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On the Convergence of Composite Implicit Iteration Process with Errors for Asymptotically Nonexpansive Mappings in the Intermediate Sense

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Abstract. In this paper we establish a necessary and sufficient condition for the strong convergence of the composite iteration process with errors to a common fixed point of the finite family of asymptotically nonexpansive mappings in the intermediate sense in a arbitrary real Banach space. We also prove several strong and weak convergence results of this implicit iterative scheme in a uniformly convex Banach space. Further we also prove that in a uniformly convex Banach space with the dual having Kadec-Klee property, the composite implicit iteration process converges weakly to a common fixed point of a finite family of asymptotically nonexpansive mappings in the intermediate sense. Our results extend several existing results.

Key Words and Phrases: composite implicit iteration process with errors; asymptotically nonexpansive mapping in the intermediate sense; Opial's condition; Kadec-Klee property; uniformly convex Banach space; common fixed point; condition($\overline{B}$); weak and strong convergence.

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1. Introduction

Let X be a normed space, C be a nonempty subset of X and let $T : C \to C$ be a given mapping. Then T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|, \text{ for all } x, y \in C \text{ and each } n \geq 1.$$

If $k_n \equiv 1$ then T is known as a nonexpansive mapping. The weaker definition [10] requires that

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

for every $x \in C$ and that T^N be continuous for some $N \geq 1$.
Bruck et al.[1] gave a definition which is somewhere between these two : T is called

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asymptotically nonexpansive mapping in the intermediate sense [1] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \leq 0.$$

T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\|, \text{ for all } x, y \in C \text{ and each } n \geq 1.$$

The above definitions make it clear that asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense and uniformly L-Lipschitzian mapping, but the converse need not be true:
Example [9]: Let $X = R, C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is asymptotically nonexpansive mapping in the intermediate sense, but it is not asymptotically nonexpansive mapping.

In 2001, Xu and Ori[20] introduced the following implicit iteration process for a finite family of N nonexpansive self mappings $\{T_i : i \in I\}$ of C (here $I = \{1, 2, .., N\}$) with $\{t_n\}$ a real sequence in $(0,1)$ and an initial point $x_0 \in C$ which is defined as follows:

$$\begin{cases} x_1 = t_1 x_0 + (1-t_1)T_1 x_1 \\ x_2 = t_2 x_1 + (1-t_2)T_2 x_2 \\ . \\ . \\ x_N = t_N x_{N-1} + (1-t_N)T_N x_N \\ x_{N+1} = t_{N+1} x_N + (1-t_{N+1})T_1 x_{N+1} \\ . \\ . \\ x_{2N} = t_{2N} x_{2N-1} + (1-t_{2N})T_N x_{2N} \\ x_{2N+1} = t_{2N+1} x_{2N} + (1-t_{2N+1})T_1 x_{2N+1} \\ . \\ . \end{cases}$$

The above process can be written in the compact form as:

$$x_n = t_n x_{n-1} + (1-t_n)T_n x_n, n \geq 1, \tag{1}$$

where $T_n = T_{n \bmod N}$. Xu and Ori they[20] proved the weak convergence of the process (1) to a common fixed point in the setting of a Hilbert space. Zhou and Chang [21] studied the modified implicit iteration with errors for a finite family of asymptotically nonexpansive mappings which in compact form can be written as

$$x_n = \alpha_n x_{n-1} + \beta_n T^n_{n(\bmod N)} x_n + \gamma_n u_n, \qquad n \geq 1, \tag{2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $\{u_n\}$ is a bounded sequence in C. Chang et al.[3] defined an implicit iteration process with error by

$$\begin{cases} x_1 = \alpha_1 x_0 + (1-\alpha_1)T_1 x_1 + v_1, \\ x_2 = \alpha_2 x_1 + (1-\alpha_2)T_2 x_2 + v_2, \\ . \\ . \\ x_N = \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N + v_N, \\ x_{N+1} = \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1} + v_{N+1}, \\ . \\ . \\ x_{2N} = \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N} + v_{2N}, \\ x_{2N+1} = \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1} + v_{2N+1}, \\ . \\ . \end{cases} \tag{3}$$

