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Inverse Sturm-Liouville Problem with Eigenparameter Dependent Boundary and Transmission Conditions

M. Shahriari

Abstract. This paper deals with the boundary value problem involving the differential equation

$$\ell y := -y'' + qy = \lambda y$$

subject to the eigenparameter dependent boundary conditions along with the following discontinuity conditions

$$y(d+0) = ay(d-0), \quad y'(d+0) = ay'(d-0) + by(d-0)$$

at a part $d \in (0, \pi)$, where $q(x)$, a, b are real, $q \in L^2(0, \pi)$ and λ is a parameter independent of x. We develop the Hochestadt's result based on transformation operator for inverse Sturm-Liouville problem with eigenparameter dependent boundary and discontinuous conditions. Furthermore, we establish a formula for $q(x) - \tilde{q}(x)$ in the finite interval, where $\tilde{q}(x)$ is an analogous function with $q(x)$.

Key Words and Phrases: inverse Sturm-Liouville problem, jump conditions, parameter dependent boundary condition, Green's function

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1. Introduction

We consider the boundary value problem

$$\ell y := -y'' + qy = \lambda y, \tag{1}$$

$$\begin{aligned} U(y) &:= y'(0) - hy(0) = 0, \\ V(y) &:= Hy'(\pi) + \lambda y(\pi) = 0 \end{aligned} \tag{2}$$

with jump conditions

$$U_1(y) := y(d+0) - ay(d-0) = 0, \; U_2(y) := y'(d+0) - ay'(d-0) - by(d-0) = 0, \tag{3}$$

where $q(x), h, H, a, b, d$ are real, $q \in L^2(0,\pi)$, $d \in (0,\pi)$, $0 < H < \infty$ and λ is the spectral parameter. We use the notation $L = L(q(x); h; H; d)$ for the problem (1)-(3).

The method of separation of variables for solving PDEs with discontinuous boundary conditions naturally leads to ODE with discontinuities inside the interval which often appears in mathematics. Inverse spectral problem consists in recovering operators from their spectral characteristics. For example, the mathematical formulation of a large variety of technical and physical problems leads to inverse problems such as identifying the density of the thing from data collected from the sets of frequencies of oscillations of the string with barrier.

The inverse spectral Sturm-Liouville problem can be regarded as having three aspects, e.g., existence, uniqueness and reconstruction of the coefficients given specific properties of eigenvalues and eigenfunctions [1]-[22]. In particular, the operator ℓ plays an important role of the one-dimensional Schrödinger operator in quantum mechanics and our transmission conditions include the case of point interactions (see e.g. the monographs [23] and [24]). Our work concerns uniqueness and other properties of potential function.

The applications of boundary value problems with discontinuity conditions inside the interval are connected with discontinuous material properties. Inverse problems with a discontinuity condition inside the interval play an important role in mathematics, mechanics, radio electronics, geophysics, and other fields of science and technology. As a rule, such problems are related to discontinuous and non-smooth properties of a medium (e.g., see [9]-[11] and [31]). In this work, we generalize the Hochstadt's result [6], refining the approach of Levinson [4] to show that precisely how much freedom q has where the λ_n and all but finitely many of the λ'_n are specified. Note that the eigenvalues λ'_n are obtained by replacing H with H_1 in (2). Nowadays there is a number of papers dedicated to inverse problems for the Sturm-Liouville operator with eigenparameter dependent boundary conditions (see [21] and [25]-[27]). There are many papers concerning problems with discontinuous conditions. One can find the similar works for discontinuous conditions in [11] and [15]-[18], and the similar works for Hochstadt's result in [12]-[14].

Remark 1. *The same result can be obtained by the same method in the more general case of the following eigenparameter dependent boundary conditions and jump conditions* (3)*:*

$$y'(0) - hy(0) = 0, \quad \lambda(y'(\pi) - H_1 y(\pi)) - H_2 y'(\pi) - H_3 y(\pi) = 0.$$

2. The Hilbert space formulation and asymptotic form of solutions and eigenvalues

In this section, we introduce the special inner product in the Hilbert space $(L_2(0,d) \oplus L_2(d,\pi)) \oplus \mathbb{C}$ and we define a linear operator A such that the considered problem (1)-(3) can be interpreted as the eigenvalue problem for A. So, we define a new Hilbert space inner product on $\mathcal{H} := (L_2(0,d) \oplus L_2(d,\pi)) \oplus \mathbb{C}$ by

$$\langle F, G \rangle_{\mathcal{H}} = |a| \int_0^d f\bar{g} + \frac{1}{|a|} \int_d^\pi f\bar{g} + \frac{1}{H|a|} R_1(f)\bar{R}_1(g), \tag{4}$$

where $F = \begin{pmatrix} f(x) \\ R_1(f) \end{pmatrix}$ and $G = \begin{pmatrix} g(x) \\ R_1(g) \end{pmatrix} \in \mathcal{H}$. In this Hilbert space we construct the operator

$$A : \mathcal{H} \to \mathcal{H} \tag{5}$$

by

$$F = \begin{pmatrix} f(x) \\ R_1(f) \end{pmatrix}, \quad \text{and} \quad AF = \begin{pmatrix} \ell f \\ -R_1'(f) \end{pmatrix}$$

with domain

$$D(A) = \left\{ \begin{array}{c} F = \begin{pmatrix} f(x) \\ R_1(f) \end{pmatrix} | f(x), f'(x) \in AC[0,d) \cup (d,\pi] \text{ and,} \\ f(d \pm 0),\ f'(d \pm 0) \text{ is defined, } \ell f \in L^2[[0,d) \cup (d,\pi]] \\ U(f) = U_1(f) = U_2(f) = 0,\ R_1(f) := f(\pi) \end{array} \right\}, \tag{6}$$

where $R_1'(f) := Hf'(\pi)$. Thus, we can rearrange the boundary value problem (1)-(3) as follows:

$$AY = \lambda Y \qquad Y := \begin{pmatrix} y(x) \\ R_1(y) \end{pmatrix} \in D(A). \tag{7}$$

