Azerbaijan Journal of Mathematics
V. 4, No 2, 2014, July
ISSN 2218-6816

A-Statistical Supremum-Infimum and A-Statistical Convergence

M. Altınok, M. Küçükaslan

Abstract. In this paper, the concept of A-statistical supremum ($\sup_A x$) and A-statistical infumum ($\inf_A x$) for real valued sequences $x = (x_n)$ are defined and studied. It is mainly shown that, the equality of $\sup_A x$ and $\inf_A x$ is necessary but not sufficient for to existence of usual limit of the sequence. On the other hand, the equality of $\sup_A x$ and $\inf_A x$ is necessary and sufficient for to existence of A-statistical limit of the real valued sequences.

Key Words and Phrases: statistical supremum, statistical infimum, statistical convergence, upper (lower) peak point

2010 Mathematics Subject Classifications: 40A05, 40C05

The concept of statistical convergence is introduced by Fast and Steinhaus in [7] and [14], respectively. The idea of this concept based on asymptotic density of the subset K of natural numbers $\mathbb{N}$ (see [3]).

Over the years, by using asymptotic density some concepts in mathematics are generalized.

Let K be a subset of natural numbers $\mathbb{N}$ and

$$K(n) := \{k \ : \ k \leq n, \ k \in \mathbb{K}\}.$$

Then, the asymptotic density of $K \subseteq \mathbb{N}$ is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n}|K(n)|$$

if the limit exists. The symbol $|K(n)|$ indicates the cardinality of the set $K(n)$.

A real or complex valued sequence $x = (x_n)$ is said to be statistically convergent to the number L, if for every $\varepsilon > 0$, the set

$$K(n,\varepsilon) := \{k \ : \ k \leq n, \ |x_k - L| \geq \varepsilon\}$$

has zero asymptotic density, i. e.,

$$\delta(K(n,\varepsilon)) := \lim_{n\to\infty} \frac{1}{n}|K(n,\varepsilon)| = 0,$$

and it is denoted by $x_n \to L(S)$.

Let $A = (a_{n,k})$ be a non-negative matrix. If $A = (a_{n,k})$ transforms all convergent sequences to convergent sequences with the same limit, then it is called regular matrix transformation. The theorem in (1.3.3 Theorem [13]) gives the conditions for a matrix to be regular:

(a) There exists a constant K such that $\sum_{k=1}^{\infty} |a_{n,k}| < K$ for all n,

(b) For every k, $\lim_{n\to\infty} a_{nk} = 0$,

(c) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$.

A-density of the set $K \subset \mathbb{N}$ is defined by

$$\delta_A(K) := \lim_{n\to\infty} \sum_{k\in K} a_{n,k}$$

if the limit exists.

The sequence $x = (x_n)$ is A-statistical convergent to $L \in \mathbb{R}$, if for every $\varepsilon > 0$ the set $K(n,\varepsilon) := \{k \ : \ k \le n, \ |x_k - L| \ge \varepsilon\}$ has A-density zero. It is denoted by $x_n \to L(A-st)$.

By using A-density, matrix characterization of statistical convergence has been given in [8]. After this study, some concepts in classical analysis has been generalized [1, 2, 4, 5, 6, 9, 10, 12], etc.

In this study, A-statistical lower and upper bound of real valued sequences will be defined. By using this concept, A-statistical supremum and A-statistical infimum will be investigated and mainly their relations between A-statistical convergence will be given.

Definition 1. *(A-Statistical Lower Bound) The point $l \in \mathbb{R}$ is called A- statistical lower bound of the sequence $x = (x_n)$, if the following*

$$\delta_A(\{k \ : \ x_k \ge l\}) = 1 \quad (or \quad \delta_A(\{k \ : \ x_k < l\}) = 0) \tag{1}$$

hold.