For each $n \geq 1$ we have $n = (k-1)N + i$, where $i = i(n) \in \{1, 2, ..., N\}, k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Then (3) can be written in the following compact form:

$$x_n \;=\; \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} x_n + v_n, \qquad n \geq 1, \tag{4}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ and $\{v_n\}$ is a bounded sequence in C where C is a nonempty closed convex subset of E satisfying $C + C \subset C$. Very recently Su and Li [15] introduced composite implicit iteration process for a finite family of strictly pseudocontractive maps which is defined as follows:

$$\begin{cases} x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n \\ y_n = \beta_n x_{n-1} + (1-\beta_n)T_n x_n, \end{cases} \tag{5}$$

where $T_n = T_{n(\bmod N)}$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0,1]$. Also Thakur [18] introduced the following composite implicit iteration process for a finite family of asymptotically nonexpansive mappings:

$$\begin{cases} x_1 = x \in C \\ x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n \\ y_n = (1-\beta_n)x_{n-1} + \beta_n T_{i(n)}^{k(n)} x_n, \qquad n \geq 1, \end{cases} \tag{6}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$.

Recently Cianciaruso et al.[5] introduced the following implicit iteration process with errors for a finite family of N self asymptotically nonexpansive mappings which is defined as follows:

$$\begin{cases} x_n = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n \\ y_n = (1-\beta_n-\delta_n)x_n + \beta_n T_{i(n)}^{k(n)} x_n + \delta_n v_n, \qquad n \geq 1, \end{cases} \tag{7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \gamma_n \leq 1, \beta_n + \delta_n \leq 1$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. It is called composite implicit iteration process with errors. For $\gamma_n = \delta_n = 0$, (7) reduces to (6).

To proceed we shall need the following well known definitions and lemmas:
A Banach space X is said to satisfy Opial's condition[11] if $x_n \rightharpoonup x$ (i.e. $x_n \to x$ weakly) and $x \neq y$ imply

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

A Banach space X is said to satisfy τ-Opial condition[1] if for every bounded $\{x_n\} \in X$ that τ-converges to $x \in X$ it holds

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for every $x \neq y$, where τ is a Hausdorff linear topology on X.
A Banach space X has the uniform τ-Opial property[1] if for each $c > 0$ there exists $r > 0$ with the property that for each $x \in X$ and each sequence $\{x_n\}$ such that $\{x_n\}$ is τ-convergent to 0 and

$$1 \leq \limsup_{n\to\infty} \|x_n\| < \infty, \|x\| \geq c$$

it holds $\limsup_{n\to\infty} \|x_n - x\| \geq 1 + r$. Clearly uniform τ-Opial condition implies τ-Opial condition. Note that a uniformly convex space which has the τ-Opial property necessarily has the uniform τ-Opial property, where τ is a Hausdorff linear topology on X.
Let T be a self-mapping of a nonempty subset C of a Banach space X. A sequence $\{x_n\}$ in C is called an almost orbit[6] of T if $\lim_{n\to\infty}[\sup_{m\geq 0} \|x_{n+m} - T^m x_n\|] = 0$
A Banach space X is said to satisfy Kadec-Klee property, if for every sequence $\{x_n\} \in X, x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply that $x_n \to x$ as $n \to \infty$. There are uniformly convex Banach spaces which neither have a Frèchet differentiable norm nor satisfy Opial's property but their duals do have the Kadec-Klee property (see [6],[8]).
Also we recall that a mapping $T : C \to C$ is called semi-compact[16] if for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ (as $n \to \infty$), there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^\star \in C$.

Lemma 1.1. ([*17*], *Lemma* 1) *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then*

(*i*) $\lim_{n\to\infty} a_n$ *exists,*

(*ii*) $\lim_{n\to\infty} a_n = 0$ *whenever* $\liminf_{n\to\infty} a_n = 0$.

Lemma 1.2. ([1]) *Suppose a Banach space X has the uniform τ-Opial property, C is a norm bounded, sequentially τ-compact subset of X and $T : C \to C$ is asymptotically nonexpansive in the weak sense. If $\{y_n\}$ is a sequence in C such that $\lim_{n\to\infty} \|y_n - z\|$ exists for each fixed point z of T and if $\{y_n - T^k y_n\}$is τ-convergent to 0 for each $k \in N$, then $\{y_n\}$ is τ-convergent to a fixed point of T.*