It is easy to see that the eigenvalues of the operator A coincide with those of the problem (1)-(3). Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be solutions of (1) under the jump conditions (3) and initial conditions

$$\varphi(0,\lambda) = 1, \quad \varphi'(0,\lambda) = h, \tag{8}$$

and

$$\psi(\pi,\lambda) = H, \quad \psi'(\pi,\lambda) = -\lambda. \tag{9}$$

By attaching a subscript 1 or 2 to functions φ and ψ, we mean to refer to the first subinterval $[0,d)$ or to the second subinterval $(d,\pi]$. For example

$$\varphi(x,\lambda) = \begin{cases} \varphi_1(x,\lambda)\ ,\ x \in [0,d) \\ \varphi_2(x,\lambda)\ ,\ x \in (d,\pi]. \end{cases}$$

By virtue of [16], problem (1) has a unique solution $\varphi_1(x,\lambda)$ or $\psi_2(x,\lambda)$, an entire function of $\lambda \in \mathbb{C}$, under the initial conditions (8) or (9). We can obtain from the linear differential equations theory that each of the Wronskians

$$\Delta_1(\lambda) := W(\varphi_1(x,\lambda), \psi_1(x,\lambda)) \tag{10}$$

and

$$\Delta_2(\lambda) := W(\varphi_2(x,\lambda), \psi_2(x,\lambda)) \tag{11}$$

are independent of x for all $x \in [0,d) \cup (d,\pi]$, respectively. It is easy to see that the equality $\Delta_2(\lambda) = a^2 \Delta_1(\lambda)$ holds for each $\lambda \in \mathbb{C}$.

Corollary 1. *The zeros of* $\Delta(\lambda) := \Delta_2(\lambda) = a^2\Delta_1(\lambda)$ *coincide. The eigenvalues of the problem (1)-(3) coincide with the zeros of the function* $\Delta(\lambda)$.

Apply the definition of A to prove the following theorem. For other similar proofs we can refer to [28], [30] and [31].

Theorem 1. *(i) The operator* A *is self-adjoint.*

(ii) By self-adjointness of A *and Corollary 1, all eigenvalues of the problem (1)-(3) are real and simple.*

(iii) The sequence φ_n *of the orthonormal eigenfunction system (1)-(3) in* $L^2(0,\pi)$ *is complete.*

Theorem 2. *Let* $\lambda = \rho^2$, $\tau := Im\rho$. *For equation (1) with boundary conditions (2) and jump conditions (3) as* $|\lambda| \to \infty$, *the following asymptotic formulas hold:*

$$\varphi(x;\lambda) = \begin{cases} \cos\rho x + (h + \frac{1}{2}\int_0^x q(t)dt)\frac{\sin\rho x}{\rho} + O(\frac{\exp|\tau|x}{\rho^2}), & x < d, \\ \\ a\cos\rho x + f_1(x)\frac{\sin\rho x}{\rho} + f_2(x)\frac{\sin\rho(2d-x)}{\rho} + O(\frac{\exp|\tau|x}{\rho^2}), & x > d, \end{cases} \tag{12}$$

$$\varphi'(x;\lambda) = \begin{cases} -\rho\sin\rho x + (h + \frac{1}{2}\int_0^x q(t)dt)\cos\rho x + O(\frac{\exp|\tau|x}{\rho}) & x < d, \\ \\ -a\rho\sin\rho x + f_1(x)\cos\rho x - f_2(x)\cos\rho(2d-x) + O(\frac{\exp|\tau|x}{\rho}) & x > d, \end{cases} \tag{13}$$

and

$$\psi(x;\lambda) = \begin{cases} \frac{1}{a}\rho\sin\rho(\pi-x) + g_1(x)\cos\rho(\pi-x) + g_2(x)\cos\rho(2d+x-\pi) \\ \qquad +O(\frac{\exp|\tau|(\pi-x)}{\rho}), & x < d, \\ \\ \rho\sin\rho(\pi-x) + (H + \frac{1}{2}\int_{\pi-x}^{\pi} q(x)dx)\cos\rho(\pi-x) + O(\frac{\exp|\tau|(\pi-x)}{\rho}), & x > d, \end{cases} \tag{14}$$

$$\psi'(x;\lambda) = \begin{cases} -\frac{1}{a}\rho^2\cos\rho(\pi-x) + \rho(g_1(x)\sin\rho(\pi-x) - g_2(x)\sin\rho(2d+x-\pi)) \\ \qquad +O(\exp(|\tau|\pi-x)), & x < d, \\ \\ -\rho^2\cos\rho(\pi-x) + \rho(H + \frac{1}{2}\int_{\pi-x}^{\pi} q(x)dx)\sin\rho(\pi-x) + O(\exp|\tau|(\pi-x)), & x > d, \end{cases} \tag{15}$$

$$f_1(x) = a\left(h + \frac{1}{2}\int_0^x q(t)dt\right) + \frac{b}{2}, \quad f_2(x) = \frac{b}{2}.$$

$$g_1(x) = \frac{1}{a}\left(H + \frac{1}{2}\int_{\pi-x}^{\pi} q(t)dt\right) - \frac{b}{2a^2}, \quad g_2(x) = -\frac{b}{2a^2}.$$

