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# On an Inverse Problem for a Semilinear Parabolic Equation in the Case of Boundary Value Problem with Nonlinear Boundary Condition

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**Abstract.** The goal of this paper is to investigate the well-posedness of the inverse problem on determination of the coefficient at a minor term of a semilinear parabolic equation in the case of nonlinear boundary condition. Additional condition is given in the nonlocal integral form. A uniqueness theorem and a "conditional" stability are proved.

**Key Words and Phrases**: inverse problem, semilinear parabolic equation, nonlinear boundary condition.
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Let $R^n$ be an $n$–dimensional real Euclidian space, $x = (x_1, ..., x_n)$ be an arbitrary point in the bounded domain $D \subset R^n$ with a sufficiently smooth boundary $\partial D$, $\Omega = D \times (0;T]$, $S = \partial D \times [0;T]$ and $0 < T$ be a fixed number.

The spaces $C^l(\cdot)$, $C^{l+\alpha}(\cdot)$, $C^{l,l/2}(\cdot)$, $C^{l+\alpha,(l+\alpha)/2}(\cdot)$, $l = 0, 1, 2$, $\alpha \in (0, 1)$ and the norms in these spaces are defined as in $[1, \text{pp.}12 - 20]$,

$$\|\cdot\|_l = \|\cdot\|_{C^l}, \quad u_t = \frac{\partial u}{\partial t}, \quad u_{x_i} = \frac{\partial u}{\partial x_i}, \quad i = \overline{1, n},$$

$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$– is a Laplace operator, $\frac{\partial u}{\partial \nu}$ is an internal conormal derivative.

We consider the following inverse problem on determining a pair of functions $\{u(x,t), c(t)\}$:

$$u_t - \Delta u + c(t)\, u = f(x,t,u), \qquad (x,t) \in \Omega \tag{1}$$

$$u(x,0) = \varphi(x), \; x \in \bar{D} = D \bigcup \partial D \tag{2}$$

$$\frac{\partial u}{\partial \nu} = \psi(x,t,u), \quad (x,t) \in S \tag{3}$$

 

$$\int_D u(x,t)\,dx = h(t)\,,\ \ t \in [0,T]\,, \tag{4}$$

where $f(x,t,p)\,, \varphi(x)\,, \psi(x,t,p)\,, h(t)$ are the given functions.

The coefficient inverse problems were studied in the papers $[2-4]$ (see also the references therein).

We make the following assumptions for the data of problem (1)-(4):

$1^0$. $f(x,t,p) \in C^{\alpha,\alpha/2}\left(\bar{\Omega}\times R^1\right)$, there exists $m_1>0$ such that for any $(x,t)\in\bar{\Omega}$ and $p_1,p_2\in R^1 : |f(x,t,p_1)-f(x,t,p_2)| \le m_1|p_1-p_2|$;

$2^0$. $\varphi(x)\in C^{2+\alpha}\left(\bar{D}\right)$ ;

$3^0$. $\psi(x,t,p) \in C^{\alpha,\alpha/2}\left(S\times R^1\right)$, there exists $m_2>0$ such that for any $(x,t)\in S$ and $p_1,p_2\in R^1 : |\psi(x,t,p_1)-\psi(x,t,p_2)| \le m_2|p_1-p_2|$;

$4^0$. $h(t)\in C^{1+\alpha}[0,T]$.

**Definition 1.** *The pair of functions* $\{c(t)\,, u(x,t)\}$ *is called the solution of problem* (1)-(4) *if*

1) $c(t)\in C(0,T]$;

2) $u(x,t)\in C^{2,1}(\Omega)\bigcap C^{1,0}\left(\bar{\Omega}\right)$;

3) *The conditions* (1)-(4) *hold for these functions, with condition* (3) *defined in the following sense:*

$$\frac{\partial u(x,t)}{\partial \nu(x,t)} = \lim_{\substack{y\to x\\ y\in\sigma}} \frac{\partial u(y,t)}{\partial \nu(x,t)},$$

*where* $\sigma$ *is any closed cone with a vertex* $x$ *in* $D\bigcup\{x\}$.