Lemma 1.3. ([13]) *Suppose that X is a uniformly convex Banach space and $0 < a \le t_n \le b < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\limsup_{n\to\infty} \|x_n\| \le r, \limsup_{n\to\infty} \|y_n\| \le r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.4. ([6], *Theorem* 5.3) *Let X be a uniformly convex Banach space such that $X^\star$ has the Kadec-Klee property and let C be a nonempty bounded closed convex subset of X. Suppose $T : C \to C$ is asymptotically nonexpansive mapping in the intermediate sense and $\{x_n\}$ is an almost orbit of T. Then $\{x_n\}$ is weakly convergent to a fixed point of T if and only if $w - \lim_{n\to\infty}(x_n - x_{n+1}) = 0$.*

Now we recall some well-known definitions:
A mapping $T : K \to K$ with nonempty fixed point set $F(T)$ in K satisfies **Condition** (I) [14] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$f(d(x, F(T))) \le \|x - Tx\| \text{ for all } x \in K.$$

A finite family of mappings $T_i : K \to K,$ for all $i = 1, 2, 3, ..., N$ with nonempty fixed point set $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ satisfies
Condition$(\overline{A})$[4] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$f(d(x, F)) \le \frac{1}{N}(\sum_{i=1}^{N} \|x - T_i x\|) \text{ for all } x \in K,$$

Condition$(\overline{B})$[4] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$f(d(x, F)) \le \max_{1\le i\le N}\{\|x - T_i x\|\} \text{ for all } x \in K,$$

Condition$(\overline{C})$[4] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that at least one of the T_i's satisfies condition (I)(*i.e.* $f(d(x, F(T))) \le \|x - T_i x\|$ for at least one $T_i, i = 1, 2, .., N$).
Clearly, if $T_i = T,$ for all $i = 1, 2, ..., N$, then Condition$(\overline{A})$ reduces to Condition(I). Also Condition$(\overline{B})$ reduces to Condition(I) if all but one of T_i's are identities. Also it contains Condition$(\overline{A})$. Furthermore, Condition$(\overline{C})$ and Condition$(\overline{B})$ are equivalent (see [4]). It is well known that every continuous and demicompact mapping must satisfy

Condition(I)[14]. Since every completely continuous mapping is continuous and demicompact so it must satisfy Condition(I). Therefore to study the strong convergence of the iterative sequence $\{x_n\}$ defined by (7) we use Condition$(\overline{B})$ instead of the complete continuity of the mappings $\{T_1, T_2, ..., T_N\}$.
Recently convergence problems of an nonimplicit iteration process to a common fixed point for a finite family of asymptotically nonexpansive mappings in the intermediate sense in uniformly convex Banach spaces have been considered by several authors (see [1], [9], [12], [2]). The purpose of this paper is to study the weak and strong convergence of the composite implicit iterative sequence $\{x_n\}$ defined by (7) to a common fixed point for a finite family of asymptotically nonexpansive mappings in the intermediate sense in Banach spaces.

2. Main Results

We begin this section with the following lemmas. Throughout this section we denote $\{1, 2, ..., N\}$ by I.

Lemma 2.1. *Let X be a Banach space, C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of N asymptotically nonexpansive self-mappings of C in the intermediate sense. Set*

$$d_n = max\{max_{1\leq i\leq N} sup_{x,y\in C}(\|T_i^n x - T_i^n y\| - \|x-y\|), 0\}\forall n \geq 1,$$

where $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence as defined in (7) with $\limsup_{n\to\infty} \alpha_n < 1$ and $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ for all $n \geq 1$. If $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.

Proof: Let $p \in F$. Since $\{u_n\}, \{v_n\}$ are bounded sequences in C, let $M = \sup_{n\geq 1}\|u_n - p\| \vee \sup_{n\geq 1}\|v_n - p\|$. Obviously $M < \infty$. Now