Moreover, the characteristic function is

$$\Delta(\lambda) = a\rho^2\cos\rho\pi + \rho[(f_1(\pi) - aH)\sin\rho\pi + f_2(\pi)\sin\rho(2d-\pi)] + O(\exp|\tau|\pi). \tag{16}$$

Proof. Let $C(x,\lambda)$ and $S(x,\lambda)$ be solutions of (1) under the initial conditions

$$C(0,\lambda)=S'(0,\lambda)=1, \qquad C'(0,\lambda)=S(0,\lambda)=0,$$

and the jump conditions (3). It is obvious that $\varphi(x,\lambda)=C(x,\lambda)+hS(x,\lambda)$. By using the asymptotic formulas of $C(x,\lambda)$ and $S(x,\lambda)$ and applying the similar calculations of [16], we obtain the asymptotic form of $\varphi(x,\lambda)$ and $\varphi'(x,\lambda)$. By changing x to $\pi-x$, one can obtain the asymptotic form of $\psi(x,\lambda)$ and $\psi'(x,\lambda)$. ◄

Applying the similar calculations of [16], we find that

$$\rho_n = n-\frac{1}{2}+\frac{\theta_n}{n-\frac{1}{2}}+\frac{\kappa_n}{n}, \quad \kappa_n=o(1) \tag{17}$$

and

$$\theta_n=\frac{(-1)^{(n+1)}}{2}\left(\omega_1+\omega_2\cos 2d(n-\frac{1}{2})\right),$$

where

$$\omega_1=a\Big(H-h-\frac{1}{2}\int_0^\pi q(t)\,dt\Big)-\frac{b}{2}, \quad \omega_2=-\frac{b}{2}.$$

3. Main result

In this section the uniqueness theorem for (1)-(3) is given. We need some lemma and technical notation to prove our main result. The boundary value problem $L=L(q;h;H;d)$ is defined with the operator $A:\mathcal{H}\to\mathcal{H}$. We now consider boundary value problems $\tilde{L}=L(\tilde{q};h;H;d)$, $L_1=L(q;h;H_1;d)$, and $\tilde{L}_1=L(\tilde{q};h;H_1;d)$, by the same approach where $H_1\neq H$, with operators $\tilde{A}$, A_1, and $\tilde{A}_1$, respectively. Suppose that $\theta(x,\lambda)$ is the solution of (1) satisfying the initial conditions $\theta(\pi,\lambda)=H_1$, $\theta'(\pi,\lambda)=\lambda$ and the jump conditions (3). Define $\phi_i(\lambda):=W(\varphi_i(x,\lambda),\theta_i(x,\lambda))$, and $\tilde{\phi}_i(\lambda):=W(\tilde{\varphi}_i(x,\lambda),\tilde{\theta}_i(x,\lambda))$, for $i=1,2$.

Lemma 1. *If $L(q;h;H_1;d)$ and $L(\tilde{q};h;H_1;d)$ have the same eigenvalues, then $\phi_i(\lambda)=\tilde{\phi}_i(\lambda)$ for $i=1,2$.*

Proof. From [33] it follows that ϕ_i and $\tilde{\phi}_i$ are entire functions of order $\frac{1}{2}$, and consequently, by Hadamard's factorization theorem [34], are determined up to a multiplicative constant by their zeros. Hence there is a constant k such that $k=\frac{\phi_i(\lambda)}{\tilde{\phi}_i(\lambda)}$. Using the asymptotic form of $\phi_i(\lambda)$ and $\tilde{\phi}_i(\lambda)$ as a similar form of (16) with H replaced by H_1 we obtain $k=1+O(\frac{1}{\rho})$. Letting $\rho\to\infty$, we obtain $k=1$ and so $\phi_i(\lambda)=\tilde{\phi}_i(\lambda)$.◄

If $\psi_n(x):=\psi(x,\lambda_n)$ is another eigenfunction of L satisfying the initial conditions (9), then $\varphi_n(x)$ and $\psi_n(x)$ are linearly dependent for each $n\in\mathbb{N}$. We have

$$\psi_n(x)=k_n\varphi_n(x), \quad x\in[0,d)\cup(d,\pi], \tag{18}$$

where k_n is a real number. Define $\tilde{\varphi}_n(x)$, $\tilde{\psi}_n(x)$ and $\tilde{k}_n$ in a similar manner. From now on, we assume that $\Lambda_0\subseteq\mathbb{N}$ is a finite set and $\Lambda=\mathbb{N}\backslash\Lambda_0$.