*The uniqueness theorem and the estimate of stability for the solutions of inverse problems occupy a central place in investigation of their well-posedness. In this paper, the uniqueness of the solution of problem* (1)-(4) *is proved under more general assumptions and the estimate characterizing the "conditional" stability of the problem is established.*

Let $\{u_i(x,t)\,, c_i(t)\}$ be the solution of problem $(1)-(4)$ corresponding to the given $f_i(x,t,u_i)\,, \varphi_i(x)\,, \psi_i(x,t,u_i)\,, h_i(t)\,,\ i=1,2.$

**Definition 2.** *A solution of problem* (1)-(4) *is called stable if for any* $\varepsilon>0$ *there is a* $\delta(\varepsilon)>0$ *such that for* $\|f_1-f_2\|_0<\delta,\ \ \|\varphi_1-\varphi_2\|_2<\delta,\ \ \|\psi_1-\psi_2\|_0<\delta,\ \ \|h_1-h_2\|_1<\delta$ *the inequality* $\|u_1-u_2\|_0+\|c_1-c_2\|_0\le\varepsilon$ *is fulfilled.*

**Theorem 1.** *Let*

*1.* $f_i,\varphi_i,\psi_i,h_i, i=1,2$ *satisfy the conditions* $1^0-4^0$, *respectively;*

*2. there exist the solutions* $\{u_i(x,t)\,,\ c_i(t)\}\,,\ i=1,2,$ *of problem* $(1)-(4)$ *in the sense of definition 1, and let they, in addition, belong to the set*

$$K_\alpha = \left\{(u,c)\,\middle|\,u(x,t)\in C^{2+\alpha,1+\alpha/2}\left(\bar{\Omega}\right),\quad c(t)\in C^\alpha[0,T]\right\}$$

*Then there exists a* $T^* > 0$ *such that for* $(x,t) \in \bar{D} \times [0,T^*]$ *the solution of problem* (1) – (4) *is unique, and the stability estimate*

$$\|u_1 - u_2\|_0 + \|c_1 - c_2\|_0 \le m_3 \left[\|f_1 - f_2\|_0 + \|\varphi_1 - \varphi_2\|_2 + \|\psi_1 - \psi_2\|_0 + \|h_1 - h_2\|_1\right] \quad (5)$$

*is valid, where* $m_3 > 0$ *depends on the data of problem* (1)-(4) *and the set* $K_\alpha$.

**Proof.** First we prove the validity of estimate (5). In order to get a uniqueness theorem, below we should suppose that perturbations of problem data are everywhere identically equal to zero. In view of (2) and the conditions of the theorem, from equation (1) for the function $c(t)$ we get

$$c(t) = \left[\int_{\partial D} \frac{\partial u}{\partial \nu} dx + \int_D f(x,t,u)\, dx - h_t(t)\right] \backslash h(t), \quad t \in [0,1], \quad (6)$$

Denote $z(x,t) = u_1(x,t) - u_2(x,t)$, $\lambda(t) = c_1(t) - c_2(t)$, $\delta_1(x,t,u) = f_1(x,t,u) - f_2(x,t,u)$, $\delta_2(x) = \varphi_1(x) - \varphi_2(x)$, $\delta_3(x,t,u) = \psi_1(x,t,u) - \psi_2(x,t,u)$, $\delta_4(t) = h_1(t) - h_2(t)$

We can verify that the functions $\{\lambda(t), \quad w(x,t) = z(x,t) - \delta_2(x)\}$ satisfy the conditions of the system

$$w_t - \Delta w = F(x,t), \qquad (x,t) \in \Omega, \quad (7)$$

$$w(x,0) = 0, x \in \bar{D}; \quad \frac{\partial w}{\partial \nu}(x,t) = \Psi(x,t), \quad (x,t) \in S \quad (8)$$

$$\lambda(t) = \int_{\partial D} \frac{\partial z}{\partial \nu} dx \backslash h_1(t) + H(t), \quad t \in [0,T], \quad (9)$$