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n-\delta_n)x_n + \beta_n T_{i(n)}^{k(n)} x_n + \delta_n v_n - p\| \\
&\leq (1-\beta_n-\delta_n)\|x_n - p\| + \beta_n\|T_{i(n)}^{k(n)} x_n - p\| + \delta_n\|v_n - p\| \\
&\leq (1-\beta_n-\delta_n)\|x_n - p\| + \beta_n\|x_n - p\| + \beta_n d_{k(n)} + \delta_n M \\
&\leq \|x_n - p\| + d_{k(n)} + \delta_n M, \qquad (8)
\end{aligned}$$

$$\begin{aligned}
\|x_n - p\| &= \|(1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n - p\| \\
&\leq (1-\alpha_n-\gamma_n)\|x_{n-1} - p\| + \alpha_n\|T_{i(n)}^{k(n)} y_n - p\| + \gamma_n\|u_n - p\| \\
&\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n\|y_n - p\| + \alpha_n d_{k(n)} + \gamma_n M \\
&\leq (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n[\|x_n - p\| + d_{k(n)} + \delta_n M] + \\
&\quad \alpha_n d_{k(n)} + \gamma_n M \\
&= (1-\alpha_n)\|x_{n-1} - p\| + \alpha_n\|x_n - p\| + 2\alpha_n d_{k(n)} +
\end{aligned}$$

$$(\delta_n + \gamma_n)M,$$

which implies that

$$\|x_n - p\| \leq \|x_{n-1} - p\| + \frac{2\alpha_n}{1-\alpha_n} d_{k(n)} + \frac{M}{1-\alpha_n}(\delta_n + \gamma_n).$$

Since $\limsup_{n\to\infty} \alpha_n < 1$, there exists $\beta < 1$ such that $\alpha_n < \beta$ for big n. So from above it follows that

$$\begin{aligned} \|x_n - p\| &\leq \|x_{n-1} - p\| + \frac{2\beta}{1-\beta} d_{k(n)} + \frac{M}{1-\beta}(\delta_n + \gamma_n) \\ &= \|x_{n-1} - p\| + \sigma_n, \end{aligned} \tag{9}$$

where $\sigma_n = \frac{2\beta}{1-\beta} d_{k(n)} + \frac{M}{1-\beta}(\delta_n + \gamma_n)$. Now $\sum_{n=1}^{\infty} \sigma_n < \infty$. Hence by Lemma 1.1 we have $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$.

Theorem 2.1. *Let X be a Banach space, C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of N asymptotically nonexpansive self-mappings of C in the intermediate sense. Set*

$$d_n = max\{max_{1\leq i\leq N} sup_{x,y\in C}(\|T_i^n x - T_i^n y\| - \|x-y\|), 0\} \forall n \geq 1,$$

where $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence as defined in (7) *with* $\limsup_{n\to\infty} \alpha_n < 1$ *and $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ for all $n \geq 1$. If $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i : i \in I\}$ if and only if* $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof: The necessary part is trivial. We only prove sufficient part. From (9) we have

$$\|x_n - p\| \leq \|x_{n-1} - p\| + \sigma_n.$$

Taking infimum over all $p \in F$, we have

$$d(x_n, F) \leq d(x_{n-1}, F) + \sigma_n.$$

Hence by Lemma 1.1 we have $\lim_{n\to\infty} d(x_n, F)$ exists. Since $\liminf_{n\to\infty} d(x_n, F) = 0$, we get $\lim_{n\to\infty} d(x_n, F) = 0$. Now

$$\begin{aligned} \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + \sigma_{n+m} \\ &\leq \|x_{n+m-2} - p\| + \sigma_{n+m-1} + \sigma_{n+m} \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &\leq \|x_n - p\| + \sum_{k=n+1}^{n+m} \sigma_k. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\lim_{n\to\infty} d(x_n, F) = 0$, there exists $N_1 \in N$ such that for all $n \geq N_1$ we have $d(x_n, F) < \frac{\epsilon}{3}$ and $\sum_{n=N_1}^{\infty} \sigma_n < \frac{\epsilon}{6}$. Therefore there exists $q \in F$ such that $d(x_{N_1}, q) < \frac{\epsilon}{3}$. From above we get

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - q\| + \|x_n - q\|$$

$$< \|x_{N_1} - q\| + \sum_{k=N_1+1}^{n+m} \sigma_k + \|x_{N_1} - q\| + \sum_{k=N_1+1}^{n} \sigma_k$$
$$< \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} x_n = x^\star$. Since C is closed, so $x^\star \in C$. Since T_i's are uniformly continuous, so $F(T_i)$'s are closed for all $i \in I$ which in turn implies that F is closed. Now note that

$$|d(x^\star, F) - d(x_n, F)| \leq \|x^\star - x_n\| \to 0 \text{ for all } n. \quad (10)$$

Since $\lim_{n\to\infty} x_n = x^\star$ and $\lim_{n\to\infty} d(x_n, F) = 0$, it follows from above that $d(x^\star, F) = 0$, that is $x^\star \in F$. Thus $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_N$.

