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Completeness of Derivative Chains for Polynomial Operator Pencil of Third Order with Multiple Characteristics

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Abstract. In this paper, we prove the completeness of a derivative chain constructed by the eigen- and adjoined vectors of polynomial operator pencils of third order with multiple characteristics corresponding to a boundary value problem on the semiaxis.

Key Words and Phrases: polynomial operator pencil, eigen- and adjoined vectors, derivative chain, resolvent, regular solvability

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1. Introduction

In the separable Hilbert space H, consider the polynomial operator pencil

$$P(\lambda) = (-\lambda E + A)(\lambda E + A)^2 + \lambda^2 A_1 + \lambda A_2, \tag{1}$$

where E is the identity operator, A is a self-adjoint positive-definite operator with compact inverse A^{-1} and A_1, A_2 are linear operators with $A_j A^{-j}$, $j = 1, 2$, bounded on H. In this case the pencil $P(\lambda)$ has a discrete spectrum.

We denote by H_θ the scale of Hilbert spaces generated by the operator$A$, i.e. $H_\theta = D(A^\theta)$, $\theta \geq 0$, $(x, y)_{H_\theta} = (A^\theta x,\ A^\theta y)$, $x, y \in D(A^\theta)$.

We introduce the following Hilbert spaces:

$$L_2(R_+; H) = \left\{ u(t) : \|u\|^2_{L_2(R_+;H)} = \int_0^{+\infty} \|u(t)\|^2_H \, dt < +\infty \right\},$$

$$W_2^3(R_+; H) = \left\{ u(t) : \|u\|^2_{W_2^3(R_+;H)} = \int_0^{+\infty} \left(\left\| \frac{d^3 u(t)}{dt^3} \right\|^2_H + \left\| A^3 u(t) \right\|^2_H \right) dt < +\infty \right\}$$

(see [1]). Here and further, the derivatives are understood in the sense of distribution theory.

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We associate to pencil (1) the boundary value problem of the form

$$P\,(d/dt)\,u\,(t)=0,\ u\,(0)=\varphi_0,\ \frac{du\,(0)}{dt}=\varphi_1,\ t\in R_+=[0,+\infty)\,, \tag{2}$$

where $u\,(t)\in W_2^3\,(R_+;H)$, $\varphi_s\in H_{5/2-s}$, $s=0,1$.

Let λ_n $(n=1,2,...)$ be the eigenvalues of the pencil $P\,(\lambda)$ in the left half-plane Π_- and $\psi_{0,n}$, $\psi_{1,n}$,..., $\psi_{m,n}$ be the eigen- and adjoined vectors corresponding to the eigenvalue λ_n:

$$P(\lambda_n)\psi_{0,n}=0,\ \ P(\lambda_n)\psi_{1,n}+P'(\lambda_n)\psi_{0,n}=0,$$

$$P(\lambda_n)\psi_{2,n}+P'(\lambda_n)\psi_{1,n}+\frac{P''(\lambda_n)}{2}\psi_{0,n}=0,$$

$$P(\lambda_n)\psi_{q,n}+P'(\lambda_n)\psi_{q-1,n}+\frac{P''(\lambda_n)}{2}\psi_{q-2,n}-\psi_{q-3,n}=0,\ q=3,...,m.$$

Then the vector-functions

$$u_{h,n}\,(t)=e^{\lambda_n t}\left(\psi_{h,n}+\frac{t}{1!}\psi_{h-1,n}+...+\frac{t^h}{h!}\psi_{0,n}\right),\ \ h=0,1,...,m,$$

belong to $W_2^3\,(R_+;H)$ and satisfy the equation $P\,(d/dt)\;u\,(t)=0$. These functions are called elementary solutions of the equation $P\,(d/dt)\;u\,(t)=0$. By means of these solutions we define the vector

$$\tilde{\psi}_{h,n}\ =\left\{\psi_{h,n}^{(0)},\,\psi_{h,n}^{(1)}\right\}\in\tilde{H}\equiv H_{5/2}\oplus H_{3/2},$$

where $\psi_{h,n}^{(s)}\equiv\frac{d^s}{dt^s}u_{h,n}\,(t)\Big|_{t=0}$, $s=0,1$, $h=0,1,...,m$. The system $\left\{\tilde{\psi}_{h,n}\right\}_{n=1}^{\infty}$ will be called the derivative chain of eigen- and adjoined vectors of the pencil $P\,(\lambda)$ generated by the boundary value problem of the form (2).

In this paper, the completeness of the system $\left\{\tilde{\psi}_{h,n}\right\}_{n=1}^{\infty}$ in the space $\tilde{H}$ is established in terms of operator coefficients of pencil (1).