The set of A-statistical lower bound of the sequence $x = (x_n)$ is denoted by $L_A(x)$:

$$L_A(x) := \{l \in \mathbb{R} \ : \ l \ satisfies \ (1)\}$$

Let us denote the set of usual lower bound of the sequence $x = (x_n)$ by $L(x)$:

$$L(x) := \{l \in \mathbb{R} \ : \ l \le x_n \ for \ all \ n \in \mathbb{N}\}.$$

From the above definition we have following simple result:

Theorem 1. *If $l \in \mathbb{R}$ is an usual lower bound of the sequence $x = (x_n)$, then $l \in \mathbb{R}$ is an A-statistical lower bound of the sequence $x = (x_n)$.*

Proof. Let us assume $l \in \mathbb{R}$ is a lower bound of the sequence $x = (x_n)$. From the definition of usual lower bound we have $l \leq x_n$ for all $n \in \mathbb{N}$. So, the set

$$\{k \ : \ x_k \geq l\} = \mathbb{N}.$$

Therefore,

$$\delta_A(\{k \ : \ x_k \geq l\}) = 1$$

holds. This shows that every usual lower bound is an A-statistical lower bound, i. e, $(L(x) \subseteq L_A(x))$. ◄

Remark 1. *The inverse of Theorem 1 is not true in general.*

Let us consider the sequence $x = (x_n) = (-\frac{1}{n})$ and $l = -\frac{1}{2} \in \mathbb{R}$. If we choose a regular matrix as

$$a_{nk} = \begin{cases} \frac{2k}{n^2}, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

It is clear that $l = -\frac{1}{2}$ is an A-statistical lower bound but it is not usual lower bound for the sequence $x = (x_n) = (-\frac{1}{n})$.

Definition 2. *(A-statistical Upper Bound) The point $m \in \mathbb{R}$ is called A-statistical upper bound of the sequence $x = (x_n)$, if the following*

$$\delta_A(\{k \ : \ x_k \leq m\}) = 1 \ (or \ \delta_A(\{k \ : \ x_k > m\}) = 0) \tag{2}$$

hold.

The set of A-statistical upper bound of the sequence $x = (x_n)$ is denoted by $U_A(x)$:

$$U_A(x) := \{m \in \mathbb{R} \ : \ m \ satisfies \ (2)\}.$$

Let us denote the set of usual upper bound of the sequence $x = (x_n)$ by $U(x)$:

$$U(x) := \{m \in \mathbb{R} \ : \ x_n \leq m \ for \ all \ n \ \in \mathbb{N}\}.$$

From the above definition we have following simple result:

Theorem 2. *If $m \in \mathbb{R}$ is an usual upper bound of the sequence $x = (x_n)$, then it is an A-statistical upper bound of the sequence $x = (x_n)$.*

Proof. Let us assume $m \in \mathbb{R}$ is an usual upper bound of the sequence $x = (x_n)$. From the definition of usual upper bound we have $x_n \leq m$ for all $n \in \mathbb{N}$. So, the set

$$\{k \ : \ x_k \leq m\} = \mathbb{N}.$$

Therefore,

$$\delta_A(\{k \ : \ x_k \leq m\}) = 1$$

holds. This shows that every usual upper bound is an A-statistical upper bound, i. e, $U(x) \subset U_A(x)$.◄

Remark 2. *The inverse of Theorem 2 is not true in general.*

Let us consider the sequence $x = (x_n) = (\frac{1}{n})$ and $l = \frac{1}{2} \in \mathbb{R}$. If we choose

$$a_{nk} = \begin{cases} \frac{2k-1}{n^2}, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

regular matrix. It is clear that $l = \frac{1}{2}$ is an A-statistical upper bound but it is not usual upper bound for the sequence $x = (x_n) = (\frac{1}{n})$.

Corollary 1. *If $l \in \mathbb{R}$ is an A-statistical lower bound of the sequence $x = (x_n)$ and $l' < l$, then $l' \in \mathbb{R}$ is also A-statistical lower bound of the sequence $x = (x_n)$.*

Proof. Assume that $l \in \mathbb{R}$ is an A-statistical lower bound of the sequence $x = (x_n)$. Then,

$$\delta_A(\{k \ : \ x_k \geq l\} = 1).$$

Since $l' < l$, then the inclusion

$$\{k \ : \ x_k \geq l\} \subset \{k \ : \ x_k \geq l'\}$$

holds. So, we have

$$1 \leq \delta_A(\{k \ : \ x_k \geq l'\}).$$

Therefore,

$$\delta_A(\{k \ : \ x_k \geq l'\}) = 1.$$

So, $l' \in \mathbb{R}$ is an A-statistical lower bound of the sequence $x = (x_n)$.◄

Corollary 2. *If $m \in \mathbb{R}$ is an A-statistical upper bound of the sequence $x = (x_n)$ and $m < m'$, then $m' \in \mathbb{R}$ is also A-statistical upper bound of the sequence $x = (x_n)$.*