Lemma 2.2. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of N asymptotically nonexpansive self-mappings of C in the intermediate sense. Put*

$$d_n = max\{max_{1\leq i\leq N} sup_{x,y\in C}(\|T_i^n x - T_i^n y\| - \|x - y\|), 0\}\forall n \geq 1,$$

where $\sum_{n=1}^{\infty} d_n < \infty$. Suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence as defined in (7) with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\limsup_{n\to\infty} \beta_n < 1$ and $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ for all $n \geq 1$. Then $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ for all $l \in I$.

Proof: Let $p \in F$. Then by Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$. Let $\lim_{n\to\infty} \|x_n - p\| = d$, for some $d \geq 0$. So $\{x_n\}$ is bounded. Since $\{u_n\}, \{v_n\}$ are bounded so $\{u_n - x_{n-1}\}, \{v_n - x_{n-1}\}$ are also bounded. Now

$$\begin{aligned}\|x_n - p\| &= \|(1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n - p\| \\ &= \|(1 - \alpha_n)(x_{n-1} - p + \gamma_n(u_n - x_{n-1})) + \alpha_n(T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1}))\|\end{aligned}$$

and

$$\|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\| \leq \|x_{n-1} - p\| + \gamma_n\|u_n - x_{n-1}\|. \quad (11)$$

Taking limsup on the both sides of (11) we get

$$\limsup_{n\to\infty} \|x_{n-1} - p + \gamma_n(u_n - x_{n-1})\| \leq \limsup_{n\to\infty} \|x_{n-1} - p\| + \limsup_{n\to\infty} \gamma_n\|u_n - x_{n-1}\| = d. \quad (12)$$

Again

$$\begin{aligned}&\|T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1})\| \\ \leq\ &\|T_{i(n)}^{k(n)} y_n - p\| + \gamma_n\|u_n - x_{n-1}\| \leq \|y_n - p\| + d_{k(n)} + \gamma_n\|u_n - x_{n-1}\|. \quad (13)\end{aligned}$$

Now from (8) we get

$$\|y_n - p\| \le \|x_n - p\| + d_{k(n)} + \delta_n M. \tag{14}$$

Taking limsup on the both sides of (14) we get

$$\limsup_{n\to\infty} \|y_n - p\| \le d. \tag{15}$$

Thus from (13) and (15) we get

$$\limsup_{n\to\infty} \|T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1})\| \le d. \tag{16}$$

Now

$$\begin{aligned} d &= \lim_{n\to\infty} \|x_n - p\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)(x_{n-1} - p + \gamma_n(u_n - x_{n-1})) \\ &+ \alpha_n(T_{i(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1}))\|. \end{aligned} \tag{17}$$

As $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, there exist $a, b \in (0,1)$ such that $0 < a \le \alpha_n \le b < 1$ for big n. Therefore by using Lemma 1.3 and (12), (16) and (17) we get

$$\lim_{n\to\infty} \|T_{i(n)}^{k(n)} y_n - x_{n-1}\| = 0. \tag{18}$$

Again from (7) and (18) it follows that

$$\begin{aligned} \|x_n - x_{n-1}\| &= \|(1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{k(n)} y_n + \gamma_n u_n - x_{n-1}\| \\ &\le \alpha_n \|T_{i(n)}^{k(n)} y_n - x_{n-1}\| + \gamma_n \|u_n - x_{n-1}\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{19}$$

So we have

$$\lim_{n\to\infty} \|x_n - x_{n+l}\| = 0 \text{ for all } l \in I. \tag{20}$$

Since

$$\|x_n - T_{i(n)}^{k(n)} y_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - T_{i(n)}^{k(n)} y_n\|,$$

by (18) and (19) we have

$$\lim_{n\to\infty} \|x_n - T_{i(n)}^{k(n)} y_n\| = 0. \tag{21}$$

Now

$$\|y_n - x_n\| = \|(1-\beta_n-\delta_n)x_n + \beta_n T_{i(n)}^{k(n)} x_n + \delta_n v_n - x_n\|$$