The first basic results in the spectral theory of abstract nonself-adjoint operators were obtained by M.V. Keldysh in [2] (for more details, see [3]). Those were the results concerning n-fold completeness of the eigen- and adjoined vectors. Later they were successfully used by many mathematicians to expand the class of operator coefficients of Keldysh operator pencil and to extend the previous results to the case of Banach space (see [4]). We especially note the results concerning the completeness of a part of the eigen- and adjoined vectors corresponding to boundary value problems on the semiaxis obtained by M.G.Gasymov in [5], [6], [7]. In the above-cited works, M.G.Gasymov proposed a new method that included obtaining different criteria for solvability of boundary value problems which imply the completeness of a part of the eigen- and adjoined vectors. His results were further developed by S.S.Mirzoev in [8], [9]. We also note the work by A.A.Shkalikov [10] with important results on the minimality and basicity for derivative chains corresponding to boundary value problems on the semiaxis. Recently, there appeared a number of works dedicated to the polynomial operator pencils with multiple characteristics which occur when studying specific model problems of mechanics (see, e.g., [11]).

2. Main results

Following M.G.Gasymov's method, we begin our study with the solvability of problem (2).

Definition 1. *Boundary value problem (2) is said to be regularly solvable if for any* $\varphi_s \in H_{5/2-s}$, $s = 0, 1$, *there exists a vector-function* $u(t) \in W_2^3(R_+; H)$ *satisfying the equation* $P(d/dt)\, u(t) = 0$ *almost everywhere in* R_+, *the boundary conditions are satisfied in the sense of relations*

$$\lim_{t \to 0} \|u(t) - \varphi_0\|_{H_{5/2}} = 0, \ \lim_{t \to 0} \left\| \frac{du(t)}{dt} - \varphi_1 \right\|_{H_{3/2}} = 0$$

and the following inequality holds:

$$\|u\|_{W_2^3(R_+;H)} \le const \left(\|\varphi_0\|_{H_{5/2}} + \|\varphi_1\|_{H_{3/2}} \right).$$

Here $u(t)$ *will be called a regular solution of boundary-value problem (2).*

The following theorem holds.

Theorem 1. *Suppose that* A *is a self-adjoint positive-definite operator, the operators* $A_j A^{-j}$, $j = 1, 2$, *are bounded on* H *and the following inequality holds:*

$$\frac{1}{\sqrt{2}\left(\sqrt{5}+1\right)^{1/2}} \left\| A_1 A^{-1} \right\|_{H \to H} + \frac{2}{3\sqrt{3}} \left\| A_2 A^{-2} \right\|_{H \to H} < 1.$$

Then boundary value problem (2) is regularly solvable.

We will outline the **Proof of Theorem 1** briefly. In the case of $A_1 = A_2 = 0$, it is easy to show that the boundary value problem (2), i.e. the problem

$$\left(-\frac{d}{dt} + A \right) \left(\frac{d}{dt} + A \right)^2 u(t) = 0, \ u(0) = \varphi_0, \ \frac{du(0)}{dt} = \varphi_1, \ t \in R_+ = [0, +\infty), \tag{3}$$

is regularly solvable. If we assume that at least one of A_j, $j = 1, 2$, is nonzero, then the regular solution of (2) must be found in the form $u(t) = u_0(t) + v(t)$, where $u_0(t)$ is a regular solution of boundary value problem (3) and $v(t) \in W_2^3(R_+; H)$. In this case, boundary value problem (2) can be reduced to the following problem with respect to $v(t)$:

$$P(d/dt)\, v(t) = f(t), \ v(0) = 0, \ \frac{dv(0)}{dt} = 0, \tag{4}$$

where $f(t) \in L_2(R_+; H)$. Proof of the theorem is completed as the regular solvability of problem (4) under the conditions of the theorem has been proved in [12].

Now we state some estimates for the resolvent of the pencil $P(\lambda)$ on the imaginary axis and on some adjacent rays which have been established in [13].

Theorem 2. *Suppose that A is a self-adjoint positive-definite operator, the operators $A_j A^{-j}$, $j = 1, 2$, are bounded on H and the following inequality holds:*

$$\frac{2}{3\sqrt{3}} \left(\| A_1 A^{-1} \|_{H \to H} + \| A_2 A^{-2} \|_{H \to H} \right) < 1.$$

Then the resolvent of the pencil $P(\lambda)$ exists on the imaginary axis and the following estimates hold:

$$\sum_{j=0}^{2} \| \lambda^{3-j} A^j P^{-1}(\lambda) \| \le const. \tag{5}$$

$$\left\| A^{\alpha} P^{-1}(\lambda) \right\| \le const\, |\lambda|^{\alpha - 3},\ 0 < \alpha < 3,\ \lambda \ne 0. \tag{6}$$