Proof. Assume that $m \in \mathbb{R}$ is an A-statistical upper bound of the sequence $x = (x_n)$. Then,

$$\delta_A(\{k \ : \ x_k \leq m\} = 1).$$

Since $m < m'$, then the inclusion

$$\{k \ : \ x_k \leq m\} \subset \{k \ : \ x_k \leq m'\}$$

holds. So, we have

$$1 \leq \delta_A(\{k \ : \ x_k \leq m'\}).$$

Therefore,

$$\delta_A(\{k \ : \ x_k \leq m'\}) = 1.$$

So, $m' \in \mathbb{R}$ is an A-statistical upper bound of the sequence $x = (x_n)$.◄

Remark 3. *If the sequence $x = (x_n)$ has an A-statistical lower (upper) bound, then it has infinitely many A-statistical lower (upper) bounds.*

Definition 3. *(A-Statistical Infimum ($\inf_A$)) A number $s \in \mathbb{R}$ is called A-statistical infimum of the sequence $x = (x_n)$ if $s \in \mathbb{R}$ is supremum of $L_A(x)$. That is*

$$\inf_A x := \sup L_A(x).$$

Definition 4. *(A-Statistical Supremum ($\sup_A$)) A number $s' \in \mathbb{R}$ is called A-statistical supremum of the sequence $x = (x_n)$ if $s' \in \mathbb{R}$ is infimum of $U_A(x)$. That is*

$$\sup_A x := \inf U_A(x).$$

Theorem 3. *Let $x = (x_n)$ be a sequence of real numbers. Then,*

$$\inf x_n \leq \inf_A x_n \leq \sup_A x_n \leq \sup x_n$$

hold.

Proof. From the definition of usual infimum we have

$$\delta_A(\{k \;:\; \inf x_n \leq x_k\}) = \delta_A(\mathbb{N}) = 1.$$

This gives $\inf x_n \in L_A(x)$. Since $\inf_A x = \sup L_A(x)$, then $\inf_A x \geq \inf x_n$ hold.
From the definition of usual supremum we have

$$\delta_A(\{k \;:\; \sup x_n \geq x_k\}) = \delta_A(\mathbb{N}) = 1.$$

This gives $\sup x_n \in U_A(x)$. Since $\sup_A x_n = \inf U_A(x)$, then $\sup_A x \leq \sup x_n$ hold. For to completion of the proof it is enough to show that the inequality

$$l \leq m \tag{3}$$

holds for any $l \in L_A(x)$ and $m \in U_A(x)$.

Let us assume (3) is not true. That is there exist a $l' \in L_A(x)$ and $m' \in U_A(x)$ such that $m' < l'$ is satisfied. Since m' is an A-statistical upper bound, then from Corollary 1 (II) l' is also A-statistical upper bound of the sequence. This is the contradiction to the assumption of l'. Therefore, $l \leq m$ hold.◀

Remark 4. *Let $A = (a_{nk})$ be a non-negative regular matrix.*

a) If $x = (x_n)$ is a constant sequence then,

$$\inf x_n = \inf_A x_n = \sup_A x_n = \sup x_n.$$

b) If we consider the sequence $x = (x_n)$ as

$$x_n = \begin{cases} x_n, & n \leq n_0, n_0 \in \mathbb{N}\ fixed \\ a, & n > n_0 \end{cases}$$

such that $x_n \leq a$ *for all* $n \in \{1,2,3,...,n_0\}$*, then*

$$\inf x_n \leq \inf_A x_n \leq \sup_A x_n = \sup x_n.$$

c) If we consider the sequence $x = (x_n)$ *as*

$$x_n = \begin{cases} x_n, & n \leq n_0, n_0 \in \mathbb{N} \ fixed \\ a, & n > n_0 \end{cases}$$

such that $x_n \geq a$ *for all* $n \in \{1,2,3,...,n_0\}$*, then*

$$\inf x_n = \inf_A x_n \leq \sup_A x_n \leq \sup x_n.$$

Theorem 4. *Let* $x = (x_n)$ *be a real valued sequence and* $A = (a_{nk})$ *be a regular matrix. Then,*

$$\delta_A(\{k \ : \ x_k \notin [\inf_A x_n, \sup_A x_n]\}) = 0$$

and

$$\delta_A(\{k \ : \ x_k \in [\inf_A x_n, \sup_A x_n]\}) = 1$$

hold.