Lemma 2. *If L_1 and $\tilde{L}_1$ have the same eigenvalues and, in addition, $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$, where λ_n and $\tilde{\lambda}_n$ are the eigenvalues of L and $\tilde{L}$, respectively, then $k_n = \tilde{k}_n$ for all $n \in \Lambda$.*

Proof. Define $\delta_i(\lambda) := W(\psi_i(x,\lambda), \theta_i(x,\lambda))$. It is easy to see that $\delta_i(\lambda)$ is independent of x. From definition of ϕ, θ and ψ it follows that

$$\begin{cases} W(\varphi_{in}(x), \psi_{in}(x)) = 0, \\ W(\varphi_{in}(x), \theta_{in}(x)) = \phi_i(\lambda_n), \end{cases} \tag{19}$$

for $i = 1, 2$. The above linear system has a unique solution

$$\varphi_{in}(x) = \frac{\psi_{in}(x)\phi_i(\lambda_n)}{\delta_i(\lambda_n)}, \tag{20}$$

and similarly we obtain

$$\tilde{\varphi}_{in}(x) = \frac{\tilde{\psi}_{in}(x)\tilde{\phi}_i(\tilde{\lambda}_n)}{\tilde{\delta}_i(\tilde{\lambda}_n)}. \tag{21}$$

From $\lambda_n = \tilde{\lambda}_n$ for all $n \in \Lambda$ and Lemma 1, we have $\phi_i \equiv \tilde{\phi}_i$. From definition of $\delta_i(\lambda)$ it follows that $\delta_2(\lambda_n) = \tilde{\delta}_2(\lambda_n)|_{x=\pi} = \lambda_n(H - H_1)$. Thus

$$k_n = \tilde{k}_n = \frac{\lambda_n(H - H_1)}{\phi_2(\lambda_n)}$$

for all $n \in \Lambda$.◄

Assume that λ is not in the spectrum of (1)-(3) and (7). Let

$$S_\lambda = (A - \lambda I)^{-1}|_D.$$

Replace A by $\tilde{A}$ and define $\tilde{S}_\lambda$ analogously. We consider the following spaces:

$$K := D(A) \ominus \{\Phi_n : n \in \Lambda_0\}, \tag{22}$$

$$\tilde{K} := D(\tilde{A}) \ominus \{\tilde{\Phi}_n : n \in \Lambda_0\}. \tag{23}$$

Define the transformation operator $T : K \to \tilde{K}$ by

$$T\Phi_n = \tilde{\Phi}_n, \tag{24}$$

where $\Phi_n = \begin{pmatrix} \varphi_n(x) \\ R_1(\varphi_n) \end{pmatrix}$ and $\tilde{\Phi}_n = \begin{pmatrix} \tilde{\varphi}_n(x) \\ R_1(\tilde{\varphi}_n) \end{pmatrix}$ for $n \in \Lambda$. By using the asymptotic form of solutions (14) and (15) it is easy to verify that T is a bounded operator. From (7) we have

$$(\lambda I - A)\Phi_n = (\lambda - \lambda_n)\Phi_n,$$

thus we obtain

$$\frac{\Phi_n}{(\lambda - \lambda_n)} = -S_\lambda \Phi_n.$$

A similar relation is obviously valid for $\tilde{\Phi}_n$.

Lemma 3. *The relation* $\tilde{S}_\lambda T = TS_\lambda$ *holds for* $\lambda \neq \lambda_n, \tilde{\lambda}_n$ *and* $n \in \mathbb{N}$.

Proof. Let $F \in K$. Then we can expand F in terms of the set Φ_n:

$$F(x) = \begin{pmatrix} f(x) \\ R_1(f) \end{pmatrix} = \sum_{\Lambda} f_n \Phi_n(x), \tag{25}$$

for $n \in \Lambda$, where $f_n = \frac{\langle F, \Phi_n \rangle_{\mathcal{H}}}{\langle \Phi_n, \Phi_n \rangle_{\mathcal{H}}}$. Let λ be in complex plane and not an eigenvalue of $A(q; h; H; d)$. Then the operator S_λ exists and can be written as

$$- S_\lambda F(x) = \sum_{\Lambda} \frac{f_n}{\lambda - \lambda_n} \Phi_n(x). \tag{26}$$

If we apply T to the above relation, we obtain

$$-TS_\lambda F(x) = \sum_{\Lambda} \frac{f_n}{\lambda - \lambda_n} \tilde{\Phi}_n(x).$$

If we apply $\tilde{S}_\lambda$ and T to (25) respectively, we obtain

$$-\tilde{S}_\lambda TF(x) = \sum_{\Lambda} \frac{f_n}{\lambda - \lambda_n} \tilde{\Phi}_n(x).$$

Then we get

$$\tilde{S}_\lambda T = TS_\lambda.$$

◀

In a general case when the operator L have eigenparameter dependent boundary and discontinuous conditions, we generalize the well-known result of Hochstadt [6]. We construct the Green's function for A by using its solutions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$. By applying the Green's function we now prove our main theorem.

Theorem 3. *If* $L(q; h; H_1; d)$ *and* $L(\tilde{q}; h; H_1; d)$ *have the same spectrum and* $\lambda_n = \tilde{\lambda}_n$ *for all* $n \in \Lambda$, *then*

$$q(x) - \tilde{q}(x) = \begin{cases} \sum_{\Lambda_0} (\tilde{y}_{1n} \varphi_{1n})'(x), & x < d, \\ \\ \sum_{\Lambda_0} (\tilde{y}_{2n} \varphi_{2n})'(x), & x > d, \end{cases} \tag{27}$$

a.e. on $[0, d) \cup (d, \pi]$, *where* $\tilde{y}_{in}$ *and* φ_{in} *for* $i = 1, 2$ *are suitable solutions of* $\tilde{l}y = \lambda_n y$ *and* $ly = \lambda_n y$, *respectively.*