where

$$F(x,t) = \delta_1(x,t,u_1) - \Delta\delta_2(x) - c_1(t)\, z(x,t) - \lambda(t)\, u_2(x,t) + f_2(x,t,u_1) - f_2(x,t,u_2),$$

$$\Psi(x,t) = \delta_3(x,t,u_1) - \frac{\partial \delta_2}{\partial \nu}(x) + \psi_2(x,t,u_1) - \psi_2(x,t,u_2),$$

$$H(t) = \left\{\left[\int_D \delta_1(x,t,u_1)\, dx + \int_D \left[f_2(x,t,u_1) - f_2(x,t,u_2)\right] dx - \delta_{4t}(t)\right] h_2(t) - \right.$$
$$\left. - \left[\int_{\partial D} \frac{\partial u_2}{\partial \nu} dx - h_{2t}(t) + \int_D f_2(x,t,u_2)\, dx\right] \delta_4(t)\right\} \left[h_1(t) \cdot h_2(t)\right]^{-1}$$

By the conditions of the theorem, if follows that there exists a classic solution of problem (7), (8) which can be represented in the following form [5, p.182]:

$$w(x,t) = \int_0^t \int_D (x,t;\xi,\tau)\, F(\xi,\tau)\, d\xi d\tau + \int_0^t \int_{\partial D} (x,t;\xi,\tau)\, \rho(\xi,\tau)\, d\xi_{\partial D} d\tau, \qquad (10)$$

where $(x,t;\xi,\tau)$ is a fundamental solution of the equation $w_t - \Delta w = 0$, $\quad d\xi = d\xi_1 ... d\xi_n$, is an element of the surface $\partial D$, and $\rho(x,t)$ is a continuous bounded solution of the following integral equation [2, p. 183]

$$\rho(x,t) = 2\int_0^t \int_D \frac{\partial (x,t;\xi,\tau)}{\partial \nu(x,t)} F(\xi,\tau)\, d\xi d\tau +$$

$$+2\int_0^t \int_{\partial D} \frac{\partial (x,t;\xi,\tau)}{\partial \nu(x,t)} \rho(\xi,\tau)\, d\xi_{\partial D} d\tau - 2\Psi(x,t)\ . \qquad (11)$$

Assume

$$\chi = \|u_1 - u_2\|_0 + \|c_1 - c_2\|_0 \, .$$

Estimate the function $|z(x,t)|$. Taking into account that $z(x,t) = w(x,t) + \delta_2(x)$, from (10) we get:

$$|z(x,t)| \leq |w(x,t)| + |\delta_2(x)| \leq |\delta_2(x)| + \int_0^t \int_D |(x,t,\xi,\tau)| \cdot |F(\xi,\tau)|\, d\xi d\tau +$$

$$+ \int_0^t \int_{\partial D} |(x,t,\xi,\tau)| \cdot |\rho(\xi,\tau)|\, d\xi_{\partial D} d\tau. \qquad (12)$$

For the expression $\int_D |(x,t,\xi,\tau)|\, d\xi$, in the second summand on the right hand side of (12), the following estimate is true:

$$\int_D |(x,t,\xi,\tau)|\, d\xi \leq m_3. \qquad (13)$$

By the requirements imposed on the problem data and on the set $K_\alpha$, the integrand function $F(x,t)$ in the second summand on the right side of (12) satisfies the estimate

$$|F(x,t)| \leq |\delta_1(x,t)| + |\Delta\delta_2(x)| + |c_1(t)|\, |z(x,t)| + |\lambda(t)|\, |u_2(x,t)| +$$

$$+ |f_2(x,t,u_1) - f_2(x,t,u_2)| \leq \|f_1 - f_2\|_0 + \|\varphi_1 - \varphi_2\|_2 + m_4 \cdot \chi, \quad (x,t) \in \bar{\Omega}, \qquad (14)$$

where $m_4 > 0$ depends on the data of problem (1)-(4) and on the set $K_\alpha$.