$$\begin{aligned}
&\leq \beta_n\|T_{i(n)}^{k(n)}x_n - x_n\| + \delta_n\|v_n - x_n\| \\
&\leq \beta_n(\|T_{i(n)}^{k(n)}x_n - T_{i(n)}^{k(n)}y_n\| + \|T_{i(n)}^{k(n)}y_n - x_{n-1}\| + \|x_{n-1} - x_n\|) \\
&\quad +\delta_n\|v_n - x_n\| \\
&\leq \beta_n(\|x_n - y_n\| + d_{k(n)} + \|T_{i(n)}^{k(n)}y_n - x_{n-1}\| + \|x_{n-1} - x_n\|) \\
&\quad +\delta_n\|v_n - x_n\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y_n - x_n\| &\leq \frac{\beta_n}{1-\beta_n}\|x_n - x_{n-1}\| + \frac{\beta_n}{1-\beta_n}d_{k(n)} + \frac{\beta_n}{1-\beta_n}\|T_{i(n)}^{k(n)}y_n - x_{n-1}\| \\
&\quad + \frac{\delta_n}{1-\beta_n}\|v_n - x_{n-1}\|. \qquad (22)
\end{aligned}$$

As $\limsup_{n\to\infty}\beta_n < 1$, there exists $\beta < 1$ such that $\beta_n < \beta$ for big n. So from (22) and by using (19), (18) we get

$$\|y_n - x_n\| \leq \frac{\beta}{1-\beta}\|x_n - x_{n-1}\| + \frac{\beta}{1-\beta}d_{k(n)} +$$

$$+\frac{\beta}{1-\beta}\|T_{i(n)}^{k(n)}y_n - x_{n-1}\| + \frac{\delta_n}{1-\beta}\|v_n - x_{n-1}\| \quad \to \quad 0 \text{ as } n \to \infty. \qquad (23)$$

Now

$$\|x_{n-1} - T_n x_n\| \leq \|x_{n-1} - T_{i(n)}^{k(n)}y_n\| + \|T_{i(n)}^{k(n)}y_n - T_n x_n\| = \sigma_n + \|T_{i(n)}^{k(n)}y_n - T_n x_n\|, \quad (24)$$

where $\sigma_n = \|x_{n-1} - T_{i(n)}^{k(n)}y_n\|$. From (18) we have $\sigma_n \to 0$ as $n \to \infty$. Since for each $n > N, n = (n-N)(\bmod N)$ and $n = (k(n)-1)N + i(n), i(n) \in \{1, 2..., N\}$, we have $k(n-N) = k(n) - 1$ and $i(n-N) = i(n)$. Then

$$\begin{aligned}
\|T_{i(n)}^{k(n)-1}y_n - x_{n-1}\| &\leq \|T_{i(n)}^{k(n)-1}y_n - T_{i(n-N)}^{k(n)-1}x_{n-N}\| + \|T_{i(n-N)}^{k(n)-1}x_{n-N} - T_{i(n-N)}^{k(n)-1}y_{n-N}\| \\
&\quad +\|T_{i(n-N)}^{k(n)-1}y_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_{n-1}\| \\
&\leq \|T_{i(n-N)}^{k(n-N)}y_n - T_{i(n-N)}^{k(n-N)}x_{n-N}\| + \|T_{i(n-N)}^{k(n-N)}x_{n-N} - T_{i(n-N)}^{k(n-N)}y_{n-N}\| \\
&\quad +\|T_{i(n-N)}^{k(n-N)}y_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_{n-1}\| \\
&\leq \|y_n - x_{n-N}\| + d_{k(n-N)} + \|x_{n-N} - y_{n-N}\| + d_{k(n-N)} + \sigma_{n-N} \\
&\quad +\|x_{(n-N)-1} - x_{n-1}\| \\
&\leq \|y_n - x_n\| + \|x_n - x_{n-N}\| + \|x_{n-N} - y_{n-N}\| + 2d_{k(n-N)} + \sigma_{n-N} \\
&\quad +\|x_{(n-N)-1} - x_n\| + \|x_n - x_{n-1}\| \to 0 \text{ as } n \to \infty. \qquad (25)
\end{aligned}$$