Theorem 3. *Let the conditions of Theorem 2 be fulfilled. Then for sufficiently small $\nu > 0$ on the rays:*

$$\Gamma_{\frac{\pi}{2} \pm \nu} = \left\{ \lambda : \lambda = r e^{i\left(\frac{\pi}{2} \pm \nu\right)},\ r > 0 \right\},\ \Gamma_{-\frac{\pi}{2} \pm \nu} = \left\{ \lambda : \lambda = r e^{-i\left(\frac{\pi}{2} \pm \nu\right)},\ r > 0 \right\}$$

the operator pencil $P(\lambda)$ is invertible and estimates (5), (6) hold.

Remark 1. *To obtain the results of Theorems 1, 2 and 3, we don't require the condition that the operator A^{-1} is compact.*

By $\sigma_{\infty}(H)$ we denote the set of compact operators acting on H.

It is known that, if $C \in \sigma_{\infty}(H)$, then $(C^* C)^{1/2}$ is a compact self-adjoint operator on H. The eigenvalues of the operator $(C^* C)^{1/2}$ will be called s-numbers of the operator C. We will arrange the non-zero s-numbers of the operator C in descending order according to their multiplicity. Denote

$$\sigma_p = \left\{ C : \ C \in \sigma_{\infty}(H)\ ;\ \sum_{k=1}^{\infty} s_k^p(C) < \infty \right\},\ 0 < p < \infty.$$

Before presenting the main result of this paper, we state the following assertion which is easily proved using Keldysh lemma (see [3]) on the expansion of the resolvent about the eigenvalues.

Lemma 1. *In order that the system $\left\{ \tilde{\psi}_{h,n} \right\}_{n=1}^{\infty}$ be complete in the space $\tilde{H}$, it is necessary and sufficient that for every vector $\xi_k \in H_{5/2-k}$, $k = 0, 1$, the holomorphy of the vector-function*

$$R(\lambda) = \sum_{k=0}^{1} \left(A^{5/2-k} P^{-1}\left(\overline{\lambda}\right) \right)^* \lambda^k A^{5/2-k} \xi_k$$

in the half-plane Π_- imply that $\xi_k = 0$, $k = 0, 1$.

The following main theorem holds.

Theorem 4. *Let the conditions of Theorem 1 be satisfied and one of the following conditions hold:*

1) $A^{-1}\in\sigma_p,\ 0<p\leq 1;$

2) $A^{-1}\in\sigma_p,\ 0<p<\infty,\ \ A_jA^{-j}\in\sigma_\infty(H),\ \ j=1,2.$

Then the system $\left\{\tilde{\psi}_{h,n}\right\}_{n=0}^{\infty}$ *is complete in the space* $\tilde{H}$.

Proof. We will prove Theorem 4 by contradiction. If the system $\left\{\tilde{\psi}_{h,n}\right\}_{n=1}^{\infty}$ is not complete in the space $\tilde{H}$, then there exists a non-zero vector $\xi=\{\xi_0,\xi_1\}\in\tilde{H}$ such that $\left(\xi,\tilde{\psi}_{h,n}\right)_{\tilde{H}}=0,\ n=1,2,\ldots$. Then it follows from Keldysh lemma [3] that the vector-function $R(\lambda)$ is holomorphic in the half-plane Π_-. Under the conditions of the theorem (according to Theorem 1), boundary value problem (2) is regularly solvable. If $u(t)$ is a regular solution of boundary value problem (2), then it can be expressed in the form

$$u(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\hat{u}(\lambda)\,e^{\lambda t}d\lambda, \tag{7}$$

where

$$\hat{u}(\lambda)=P^{-1}(\lambda)\sum_{r=0}^{2}B_r u^{(2-r)}(0),$$

with

$$B_0=-E,\ B_1=-\lambda E+Q,\ B_2=-\lambda^2E+\lambda Q+A_2+A^2,\ Q=-A+A_1.$$

Next, as in [6], for $t>0$ in (7) we can change the contour of integration to

$$\Gamma_{\pm\theta}=\left\{\lambda:\lambda=re^{\pm i\left(\frac{\pi}{2}+\theta\right)},\ r>0\right\}.$$