Proof. By using the techniques of [28] for $\mathcal{F}(x) = (f(x),\ f_1)^T \in \mathcal{H}$, we can prove that the problem

$$y'' + (\lambda - q(x))y = f(x), \quad x \in (0, d) \cup (d, \pi) \tag{28}$$

$$y'(0) - hy(0) = 0, \quad Hy'(\pi) + \lambda y(\pi) = f_1, \tag{29}$$

with jump conditions (3) has the unique solution $y(x,\lambda)$, which can be represented as

$$y(x,\lambda)=\begin{cases}\frac{\psi_1(x,\lambda)}{\Delta_1(\lambda)}\int_0^x\varphi_1(t,\lambda)f(t)dt+\frac{\varphi_1(x,\lambda)}{\Delta_1(\lambda)}\big(\int_x^d\psi_1(t,\lambda)f(t)dt\\ \qquad +\frac{1}{a^2}\int_d^\pi\psi_2(t,\lambda)f(t)dt+\frac{1}{a^2}f_1\big), & 0<x<d,\\ \frac{\psi_2(x,\lambda)}{\Delta_2(\lambda)}\big(a^2\int_0^d\varphi_1(t,\lambda)f(t)dt+\int_d^x\varphi_2(t,\lambda)f(t)dt\big)\\ \qquad +\frac{\varphi_2(t,\lambda)}{\Delta_2(\lambda)}\big(\int_x^\pi\psi_2(t,\lambda)f(t)dt+f_1\big), & d<x<\pi.\end{cases}\tag{30}$$

By considering

$$G(x,t,\lambda)=\begin{cases}|a|\frac{\psi(x,\lambda)\varphi(t,\lambda)}{\Delta(\lambda)}, & 0\le t\le x\le\pi,\\ |a|\frac{\varphi(x,\lambda)\psi(t,\lambda)}{\Delta(\lambda)}, & 0\le x\le t\le\pi,\end{cases}\tag{31}$$

where $x\neq d$ and $t\neq d$, the formula (30) is reduced to

$$y(x,\lambda)=|a|\int_0^d G(x,t,\lambda)f(t)dt+\frac{1}{|a|}\int_d^\pi G(x,t,\lambda)f(t)dt+f_1\frac{\varphi(x,\lambda)}{\Delta(\lambda)}.\tag{32}$$

Let $n\in\Lambda$ and λ not be an eigenvalue of A. Then $-S_\lambda\Phi_n=\mathcal{G}_n$, where

$$\mathcal{G}_n(x)=\begin{pmatrix}g_n(x)\\ R_1(g_n)\end{pmatrix}=\begin{pmatrix}|a|\int_0^d G(x,t,\lambda)\varphi_{1n}(t)dt+\frac{1}{|a|}\int_d^\pi G(x,t,\lambda)\varphi_{2n}(t)dt\\ \frac{R_1(\varphi_n)}{\lambda-\lambda_n}\end{pmatrix}\tag{33}$$

and the function $G(x,t,\lambda)$ is as defined in (31). It is easy to verify that $g_n(x)$ obeys the boundary conditions (2), (3) and that

$$\Phi_n=(\lambda I-A)\mathcal{G}_n.$$

Using the asymptotic form of $\varphi(x,\lambda)$, $\psi(x,\lambda)$, and $\Delta(\lambda)$ for sufficiently large ρ and for $\rho\neq\rho_n$ we deduce that the Green's function $G(x,t,\lambda)$ is bounded. $G(x,t,\lambda)$ is a meromorphic function with the eigenvalues λ_k as its poles [29]. Let C_n be a sequence of circles about the origin intersecting the positive λ-axis between λ_n and λ_{n+1}. We have

$$\lim_{n\to\infty}\int_{C_n}\frac{G(x,t,\mu)}{\lambda-\mu}d\mu=0,\qquad \lambda\in\text{int }C_n.\tag{34}$$

From residue integration it follows that

$$\frac{1}{2\pi i}\int_{C_n}\frac{G(x,t,\mu)}{\lambda-\mu}d\mu=-G(x,t,\lambda)+\sum_{i=0}^{n}\frac{\varphi_i(x<)\psi_i(x>)}{\dot\Delta(\lambda_i)(\lambda-\lambda_i)},\tag{35}$$

where $\dot\Delta(\lambda_i)=\frac{d}{d\lambda}\Delta(\lambda)|_{\lambda=\lambda_i}$. From (34), (35) and the Mittag-Leffler expansion for $G(x,t,\lambda)$ we obtain

$$G(x,t,\lambda)=\sum_{i=0}^{\infty}\frac{\varphi_i(x<)\psi_i(x>)}{\dot\Delta(\lambda_i)(\lambda-\lambda_i)},\tag{36}$$

where for simplicity $x <:= min\{x,t\}$ and $x >:= max\{x,t\}$ and $\varphi_i(x <)$, $\psi_i(x >)$ are eigenfunctions corresponding to the eigenvalues λ_i. Therefore for $(f(x),\ R_1(f))^T \in K$, from (22), (31), (33), and Lemma 2 we have

$$\begin{pmatrix} y(x) \\ R_1(y) \end{pmatrix} = S_\lambda \begin{pmatrix} f(x) \\ R_1(f) \end{pmatrix} = S_\lambda F(x) =$$
$$\begin{pmatrix} \begin{cases} a^2 \frac{\psi_1(x)}{\Delta_2(\lambda)} \int_0^x \varphi_1(t) f(t) dt + \frac{\varphi_1(x)}{\Delta_2(\lambda)} \Big(a^2 \int_x^d \psi_1(t) f(t) dt + \int_d^\pi \psi_2(t) f(t) dt \Big), & x < d \\ \frac{\psi_2(x)}{\Delta_2(\lambda)} \big(a^2 \int_0^d \varphi_1(t) f(t) dt + \int_d^x \varphi_2(t) f(t) dt \big) + \frac{\varphi_2(x)}{\Delta_2(\lambda)} \int_x^\pi \psi_2(t) f(t) dt, & d < x < \pi \end{cases} \\ R_1(y) \end{pmatrix}$$