Proof. Let us assume for simplicity $\inf_A x_n = l$ and $\sup_A x_n = m$. That is $l = \sup L_A(x)$ and $m = \inf U_A(x)$. From the definition of infimum and supremum we have $l - \varepsilon \in L_A(x)$, $m + \varepsilon \in U_A(x)$ and

$$[l, m] \subset [l - \varepsilon, m + \varepsilon] \tag{4}$$

It is clear from (4) that we have

$$\begin{aligned} \delta_A(\{k \ : \ x_k \notin [l, m]\}) &\leq \delta_A(\{k \ : \ x_k \notin [l - \varepsilon, m + \varepsilon]\}) = \\ &= \delta_A(\{k \ : \ x_k < l - \varepsilon\}) + \delta_A(\{k \ : \ x_k > m + \varepsilon\}) \end{aligned} \tag{5}$$

Since $\delta_A(\{k \ : \ x_k < l - \varepsilon\}) = 0$ and $\delta_A(\{k \ : \ x_k > m + \varepsilon\}) = 0$, then from (5) we have

$$\delta_A(\{k \ : \ x_k \notin [\inf_A x_n, \sup_A x_n]\}) = 0.$$

It is clear that the following equality

$$\{k \ : \ x_k \in [\inf_A x_n, \sup_A x_n]\} = \mathbb{N} - \{k \ : \ x_k \notin [\inf_A x_n, \sup_A x_n]\}$$

hold and we have

$$\delta_A(\{k \ : \ x_k \in [\inf_A x_n, \sup_A x_n]\}) = \delta_A(\mathbb{N}) - \delta_A(\{k \ : \ x_k \notin [\inf_A x_n, \sup_A x_n]\}).$$

This gives the desired result.◄

Theorem 5. *If* $\lim_{n\to\infty} x_n = l$*, then* $\sup_A x_n = \inf_A x_n = l$*.*

Proof. Assume $\lim_{n\to\infty} x_n = l$, i.e,
For every $\varepsilon > 0$, there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - l| < \varepsilon, \tag{6}$$

hold for all $n \geq n_0$. Therefore, the following inclusion deduced from (6)

$$\mathbb{N} - \{1, 2, 3, ..., n_0\} \subset \{k \ : \ x_k \geq l - \varepsilon\}, \tag{7}$$

$$\mathbb{N} - \{1, 2, 3, ..., n_0\} \subset \{k \ : \ x_k \leq l + \varepsilon\}. \tag{8}$$

By using (7) and (8) we obtain

$$\delta_A(\{k \ : \ x_k \geq l - \varepsilon\}) = 1,$$

and

$$\delta_A(\{k \ : \ x_k \leq l + \varepsilon\}) = 1.$$

This discussion gives

$$l - \varepsilon \in L_A(x), \quad l + \varepsilon \in U_A(x)$$

for all $\varepsilon > 0$ such that

$$L_A(x) = (-\infty, l) \textit{ and } U_A(x) = (l, \infty).$$

Therefore,

$$\inf_A x_n = \sup(-\infty, l) = l = \inf(l, \infty) = \sup_A x_n$$

is obtained.◀

Remark 5. *Theorem 5 stays true when* $l = \mp\infty$*.*

Proof. We shall give the proof only for $l = +\infty$. Lets take arbitrary $M \in \mathbb{R}$. From the assumption, $\exists n_0 \equiv n_0(M) \in \mathbb{N}$ such that $x_n > M$ for all $n > n_0$. So,

$$\delta_A(\{k \ : \ x_k \geq M\}) \geq \delta_A(\mathbb{N} - \{1, 2, ..., n_0\}) = 1$$

and

$$\delta_A(\{k \ : \ x_k \leq M\}) \leq \delta_A(\{1, 2, ..., n_0\}) = 0.$$

Therefore, $M \in L_A(x)$, $M \notin U_A(x)$, i.e. $L_A(x) = (-\infty, +\infty)$ and $U_A(x) = \varnothing$.
Hence, $\inf_A x = \sup L_A(x) = \infty$, $\sup_A x = \inf U_A(x) = \inf_A \varnothing = \infty$.◀

The following Corollary is a simple consequence of Theorem 5. So, the proof is omitted here.