Since every T_i is uniformly continuous, it follows from (25) that

$$\lim_{n\to\infty}\|T_{i(n)}^{k(n)}y_n - T_n x_{n-1}\| = 0. \qquad (26)$$

Again by uniform continuity of the mappings and by (19) and (26) it follows that

$$\|T_{i(n)}^{k(n)}y_n - T_nx_n\| \leq \|T_{i(n)}^{k(n)}y_n - T_nx_{n-1}\| + \|T_nx_{n-1} - T_nx_n\| \to 0 \text{ as } n \to \infty. \tag{27}$$

From (24) and (27) it follows that

$$\lim_{n\to\infty} \|x_{n-1} - T_nx_n\| = 0. \tag{28}$$

From (19) and (28) we get

$$\|x_n - T_nx_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_nx_n\| \to 0 \text{ as } n \to \infty. \tag{29}$$

Now for all $l \in I$

$$\|x_n - T_{n+l}x_n\| \leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\|. \tag{30}$$

So by (30), (29) and (20) and uniform continuity of the mappings, it follows that $\lim_{n\to\infty} \|x_n - T_{n+l}x_n\| = 0$, for all $l \in I$. Consequently we have

$$\lim_{n\to\infty} \|x_n - T_lx_n\| = 0, \text{ for } l \in I. \tag{31}$$

This completes the proof of the Lemma.

Theorem 2.2. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of N asymptotically nonexpansive self-mappings of C in the intermediate sense. Set*

$$d_n = max\{max_{1\leq i\leq N} sup_{x,y\in C}(\|T_i^n x - T_i^n y\| - \|x - y\|), 0\} \forall n \geq 1,$$

where $\sum_{n=1}^{\infty} d_n < \infty$. Suppose that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by (7) with $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\limsup_{n\to\infty} \beta_n < 1$ and $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ for all $n \geq 1$. If $\{T_i : i \in I\}$ satisfies Condition($\overline{B}$), then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_N$.

Proof: By Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$. Let $\lim_{n\to\infty} \|x_n - p\| = d$, for some $d \geq 0$. If $d = 0$ then there is nothing to prove. Let $d > 0$. Now by Lemma 2.2 we get $\lim_{n\to\infty} \|x_n - T_lx_n\| = 0$, for all $l \in I$. As in the proof of Theorem2.1, we have that $\lim_{n\to\infty} d(x_n, F)$ exists. Again as $\{T_i : i \in I\}$ satisfies Condition($\overline{B}$), we have that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $f(0) = 0$ and $\lim_{n\to\infty} d(x_n, F)$ exists, we have $\lim_{n\to\infty} d(x_n, F) = 0$. Then the theorem follows from Theorem 2.1.

Theorem 2.3. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X. Let $\{T_i : i \in I\}$ be a finite family of N asymptotically nonexpansive self-mappings of C in the intermediate sense. Set*

$$d_n = max\{max_{1\leq i\leq N} sup_{x,y\in C}(\|T_i^n x - T_i^n y\| - \|x - y\|), 0\} \forall n \geq 1,$$

where $\sum_{n=1}^{\infty} d_n < \infty$. *Suppose that* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be defined by* (7) *with* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\limsup_{n\to\infty} \beta_n < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ *for all* $n \geq 1$. *If any one of the mappings* $\{T_1, T_2, ..., T_N\}$ *is semicompact, then* $\{x_n\}$ *converges strongly to a common fixed point of* $T_1, T_2, ..., T_N$.

Proof: By hypothesis, there exists one mapping, say T_1, of $\{T_1, T_2, ..., T_N\}$ which is semicompact. Now by Lemma 2.2 we have $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$, for all $l \in I$. Therefore $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$, and since T_1 is semicompact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^\star \in C$. From (31) we get

$$\|x^\star - T_l x^\star\| = \lim_{n_j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0, \text{ for all } l \in \{1, 2, .., N\}. \tag{32}$$

From (32) it follows that $x^\star \in F$. By Lemma 2.1 $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$. Since $x^\star \in F$, so $\lim_{n\to\infty} \|x_n - x^\star\|$ exists. Again since $\{x_{n_j}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_j} \to x^\star$, so it follows that $x_n \to x^\star$ as $n \to \infty$. Thus $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_N$.