$$= \begin{pmatrix} \begin{cases} \sum_\Lambda \frac{a^2 \psi_{1n}(x) \int_0^x \varphi_{1n}(t) f(t) dt + \varphi_{1n}(x) \big(a^2 \int_x^d \psi_{1n}(t) f(t) dt + \int_d^\pi \psi_{2n}(t) f(t) dt \big)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}, & x < d \\ \sum_\Lambda \frac{\psi_{2n}(x) (a^2 \int_0^d \varphi_{1n}(t) f(t) dt + \int_d^x \varphi_{2n}(t) f(t) dt) + \varphi_{2n}(x) \int_x^\pi \psi_{2n}(t) f(t) dt}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}, & d < x < \pi \end{cases} \\ \sum_\Lambda \frac{f_n R_1(\varphi_n)}{\lambda - \lambda_n} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{cases} \sum_\Lambda \frac{k_n \varphi_{1n}(x) \big(a^2 \int_0^d \varphi_{1n}(t) f(t) dt + \int_d^\pi \varphi_{2n}(t) f(t) dt \big)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}, & x < d \\ \sum_\Lambda \frac{k_n \varphi_{2n}(x) \big(a^2 \int_0^d \varphi_{1n}(t) f(t) dt + \int_d^\pi \varphi_{2n}(t) f(t) dt \big)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}, & d < x < \pi \end{cases} \\ \sum_\Lambda \frac{f_n R_1(\varphi_n)}{\lambda - \lambda_n} \end{pmatrix}. \tag{37}$$

By applying T to both sides of (37), we see that

$$TS_\lambda F(x) = \begin{pmatrix} \begin{cases} \sum_\Lambda \frac{k_n \tilde{\varphi}_{1n}(x) \Big(a^2 \int_0^d \varphi_{1n}(t) f(t) dt + \int_d^\pi \varphi_{2n}(t) f(t) dt \Big)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}, & x < d \\ \sum_\Lambda \frac{k_n \tilde{\varphi}_{2n}(x) \Big(a^2 \int_0^d \varphi_{1n}(t) f(t) dt + \int_d^\pi \varphi_{2n}(t) f(t) dt \Big)}{\dot{\Delta}(\lambda_n)(\lambda - \lambda_n)}, & d < x < \pi \end{cases} \\ \sum_\Lambda \frac{f_n R_1(\tilde{\varphi}_n)}{\lambda - \lambda_n} \end{pmatrix}. \tag{38}$$

Define

$$U(x) := \begin{pmatrix} \begin{cases} \frac{a^2 \tilde{\psi}_1(x) \int_0^x \varphi_1(t) f(t) dt + \tilde{\varphi}_1(x) \big(a^2 \int_x^d \psi_1(t) f(t) dt + \int_d^\pi \psi_2(t) f(t) dt \big)}{\Delta(\lambda)}, & x < d \\ \frac{\tilde{\psi}_2(x) \big(a^2 \int_0^d \varphi_1(t) f(t) dt + \int_d^x \varphi_2(t) f(t) dt \big) + \tilde{\varphi}_2(x) \int_x^\pi \psi_2(t) f(t) dt}{\Delta(\lambda)}, & d < x < \pi \end{cases} \\ R_1(u) \end{pmatrix}. \tag{39}$$

By the Mittag-Leffler expansion for $U(x)$, we have

$$U(x)=\begin{pmatrix} \begin{cases} \sum_{\Lambda_0}\frac{a^2\tilde{w}_{1n}(x)\int_0^x\varphi_{1n}(y)f(y)dy+\tilde{z}_{1n}(x)\left(a^2\int_x^d\psi_{1n}(t)f(t)dt+\int_d^\pi\psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}+ \\ \sum_{\Lambda}\frac{a^2\tilde{\psi}_{1n}(x)\int_0^x\varphi_{1n}(t)f(t)dt+\tilde{\varphi}_{1n}(x)\left(a^2\int_x^d\psi_{1n}(t)f(t)dt+\int_d^\pi\psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, x<d \\ \sum_{\Lambda_0}\frac{\tilde{w}_{2n}(x)\left(a^2\int_0^d\varphi_{1n}(t)f(t)dt+\int_x^d\varphi_{2n}(t)f(t)dt\right)+\tilde{z}_{2n}(x)\int_x^\pi\psi_{2n}(t)f(t)dt}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}+ \\ \sum_{\Lambda}\frac{\tilde{\psi}_{2n}(x)\left(a^2\int_0^d\varphi_{1n}(t)f(t)dt+\int_x^d\varphi_{2n}(t)f(t)dt\right)+\tilde{\varphi}_{2n}(x)\int_x^\pi\psi_{2n}(t)f(t)dt}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, d<x<\pi \end{cases} \\ \\ \sum_{n\in\mathbb{N}}\frac{f_nR_1(\tilde{\varphi}_n)}{\lambda-\lambda_n} \end{pmatrix}. \tag{40}$$