Corollary 3. *Let* $x = (x_n)$ *be a real valued sequence. The following statements are true.*
I) If the sequence $x = (x_n)$ *is monotone increasing, then* $\inf_A x_n = \sup x_n$*.*
II) If the sequence $x = (x_n)$ *is monotone decreasing, then* $\sup_A x_n = \inf x_n$*.*

Remark 6. *Let $A = (a_{nk})$ be a non-negative regular matrix.*

a) If $x = (x_n)$ is monotone increasing, then

$$\inf x_n \leq \inf_A x_n = \sup_A x_n = \sup x_n.$$

b) If $x = (x_n)$ is monotone decreasing, then

$$\inf x_n = \inf_A x_n = \sup_A x_n \leq \sup x_n.$$

Remark 7. *The inverse of Theorem 5 is not true.*

For to see this let us consider the sequence $x = (x_n)$ as

$$x_n = \begin{cases} 1, & n = k^2, k = 1, 2, ..., \\ 0, & otherwise \end{cases}$$

and the matrix $a = (a_{nk})$ as

$$a_{nk} = \begin{cases} \frac{2k}{n^2}, & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

It is clear that $\sup_A x_n = \inf_A x_n = 0$ but the sequence is not convergent to 0.

Theorem 6. $\lim_{n\to\infty} x_n = l(A - st)$ *if and only if* $\sup_A x_n = \inf_A x_n = l$.

Proof. " $\Rightarrow$ " Assume that $\lim_{n\to\infty} x_n = l(A - st)$. We have for every $\varepsilon > 0$,

$$\delta_A(\{k \ : \ k \leq n, \ |x_k - l| \geq \varepsilon\}) = 0 \tag{9}$$

hold. Since,

$$\{k \ : \ k \leq n, |x_k - l| \geq \varepsilon\} = \{k \ : \ k \leq n, x_k \geq l + \varepsilon\} \cup \{k \ : \ k \leq n, x_k \leq l - \varepsilon\}$$

and from (9) we have

$$\delta_A(\{k \ : \ x_k \geq l + \varepsilon\}) = 0 \tag{10}$$

and

$$\delta_A(\{k \ : \ x_k \leq l - \varepsilon\}) = 0. \tag{11}$$

Also, from (9) we have

$$\delta_A(\{k \ : \ k \leq n, \ |x_k - l| < \varepsilon\}) = 1 \tag{12}$$

and (12) gives

$$\delta_A(\{k \ : \ x_k < l + \varepsilon\}) = 1 \tag{13}$$

and

$$\delta_A(\{k \ : \ x_k > l - \varepsilon\}) = 1. \tag{14}$$

The equation (10) and (13) gives $l+\varepsilon$ is an A-statistical upper bound and (11) and (14) gives $l-\varepsilon$ is an A-statistical lower bound for the sequence.

So,

$$L_A(x) = (-\infty, l) \text{ and } U_A(x) = (l, \infty).$$

Therefore, we have

$$\sup L_A(x) = l, \ \inf U_A(x) = l.$$

" $\Leftarrow$ " Assume that

$$\sup_A x_n = \inf_A x_n = l.$$

That is

$$l = \sup L_A(x) = \inf U_A(x).$$

From the definition of supremum and infimum, there exists at least one element $l' \in L_A(x)$ and $l'' \in U_A(x)$ for all $\varepsilon > 0$ such that the inequality

$$l - \varepsilon < l' \text{ and } l'' < l + \varepsilon$$

hold. Since l' is an A-statistical lower bound then we have following inclusion

$$\{k \ : \ x_k \geq l + \varepsilon\} \subset \{k \ : \ x_k \geq l'\}.$$

So,

$$\delta_A(\{k \ : \ x_k \geq l + \varepsilon\}) = 0 \tag{15}$$

Since l'' is an A-statistical upper bound then we have following inclusion

$$\{k \ : \ x_k \leq l - \varepsilon\} \subset \{k \ : \ x_k \leq l''\}.$$

So,

$$\delta_A(\{k \ : \ x_k \leq l - \varepsilon\}) = 0. \tag{16}$$

From the equations (15), (16) and

$$\{k \ : \ |x_k - l| \geq \varepsilon\} = \{k \ : \ x_k \geq l + \varepsilon\} \cup \{k \ : \ x_k \leq l - \varepsilon\},$$

we have

$$\delta_A(\{k \ : \ |x_k - l| \geq \varepsilon\}) = 0.$$

Therefore, the sequence $x = (x_n)$ is A-statistical convergent to $l \in \mathbb{R}$.◄

