^k\,\big\|G_\alpha\vec{f_\infty}(x)\big\|_{l_2}.\end{aligned}$$

By Lemma 3 and from the inequality (10), we can get

$$\begin{aligned}\|A(\cdot)\|_{L^p_w(B)} &= \left(\int_B|b(x)-b_{B,w}|^{kp}\left(\big\|G_\alpha\vec{f_\infty}(x)\big\|_{l_2}\right)^p w(x)dx\right)^{\frac{1}{p}}\\ &\le \left(\int_B|b(x)-b_{B,w}|^{kp}\,w(x)dx\right)^{\frac{1}{p}}\int_{2r}^{\infty}\|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)}\left(w(B(x_0,t))\right)^{-\frac{1}{p}}\frac{dt}{t}\\ &\le \|b\|_*^k w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)}\left(w(B(x_0,t))\right)^{-\frac{1}{p}}\frac{dt}{t}.\end{aligned}$$

For $B(x)$, since $|y-x|<t$, we get $|x-z|<2t$. Thus, by Minkowski's inequality,

$$B(x)\le\Big\|\Big(\int\int_{\Gamma(x)}\Big|\int_{|x-z|<2t}|b_{B,w}-b(z)|^k\,\vec{f_\infty}(z)dz\Big|^2\frac{dydt}{t^{3n+1}}\Big)^{\frac{1}{2}}\Big\|_{l_2}$$

$$\lesssim \Big(\int_0^\infty \Big| \int_{|x-z|<2t} |b_{B,w} - b(z)|^k \left\| \vec{f}_\infty(z) \right\|_{l_2} dz \Big|^2 \frac{dt}{t^{2n+1}} \Big)^{\frac{1}{2}}$$
$$\leq \int_{|x_0-z|>2r} |b_{B,w} - b(z)|^k \left\| \vec{f}(z) \right\|_{l_2} \frac{dz}{|x-z|^n}.$$

For $B(x)$, using the inequality $|z-x| \geq \frac{1}{2}|z-x_0|$, we have

$$\begin{aligned} B(x) &\lesssim \int_{|x_0-z|>2r} |b(z) - b_{B,w}|^k \left\| \vec{f}(z) \right\|_{l_2} \frac{dz}{|x_0-z|^n} \\ &\lesssim \int_{|x_0-z|>2r} |b(z) - b_{B,w}|^k \left\| \vec{f}(z) \right\|_{l_2} \int_{|x_0-z|}^\infty \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^\infty \int_{2r\leq|x_0-z|\leq t} |b(z) - b_{B,w}|^k \left\| \vec{f}(z) \right\|_{l_2} dz \frac{dt}{t^{n+1}}. \end{aligned}$$

Applying Hölder's inequality and by Lemma 3, we get

$$\begin{aligned} \|B(\cdot)\|_{L^p_w(B)} &\lesssim w(B)^{\frac{1}{p}} \int_{2r}^\infty \left(\int_{B(x_0,t)} |b(z) - b_{B,w}|^{kp'} w(z)^{1-p'} dz \right)^{\frac{1}{p'}} \| \|\vec{f}(\cdot)\|_{l_2} \|_{L^p_w(B(x_0,t))} \frac{dt}{t^{n+1}} \\ &\lesssim \|b\|_*^k \, w(B)^{\frac{1}{p}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^k \|w^{-1/p}\|_{L_{p'}(B(x,t))} \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} \frac{dt}{t^{n+1}} \\ &\lesssim \|b\|_*^k w(B)^{\frac{1}{p}} \int_{2r}^\infty \ln^k \left(e + \frac{t}{r}\right) \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} w(B(x_0,t))^{-1/p} \frac{dt}{t}. \end{aligned}$$

Thus,

$$\left\| [b,\ G_\alpha]^k \vec{f} \right\|_{L^p_w\left(B,l_2\right)} \lesssim \|b\|_*^k \, w(B)^{\frac{1}{p}} \int_{2r}^\infty \ln^k \left(e + \frac{t}{r}\right) \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} w(B(x_0,t))^{-1/p} \frac{dt}{t}.$$

Proof of Theorem 3

By change of variables, we obtain

$$\begin{aligned} &\|[b,G_\alpha]^k \vec{f}\|_{M^{p,\varphi_2}_w(l_2)} \\ &\lesssim \|b\|_*^k \sup_{x_0\in R^n, r>0} \varphi_2(x_0,r)^{-1} \int_{2r}^\infty \ln^k \left(e + \frac{t}{r}\right) \|\vec{f}\|_{L^p_w\left(B(x_0,t),l_2\right)} w(B(x_0,t))^{-1/p} \frac{dt}{t} \\ &\lesssim \|b\|_*^k \sup_{x_0\in R^n, r>0} \varphi_2(x_0,r)^{-1} \int_0^{r^{-1}} \ln^k \left(e + \frac{1}{tr}\right) \|\vec{f}\|_{L^p_w\left(B(x_0,t^{-1}),l_2\right)} w(B(x_0,t^{-1}))^{-\frac{1}{p}} \frac{dt}{t} \\ &= \sup_{x\in R^n,\, r>0} \|b\|_*^k \, \varphi_2(x_0,r^{-1})^{-1} \, r \, \frac{1}{r} \int_0^r \ln^k \left(e + \frac{r}{t}\right) \|\vec{f}\|_{L^p_w\left(B(x_0,t^{-1}),l_2\right)} w(B(x_0,t^{-1}))^{-\frac{1}{p}} \frac{dt}{t} \\ &\lesssim \|b\|_*^k \sup_{x_0\in R^n, r>0} \varphi_1(x_0,r^{-1})^{-1} w(B(x_0,r^{-1}))^{-\frac{1}{p}} \|\vec{f}\|_{L^p_w\left(B(x_0,r^{-1}),l_2\right)} \\ &= \|b\|_*^k \sup_{x_0\in R^n, r>0} \varphi_1(x_0,r)^{-1} w(B(x_0,r))^{-\frac{1}{p}} \left\|\vec{f}\right\|_{L^p_w(B(x_0,r),l_2)} \end{aligned}$$

$= \|b\|_*^k \, \|\vec{f}\|_{M_w^{p,\varphi_1}(l_2)}$.

By using the argument similar to that of the above proofs and that of Theorem 2, we can also show the boundedness of $[b, g^*_{\lambda,\alpha}]^k$.

ACKNOWLEDGEMENTS. The research of V. Guliyev were partially supported by the grant of Ahi Evran University, Scientific Research Projects (PYO.FEN.4003.13.003) and (PYO.FEN.4003-2.13.007).

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Vagif S. Guliyev
Department of Mathematics, Ahi Evran University, Kirsehir, Turkey.
Institute of Mathematics and Mechanics, Baku, Azerbaijan
E-mail: vagif@guliyev.com

Mehriban N. Omarova
Department of Computer Science and Engineering, Changshu Institute of Technology, Changshu 215500, China
E-mail: mehriban_omarova@yahoo.com

Received 10 May 2012
Accepted 29 October 2013