The second term in the above expression is $TS_\lambda F$, as given in (38). In the first term, $\tilde{w}_n(x)$ represents $\tilde{\psi}(x,\lambda)$ and $\tilde{z}_n(x)$ represents $\tilde{\varphi}(x,\lambda)$ evaluated at λ_n. Hence

$$\tilde{S}_\lambda TF(x)=U(x)-$$
$$\begin{pmatrix} \begin{cases} \sum_{\Lambda_0}\frac{a^2\tilde{w}_{1n}(x)\int_0^x\varphi_{1n}(t)f(t)dt+\tilde{z}_{1n}(x)\left(a^2\int_x^d\psi_{1n}(y)f(y)dy+\int_d^\pi\psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, x<d \\ \sum_{\Lambda_0}\frac{\tilde{w}_{2n}(x)\left(a^2\int_0^d\varphi_{1n}(t)f(t)dt+\int_x^d\varphi_{2n}(t)f(t)dt\right)+\tilde{z}_{2n}(x)\int_x^\pi\psi_{2n}(t)f(t)dt}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}, d<x<\pi \end{cases} \\ \\ \sum_{\Lambda_0}\frac{f_nR_1(\tilde{\varphi}_n)}{\lambda-\lambda_n} \end{pmatrix}. \tag{41}$$

The right and left hand sides of (41) are in the domain $\tilde{S}_\lambda$. Therefore, both sides of (41) are continuous. By using (37) and differentiating the right-hand side of (41), for $x<d$ we obtain

$$\Big[\frac{a^2\tilde{\psi}_1'(x)\int_0^x\varphi_1(t)f(t)dt+\tilde{\varphi}_1'(x)\left(a^2\int_x^d\psi_1(t)f(t)dt+\int_d^\pi\psi_2(t)f(t)dt\right)}{\Delta(\lambda)}$$
$$-\sum_{\Lambda_0}\frac{a^2\tilde{w}_{1n}'(x)\int_0^x\varphi_{1n}(t)f(t)dt+\tilde{z}_{1n}'(x)\left(a^2\int_x^d\psi_{1n}(t)f(t)dt+\int_d^\pi\psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}\Big]$$
$$+\left[\frac{\tilde{\psi}_1(x)\varphi_1(x)-\tilde{\varphi}_1(x)\psi_1(x)}{\Delta(\lambda)}-\sum_{\Lambda_0}\frac{\tilde{w}_{1n}(x)\varphi_{1n}(x)+\tilde{z}_{1n}(x)\psi_{1n}(x)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}\right]f(x).$$

An inspection of the term in the second set of braces shows that it vanishes identically. To verify that, one merely computes the residue at each λ_n and observes that it becomes zero. By differentiating the expression in the braces in the last relation, from (41) we obtain

$$Tf(x)=\left[\frac{\tilde{\psi}_1(x)\varphi_1(x)-\tilde{\varphi}_1(x)\psi_1(x)}{\Delta(\lambda)}-\sum_{\Lambda_0}\frac{\tilde{w}_{1n}(x)\varphi_{1n}(x)+\tilde{z}_{1n}(x)\psi_{1n}(x)}{\dot{\Delta}(\lambda_n)(\lambda-\lambda_n)}\right]f(x)$$
$$-\sum_{\Lambda_0}\frac{a^2\tilde{w}_{1n}(x)\int_0^x\varphi_{1n}(t)f(t)dt+\tilde{z}_{1n}(x)\left(a^2\int_x^d\psi_{1n}(t)f(t)dt+\int_d^\pi\psi_{2n}(t)f(t)dt\right)}{\dot{\Delta}(\lambda_n)}. \tag{42}$$

The operator T is independent of λ. To compute the value of the expression in the braces in (42) we let $\lambda \to \infty$. Using the asymptotic formulas, we see that the term in the braces is reduced to unity. To simplify the second term in (42) we recall that $\psi_{1n} = k_n\varphi_{1n}$, $\psi_{2n} = k_n\varphi_{2n}$. From (4) we have

$$|a|\int_0^d \psi_{1n}(y)f(y)dy + \frac{1}{|a|}\int_d^\pi \psi_{2n}(t)f(t)dt + \frac{1}{H|a|}\psi_{2n}(\pi)R_1(f) = 0.$$

Thus from (9),

$$|a|\int_0^d \psi_{1n}(t)f(t)dt + \frac{1}{|a|}\int_d^\pi \psi_{2n}(t)f(t)dt + \frac{1}{|a|}R_1(f) = 0.$$

Then for $x < d$ and from (25),

$$Tf(x) = f(x) - \frac{1}{2}\sum_{\Lambda_0}\tilde{y}_{1n}(x)\int_0^x \varphi_{1n}(t)f(t)dt + \sum_{\Lambda_0}\frac{f_n k_n \tilde{z}_{1n}(x)R_1(\varphi_n)}{\dot{\Delta}(\lambda_n)}, \tag{43}$$

where

$$\frac{1}{2}\tilde{y}_{1n}(x) = a^2\frac{\tilde{w}_{1n}(x) - k_n\tilde{z}_{1n}(x)}{\dot{\Delta}(\lambda_n)},$$

and for $x > d$, by applying the similar computation we obtain

$$Tf(x) = f(x) + \frac{1}{2}\sum_{\Lambda_0}\tilde{y}_{2n}(x)\int_x^\pi \varphi_{2n}(t)f(t)dt + \sum_{\Lambda_0}\frac{f_n \tilde{w}_{2n}(x)R_1(\varphi_n)}{\dot{\Delta}(\lambda_n)},$$

where

$$\frac{1}{2}\tilde{y}_{2n}(x) = \frac{\tilde{w}_{2n}(x) - k_n\tilde{z}_{2n}(x)}{\dot{\Delta}(\lambda_n)}.$$