As a result, for $t>0$ we obtain:

$$\sum_{k=0}^{1}\left(\frac{d^k u(t)}{dt^k},\xi_k\right)_{H_{5/2-k}}=$$

$$\frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}}\sum_{k=0}^{1}\left(A^{5/2-k}P^{-1}(\lambda)\lambda^k\sum_{r=0}^{2}B_r u^{(2-r)}(0),A^{5/2-k}\xi_k\right)e^{\lambda t}d\lambda=$$

$$\frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}}\sum_{r=0}^{2}\left(B_r u^{(2-r)}(0),\ R(\bar{\lambda})\right)e^{\lambda t}d\lambda=\frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}}g(\lambda)\,e^{\lambda t}d\lambda,$$

where

$$g(\lambda)=\sum_{r=0}^{2}\left(B_r u^{(2-r)}(0),\ R(\bar{\lambda})\right).$$

Now, using the estimates for the resolvent of pencil (1) (Theorems 2 and 3) in case 1), and Keldysh theorem [3] with applying the Phragmén-Lindelöf theorem in case 2), we find that $g(\lambda)$ is a polynomial. And since for $t > 0$

$$\frac{1}{2\pi i}\int_{\Gamma_{\pm\theta}} g(\lambda)\, e^{\lambda t} d\lambda = 0,$$

we obtain for $t > 0$

$$\sum_{k=0}^{1}\left(\frac{d^k u(t)}{dt^k},\ \xi_k\right)_{H_{5/2-k}} = 0.$$

Passing here to the limit as $t \to 0$, we have

$$\sum_{k=0}^{1}\left(\varphi_k, \xi_k\right)_{H_{5/2-k}} = 0.$$

Since the choice of the vectors φ_k, $k = 0, 1$, is arbitrary, then $\xi_k = 0$, $k = 0, 1$, and therefore $\xi = 0$. We obtain a contradiction. The theorem is proved. ◄

References

[1] J.L. Lions and E. Magenes. *Non-Homogeneous Boundary Value Problems and Applications*. Paris: Dunod, 1968 (Moscow: Mir, 1971; Berlin: Springer, 1972).

[2] M.V. Keldysh. On eigenvalues and eigenfunctions of some classes of nonself-adjoint equations. *Dokl. Akad. Nauk SSSR*, 77(1):11–14, 1951, (Russian).

[3] M.V. Keldysh. On the completeness of the eigenfunctions of some classes of nonselfadjoint linear operators. *Russian Mathematical Surveys*, 26(4):15–44, 1971 (published in *Uspekhi Mat. Nauk*, 26(4(160)):15–41, 1971).

[4] G.V. Radzievskii. The problem of the completeness of root vectors in the spectral theory of operator-valued functions. *Russian Mathematical Surveys*, 37(2):91–164, 1982 (published in *Uspekhi Mat. Nauk*, 37(2(224)):81–145, 1982).

[5] M.G. Gasymov. On the theory of polynomial operator pencils. *Soviet Mathematics. Doklady*, 12:1143–1147, 1972 (published in *Dokl. Akad. Nauk SSSR*, 199(4):747–750, 1971).

[6] M.G. Gasymov. The multiple completeness of part of the eigen- and associated vectors of polynomial operator bundles. *Izv. Akad. Nauk Arm. SSR, ser. matem.*, 6(2-3):131–147, 1971, (Russian).

[7] M.G. Gasymov. Multiple completeness with finite-dimensional deficiency of part of the eigen- and associated vectors of operator bundles. *In: Funkts. Anal., Teoriya Funkts. Prilozh.* - Mahachkala, 3(1):55–62, 1976, (Russian).

[8] S.S. Mirzoev. Conditions for the correct solvability of boundary value problems for operator-differential equations. *Soviet Mathematics. Doklady*, 28:629–632, 1983 (published in *Dokl. Akad. Nauk SSSR*, 273(2):292–295, 1983).

[9] S.S. Mirzoev. Multiple completeness of root vectors of polynomial operator pencils corresponding to boundary-value problems on the semiaxis. *Functional Analysis and Its Applications*, 17(2):151–153, 1983 (published in *Funkts. Anal. Prilozh.*, 17(2):84–85, 1983).

[10] A.A. Shkalikov. Elliptic equations in Hilbert space and associated spectral problems. *J. Soviet Math.*, 51(4):2399–2467, 1990 (published in *Trudy Sem. Petrovsk.*, 14:140–224, 1989).

[11] A.R. Aliev and A.S. Mohamed. On the well-posedness of a boundary value problem for a class of fourth-order operator-differential equations. *Differential Equations*, 48(4):596–598, 2012 (published in *Differentsial'nye Uravneniya*, 48(4):587–589, 2012).

[12] A.R. Aliev and A.L. Elbably. Well-posedness of a boundary value problem for a class of third order operator-differential equations. *Boundary Value Problems*, 2013 (140):1–15, 2013.

[13] A.L. Elbably. Properties of the resolvent of a polynomial operator pencil of third order with multiple characteristics. *News of Baku University, ser. of phys.-math. sciences*, 1:93–99, 2012.

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