Theorem 2.4. *Let* X *be a uniformly convex Banach space satisfying Opial's condition,* C *be a nonempty closed convex subset of* X. *Let* $\{T_i : i \in I\}$ *be a finite family of* N *asymptotically nonexpansive self-mappings of* C *in the intermediate sense. Set*

$$d_n = max\{max_{1\leq i\leq N} sup_{x,y\in C}(\|T_i^n x - T_i^n y\| - \|x - y\|), 0\} \forall n \geq 1,$$

where $\sum_{n=1}^{\infty} d_n < \infty$. *Suppose that* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be defined by* (7) *with* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\limsup_{n\to\infty} \beta_n < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ *for all* $n \geq 1$. *Then* $\{x_n\}$ *converges weakly to a common fixed point of* $T_1, T_2, ..., T_N$.

Proof: By Lemma 2.2 we get $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$, for all $l \in I$. So by the uniform continuity of T_1 we get $\lim_{n\to\infty} \|x_n - T_1^m x_n\| = 0$ for all $m \in N$. Then by applying Lemma 1.2 with the τ-topology taken as a weak topology we get the following conclusion: By Lemma 1.2 there exists $z_1 \in F(T_1)$ such that $x_n \rightharpoonup z_1 (x_n \to z_1$ weakly) as $n \to \infty$. Similarly by Lemma 1.2 there exists $z_2 \in F(T_2)$ such that $x_n \rightharpoonup z_2$ as $n \to \infty$ and $z_3 \in F(T_3)$ such that $x_n \rightharpoonup z_3$ as $n \to \infty$ and ..... $z_N \in F(T_N)$ such that $x_n \rightharpoonup z_N$ as $n \to \infty$. Since weak limit is unique so we must have $z_1 = z_2 = z_3 = ... = z_N \in \bigcap_{i=1}^{N} F(T_i) = F$. Thus $\{x_n\}$ converges weakly to a common fixed point of $T_1, T_2, ..., T_N$. This completes the proof.

Theorem 2.5. *Let* X *be a uniformly convex Banach space such that* $X^\star$ *has the Kadec-Klee property and* C *be a nonempty closed convex subset of* X. *Let* $\{T_i : i \in I\}$ *be a finite family of* N *asymptotically nonexpansive self-mappings of* C *in the intermediate sense. Set*

$$d_n = max\{max_{1\leq i\leq N} sup_{x,y\in C}(\|T_i^n x - T_i^n y\| - \|x - y\|), 0\} \forall n \geq 1,$$

where $\sum_{n=1}^{\infty} d_n < \infty$. *Suppose that* $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Let* $\{x_n\}$ *be defined by* (7) *with* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, $\limsup_{n\to\infty} \beta_n < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty, \sum_{n=1}^{\infty} \delta_n < \infty$ *for all* $n \geq 1$. *If* $\{T_1, T_2, ..., T_N\}$ *satisfy Condition* $(\overline{B})$, *then* $\{x_n\}$ *converges weakly to some common fixed point of* $\{T_1, T_2, ..., T_N\}$.

Proof: By Lemma 2.2 we get $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, for all $i \in I$. So by the uniform continuity of T_i we get

$$\lim_{n\to\infty} \|x_n - T_i^m x_n\| = 0 \text{ for any } m \geq 1. \tag{33}$$

Since $\{T_1, T_2, ..., T_N\}$ satisfy Condition $(\overline{B})$, so as in the proof of Theorem 2.2 it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Then, as shown in the proof of Theorem 2.1, it follows that $\{x_n\}$ is a Cauchy sequence. So for any $m \in N$ we have

$$\|x_{n+m} - x_n\| \to 0 \text{ as } n \to \infty. \tag{34}$$

From (33) and (34) we get

$$\|x_{n+m} - T_i^m x_n\| \to 0 \text{ as } n \to \infty,$$

which in other words implies that

$$\lim_{n\to\infty} [\sup_{m\geq 0} \|x_{n+m} - T_i^m x_n\|] = 0. \tag{35}$$

So from (35) it follows that $\{x_n\}$ is almost orbit of T_i for all $i \in I$. Also from (19) we have that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. So $\{x_{n+1} - x_n\}$ is strongly convergent to 0. Therefore $\{x_{n+1} - x_n\}$ is weakly convergent to 0. Thus by Lemma 1.4 we conclude that $\{x_n\}$ is weakly convergent to a fixed point of T_i. Since weak limit is unique so we must have that $\{x_n\}$ converges weakly to a common fixed point of $\{T_1, T_2, ..., T_N\}$. This completes the proof.

Remark 2.1. *Our results generalize results of [5], Theorem 3.4 of [16].*

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