Now, from Lemma 3, we conclude that

$$\tilde{A}TF = TAF. \tag{44}$$

Suppose that $F = \Phi_n$ $(n \in \Lambda)$. Then we get $f_m = \frac{\langle\Phi_m,\Phi_n\rangle_{\mathcal{H}}}{\langle\Phi_n,\Phi_n\rangle_{\mathcal{H}}} = 0$, for $m \in \Lambda_0$. Using (44) we get

$$\begin{aligned}\tilde{A}T\Phi_n =& \tilde{A}\begin{pmatrix} \begin{cases} \varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0}\tilde{y}_{1m}\int_0^x \varphi_{1m}(t)\varphi_{1n}(t)dt, & x < d \\ \varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0}\tilde{y}_{2m}(x)\int_x^\pi \varphi_{2m}(t)\varphi_{2n}(t)dt, & d < x < \pi \end{cases} \\ \\ R_1(\tilde{\varphi}_n) \end{pmatrix} \\ =& \begin{pmatrix} \begin{cases} -\varphi_{1n}'' + \tilde{q}\varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0}\tilde{\ell}\left(\tilde{y}_{1m}\int_0^x \varphi_{1m}(t)\varphi_{1n}(t)dt\right), & x < d \\ -\varphi_{2n}'' + \tilde{q}\varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0}\tilde{\ell}\left(\tilde{y}_{2m}(x)\int_x^\pi \varphi_{2m}(t)\varphi_{2n}(t)dt\right), & d < x < \pi \end{cases} \\ \\ -R_1'(\tilde{\varphi}_n) \end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} \begin{cases} -\varphi_{1n}'' + \tilde{q}\varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0} \tilde{\ell}\tilde{y}_{1m} \int_0^x \varphi_{1m}(t)\varphi_{1n}(t)dt \\ \qquad + \frac{1}{2}\sum_{\Lambda_0} 2\tilde{y}_{1m}'(\varphi_{1m}\varphi_{1n}) + \tilde{y}_{1m}(\varphi_{1m}\varphi_{1n})', & x < d \\ -\varphi_{2n}'' + \tilde{q}\varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0} \tilde{\ell}\tilde{y}_{2m} \int_x^\pi \varphi_{2m}(t)\varphi_{2n}(t)dt \\ \qquad + \frac{1}{2}\sum_{\Lambda_0} (-2)\tilde{y}_{2m}'(\varphi_{2m}\varphi_{2n}) + \tilde{y}_{2m}(\varphi_{2m}\varphi_{2n})', & d < x < \pi \end{cases} \\ -R_1'(\tilde{\varphi}_n) \end{pmatrix} \tag{45}$$

and

$$TA\Phi_{1n} = \begin{pmatrix} \begin{cases} -\varphi_{1n}'' + q\varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{1m} \int_0^x \varphi_{1m}\ell\varphi_{1n}, & t < d \\ -\varphi_{2n}'' + q\varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{2m} \int_x^\pi \varphi_{2m}\ell\varphi_{2n}, & d < x < \pi \end{cases} \\ -R_1'(\tilde{\varphi}_n) \end{pmatrix}$$

$$= \begin{pmatrix} \begin{cases} -\varphi_{1n}'' + q\varphi_{1n} - \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{1m} \int_0^x \varphi_{1n}\ell\varphi_{1m} \\ \qquad - \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{1m}(\varphi_{1n}\varphi_{1m}' - \varphi_{1m}\varphi_{1n}'), & x < d \\ -\varphi_{2n}'' + q\varphi_{2n} + \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{2m} \int_x^\pi \varphi_{2n}\ell\varphi_{2m} \\ \qquad + \frac{1}{2}\sum_{\Lambda_0} \tilde{y}_{2m}(\varphi_{2n}\varphi_{2m}' - \varphi_{2m}\varphi_{2n}'), & d < x < \pi \end{cases} \\ -R_1'(\tilde{\varphi}_n) \end{pmatrix}. \tag{46}$$

Note that

$$\begin{aligned} \sum_{\Lambda_0} \tilde{y}_{1m} \int_0^x \varphi_{1n}\ell\varphi_{1m} &= \sum_{\Lambda_0} \tilde{y}_{1m} \int_0^x \lambda_m\varphi_{1m}\varphi_{1n} \\ &= \sum_{\Lambda_0} \lambda_m\tilde{y}_{1m} \int_0^x \varphi_{1m}\varphi_{1n} \\ &= \sum_{\Lambda_0} \tilde{\ell}\tilde{y}_{1m} \int_0^x \varphi_{1m}\varphi_{1n} \end{aligned}$$

and

$$\sum_{\Lambda_0} \tilde{y}_{2m} \int_x^\pi \varphi_{2n}\ell\varphi_{2m} = \sum_{\Lambda_0} \tilde{\ell}\tilde{y}_{2m} \int_x^\pi \varphi_{2m}\varphi_{2n}.$$

Using (44) we find that

$$q(x) - \tilde{q}(x) = \begin{pmatrix} \begin{cases} \sum_{\Lambda_0} (\tilde{y}_{1m}\varphi_{1m})', & x < d \\ \sum_{\Lambda_0} (\tilde{y}_{2m}\varphi_{2m})', & d < x < \pi \end{cases} \\ 0 \end{pmatrix}$$

If Λ_0 is empty, then T is a unitary operator and $A = \tilde{A}$. Hence $q = \tilde{q}$.◀

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Mohammad Shahriari
Department of Mathematics, Faculty of Science, University of Maragheh, P.O. Box 55181-83111, Maragheh, Iran.
E-mail: shahriari@tabrizu.ac.ir

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