Definition 5. *The real valued sequences $x = (x_n)$ and $y = (y_n)$ are called A-statistical equivalent if the A-density of the set $H = \{k : x_k \neq y_k\}$ is zero. It is denoted by $x \asymp y$.*

Theorem 7. *If the sequence $x = (x_n)$ and $y = (y_n)$ are equivalent, then*

$$\inf_A x_n = \inf_A y_n \text{ and } \sup_A x_n = \sup_A y_n.$$

Proof. Since the sequence $x = (x_n)$ and $y = (y_n)$ are equivalent, then the set $H = \{k : x_k \neq y_k\}$ has zero A-density. Let us consider an arbitrary element $l \in L_A(x)$. The element $l \in \mathbb{R}$ is an A-statistical lower bound of the sequence $x = (x_n)$, then we have

$$\delta_A(\{k : x_k < l\}) = 0 \text{ and } \delta_A(\{k : x_k \geq l\}) = 1.$$

From the following inclusion

$$\{k : y_k < l\} = \{k : x_k \neq y_k < l\} \cup \{k : x_k = y_k < l\} \subset H \cup \{k : x_k = y_k < l\}$$

we have

$$\begin{aligned} 0 \leq \delta_A(\{k : y_k < l\}) &= \delta_A(\{k : x_k \neq y_k < l\}) + \delta_A(\{k : x_k = y_k < l\}) \\ &\leq \delta_A(H) + \delta_A(\{k : x_k = y_k < l\}) = 0 + 0 = 0. \end{aligned} \tag{17}$$

Since the inclusion

$$\begin{aligned} \{k : y_k \geq l\} &= \{k : x_k \neq y_k \geq l\} \cup \{k : x_k = y_k \geq l\} \\ &\supset \{k : x_k = y_k < l\} \end{aligned}$$

then we have

$$1 = \delta_A(\{k : y_k \geq l\}) \geq \delta_A(\{k : x_k = y_k \geq l\}) = 1. \tag{18}$$

From (17) and (18), the element $l \in \mathbb{R}$ is an A-statistical lower bound of the sequence $y = (y_n)$. That is, $L_A(x) \subset L_A(y)$.

If we consider an arbitrary element $l \in L_A(y)$, it can be easily obtained that $l \in L_A(x)$. Then $L_A(y) \subset L_A(x)$. Therefore,

$$L_A(y) = L_A(x)$$

hold. Since $\sup L_A(y) = \sup L_A(x)$, then $\inf_A x = \inf_A y$ is obtained.

By using the same argument as above it can be obtained $\sup_A x = \sup_A y$.◄

Definition 6. *(Upper or Lower Peak Point) [11] The point x_k is called upper(or lower) peak point of the sequence $x = (x_n)$ if the inequality $x_k \geq x_l$ (or $x_k \leq x_l$) holds for all $l \geq k$.*

Theorem 8. *Let $x = (x_n)$ be a real valued sequence. If the element x_{n_0} is an upper(or lower) peak point of (x_n), then the element x_{n_0} is an A-statistical upper (or A-statistical lower) bound.*

Proof. Assume the point x_{n_0} is an upper peak point of the sequence $x = (x_n)$ that $x_k \leq x_{n_0}$ holds for all $k \geq n_0$. So, the following inclusion

$$\{k : x_k \leq x_{n_0}\} \supset \mathbb{N} - \{1, 2, ..., n_0\}$$

holds. From this inclusion and the properties of asymptotic density we have

$$1 \leq \delta_A(\{k : x_k \leq x_{n_0}\}) = 1.$$

This give us the point x_{n_0} is an A-statistical upper bound of the sequence $x = (x_n)$.◄

Theorem 9. *Let $x = (x_n)$ be a real valued sequence and $A = (a_{nk})$, $B = (b_{nk})$ be regular matrix. If the condition*

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| = 0$$

hold. Then

$$\inf_A x_n = \inf_B x_n \text{ and } \sup_A x_n = \sup_B x_n.$$

Proof. Let $K_1 = \{k : x_k < m\}$, $K_2 = \{k : x_k \geq m\}$ be subsets of natural numbers $\mathbb{N}$ for all $m \in \mathbb{R}$. For $K = K_1$(or K_2)

$$\begin{aligned} |\delta_A(K) - \delta_B(K)| &= |\lim_{n\to\infty} \sum_{k\in K} a_{nk} - \sum_{k\in K} b_{nk}| \\ &\leq \lim_{n\to\infty} \sum_{k\in K} |a_{nk} - b_{nk}| \\ &\leq \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| \end{aligned}$$

hold. Namely $\delta_A(K) = \delta_B(K)$. So $\inf_A x_n = \inf_B x_n$.