The expression $\int_{\partial D} |(x,t;\xi,\tau)|\, d\xi_{\partial D}$ in the third summand on the right side of (12) satisfies the estimate

$$\int_{\partial D} |(x,t;\xi,\tau)|\, d\xi_{\partial D} \leq m_5. \qquad (15)$$

Taking into account (11), the conditions of the theorem, definition of the set $K_\alpha$ and the following estimate [5, p. 20]:

$$\int_D \left| \frac{\partial (x,t;\xi,\tau)}{\partial \nu (x,t)} \right| d\xi \leq m_6 (t-\tau)^{-\mu}, \qquad \frac{1}{2} < \mu < 1$$

for the function $\rho(x,t)$ we get

$$|\rho(x,t)| \leq m_7 \left[ \|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi \right] + m_8 \|\rho\| \cdot t^{1-\mu}, \ (x,t) \in S,$$

where $m_7, \ m_8 > 0$ depend on the data of problem (1)-(4) and on the set $K_\alpha$.

The last inequality is fulfilled for all $(x,t) \in \partial D \times [0,T]$, therefore the following estimate is true:

$$\|\rho\|_0 \leq m_7 \left[ \|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi \right] + m_8 t^{1-\mu} \|\rho\|_0 .$$

Let $0 < T_1 \leq T$ be a number such that $m_8 T^{1-\mu} < 1$. Then for all $(x,t) \in \partial D \times [0,T_1]$ we have

$$\|\rho\|_0 \leq m_9 \left[ \|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \chi \right], \tag{16}$$

where $m_9 > 0$ depends on the data of problem (1)-(4) and the set $K_\alpha$.

Taking into account the inequalities (13), (14), (15) and (16), from (12) for $|z(x,t)|$ we get:

$$|z(x,t)| \leq m_{10} \left[ \|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 \right] + m_{11} \chi \cdot t, \quad (x,t) \in \bar{\Omega}, \tag{17}$$

where $m_{10}, \ m_{11} > 0$ depend on the data of problem (1)-(4) and on the set $K_\alpha$.

Now estimate the function $|\lambda(t)|$. From (9) it follows

$$|\lambda(t)| \leq \int_{\partial D} \left| \frac{\partial z}{\partial \nu} \right| dx \cdot \left| h_1(t)^{-1} \right| + |H(t)| .$$

Similar to (17), we get

$$\left| \frac{\partial z}{\partial \nu} \right| \leq m_{13} \left[ \|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 \right] + m_{14} \chi t, \ (x,t) \in \bar{\Omega}, \tag{18}$$

where $m_{13}, m_{14} > 0$ depend on the data of the problem (1)-(4) and on the set $K_\alpha$.

Taking into account the conditions of the theorem, definition of the set $K_\alpha$, inequality (18) and expression for $H(t)$, from the last inequality we get:

$$|\lambda(t)| \leq m_{15} \left[ \|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_2 \right] + m_{16} \chi \cdot t, \quad t \in [0,T], \tag{19}$$

where $m_{15}, m_{16} > 0$ depend on the data of the problem (1)-(4) and on the set $K_\alpha$.

Inequalities (18) and (19) are satisfied for any values of $(x,t) \in \bar{D} \times [0,T_1]$.

Consequently, combining these inequalities, we get

$$\chi \leq m_{17}\left[\|\delta_1\|_0 + \|\delta_2\|_2 + \|\delta_3\|_0 + \|\delta_4\|_2\right] + m_{18}t\chi, \tag{20}$$

where $m_{17},\ m_{18} > 0$ depend on the data of problem (1)-(4) and the set $K_\alpha$.

Let $T_2$ $(0 < T_2 \leq T)$ be a number such that $m_{18}T_2 < 1$. Then from (19) we get that for $(x,t) \in \bar{D} \times [0, T^*]$, $T^* = \min(T_1, T_2)$, the stability estimate for the solution of problem (1)-(4) is true.

Uniqueness of the solution of problem (1)-(4) follows from the estimate (5) for $f_1(x,t,u) = f_2(x,t,u)$, $\varphi_1(x) = \varphi_2(x)$, $\psi_1(x,t,u) = \psi_2(x,t,u)$, $h_1(t) = h_2(t)$.

The theorem is completely proved.◀

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