Remark 8. *The inverse of Theorem 9 is not true.*

For to see this let us consider the sequence $x = (x_n)$ where

$$x_n = \begin{cases} 1, & n = k^2, k = 1, 2, ..., \\ 0, & otherwise, \end{cases}$$

and the matrices $A = (a_{n,k})$ and $B = (b_{n,k})$ as

$$a_{nk} = \begin{cases} \frac{n}{3(n+1)}, & k = n^2, \\ 1 - \frac{n}{3(n+1)}, & k = n^2 + 1, \\ 0, & otherwise, \end{cases}$$

and

$$b_{nk} = \begin{cases} \frac{n}{5(n+1)}, & k = n^2, \\ 1 - \frac{n}{5(n+1)}, & k = n^2 + 1, \\ 0, & otherwise. \end{cases}$$

The matrices A, B are non-negative and regular. This sequence and matrices A and B has been considered in [4]. It is clear that $L_A(x) = (-\infty, 0]$, $L_B(x) = (-\infty, 0]$, $U_A(x) = (0, \infty)$ and $U_B(x) = (0, \infty)$. Therefore,

$$\sup L_A(x) = \sup L_B(x) = 0$$

and

$$\inf U_A(x) = \inf U_B(x) = 0.$$

That is, $\sup_A x = \sup_B x$, $\inf_A x = \inf_B x$ hold. Unfortunately, the condition given theorem doesn't hold for the matrices A and B. The other case is obtained by similar way.◄

References

[1] O I Agratini. A-statistical convergence of a class of integral operators. *Appl. Math. Inf. Sci.*, 6(2): 325-328, 2012.

[2] M Altınok and M Küçükaslan. A-statistical convergence and A-statistical monotonicity. *Applied Mathematics E-Notes*, 13: 249-260, 2013.

[3] R C Buck. The measure theoretic approch to density. *Amer. J. Math.*, 68:560-580, 1946.

[4] K Demirci. On A-statistical cluster points. *Glasnic Matematicki*, 37:293-301, 2001.

[5] O Duman. A-statistical convergence of sequences of convolution operators. *Taiwanese Journal of Mathematics*, 12(2):523-536, 2008.

[6] O Duman and M K Khan. A-statistical convergence of approximating operators. *Mathematical Inequalities and applications*, 6(4):689-699, 2003.

[7] H Fast. Sur la convergece statistique. *Colloq. Math.*, 2:241-244, 1951.

[8] J Fridy and H I Miller. A matrix characterization of statistical convergence. *Analysis*,11:59-66, 1991.

[9] J Fridy and C Orhan. Lacunary statistical summability. *J. Math. Anal. Appl.*, 173:497-504, 1993.

[10] M Küçükaslan and M Yılmaztürk. Deferred statistical convergence. *Kyunpook Mathematical Journal*, 2013, (Submitted).

[11] E Kaya, M Küçükaslan and R Wagner. On Statistical Convergence and Statistical Monotonicity. *Annales Univ. Sci. Budapest. Sect. Comp.*, 39:257-270, 2013.

[12] M Mursaleen. λ-statistical convergence. *Mathematica Slovaca*, 50(1):111-115, 2000.

[13] G M Peterson. *Regular matrix transformation.* McGraw-hill Pub. Company limited London, 1966.

[14] H Steinhaus. Sur la convergence ordinaire et la convergence asymtotique. *Colloq. Math.*, 2:73-74, 1951.

Maya Altınok
Mersin University, Faculty of Sciences and Arts, Department of Mathematics, Mersin, Turkey
E-mail: mayaaltinok@mersin.edu.tr

Mehmet Küçükaslan
Mersin University, Faculty of Sciences and Arts, Department of Mathematics, Mersin, Turkey
E-mail: mkucukaslan@mersin.edu.tr, mkkaslan@gmail.com

Received 08 September 2013
Accepted 28 November 2013