Azerbaijan Journal of Mathematics
V. 4, No 1, 2014, January
ISSN 2218-6816

Direct and Inverse Problem for the Sturm–Liouville Operator with Eigenparameter-Dependent Boundary Conditions with Two Interior Discontinuities

Isa Dehghani*, Aliasghar J. Akbarfam

Abstract. In this paper, we study the Sturm–Liouville operator with two interior discontinuities and with spectral parameter linearly contained in one of the boundary conditions. Spectral properties of the eigenvalues and norming constants of this operator are investigated. Moreover, the Weyl solution and the Weyl function for this operator are defined. We prove uniqueness theorems for the solution of the inverse problem of reconstruction of the operator from the Weyl function, from the spectral data and from two spectra.

Key Words and Phrases: Discontinuous Sturm–Liouville problems, eigenparameter-dependent boundary conditions, jump conditions, Weyl function, inverse problem

2000 Mathematics Subject Classifications: 34A55, 34B24, 34L05

1. Introduction

In this paper, we will study the discontinuous Sturm–Liouville boundary value problem L consisting of the differential equation

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in J := [0, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, \pi], \tag{1.1}$$

with boundary conditions at $x = 0$ and $x = \pi$,

$$U(y) := y'(0) - hy(0) = 0, \tag{1.2}$$
$$V(y) := (\lambda - H_1)y'(\pi) + (\lambda H - H_2)y(\pi) = 0, \tag{1.3}$$

and jump conditions at the points of discontinuities $x = \xi_1$ and $x = \xi_2$,

$$l_1(y) := y(\xi_1 + 0) - \alpha_1 y(\xi_1 - 0) = 0, \tag{1.4}$$
$$l_2(y) := y'(\xi_1 + 0) - \alpha_1^{-1} y'(\xi_1 - 0) - \alpha_2 y(\xi_1 - 0) = 0, \tag{1.5}$$
$$l_3(y) := y(\xi_2 + 0) - \beta_1 y(\xi_2 - 0) = 0 \tag{1.6}$$
$$l_4(y) := y'(\xi_2 + 0) - \beta_1^{-1} y'(\xi_2 - 0) - \beta_2 y(\xi_2 - 0) = 0, \tag{1.7}$$

*Corresponding author.

where $q(x) \in L_2(0,\pi)$ is a real-valued function, $\lambda \in \mathbb{C}$ is a spectral parameter, h, H, H_1, H_2, α_i and β_i $(i=1,2)$ are real numbers; $\alpha_1 > 0$, $|\alpha_1 - 1| + |\alpha_2| > 0$, $\beta_1 > 0$ and $|\beta_1 - 1| + |\beta_2| > 0$. We assume that $\rho := HH_1 - H_2 > 0$.

Direct and inverse problems for Sturm–Liouville operators with spectral parameter linearly contained in the boundary conditions and without discontinuities has been thoroughly studied. In [14, 31] an operator-theoretic formulation of the problems of the form (1.1)–(1.3) has been given. Oscillation and comparison results have been obtained in [6, 7, 19]. Basic properties and eigenfunction expansions have been considered in [17, 20, 21, 33]. Inverse spectral problems have been investigated in [8, 9, 12, 15].

Boundary value problems with discontinuities inside the interval have been extensively studied. Sturm–Liouville problems both with eigenparameter dependent and independent boundary conditions and with discontinuities inside an interval have been considered in [1, 2, 3, 16, 23, 28, 32, 34, 35] and other works.

Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural sciences. As a rule, such problems are connected with discontinuous material properties. The inverse problem of the reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disciplines ranging from engineering to the geosciences.

Various mathematical and physical applications of discontinuous boundary value problems are found in mathematics for investigating spectral properties of some classes of differential, integrodifferential and integral operators, in the theory of heat and mass transfer, in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics, and in geophysical models for oscillations of the Earth(see [4, 25, 26]).

In this paper, we study direct and inverse problem for the discontinuous boundary value problem L. In Section 2, the operator-theoretical formulation of the problem presented. In Section 3, spectral properties of the eigenvalues and norming constants of the problem is investigated. In Section 4, we define the Weyl Solution and the Weyl function of the problem. In Section 5, uniqueness theorems for the solution of the inverse problem from the Weyl function, from the spectral data, and from two spectra are proved.

2. The operator equation formulation

In this section, we introduce a linear operator A in a suitable Hilbert space such that the considered problem L can be interpreted as the eigenvalue problem of this operator.

Let the inner product in the Hilbert space $\mathcal{H} = L_2(0,\pi) \oplus \mathbb{C}$ be defined by

$$\langle F, G\rangle = \int_0^\pi f(x)\overline{g(x)}dx + \frac{1}{\rho} f_1 \overline{g}_1,$$

where

$$F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix}, \quad G = \begin{pmatrix} g(x) \\ g_1 \end{pmatrix} \in \mathcal{H}.$$

For convenience we will use the notations

$$R(y) := H_1 y'(\pi) + H_2 y(\pi), \quad R'(y) := y'(\pi) + Hy(\pi).$$

We define an operator A acting in $\mathcal{H}$ such that

$$AF := \begin{pmatrix} \ell f \\ R(f) \end{pmatrix}$$

with

$$\begin{aligned} D(A) := \Big\{ F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in \mathcal{H} \Big| f, f' \in AC_{loc}(J), \ \text{and have finite} \\ \text{one-hand sided limits } f(\xi_i \pm 0) \text{ and } f'(\xi_i \pm 0),\ i = 1, 2, \\ \ell f \in L_2(0,\pi),\ U(f) = 0,\ l_j(f) = 0,\ j = \overline{1,4},\ f_1 = R'(f) \Big\}. \end{aligned}$$

Thus, we can pose the discontinuous boundary value problem L as

$$AY = \lambda Y, \quad Y := \begin{pmatrix} y(x) \\ R'(y) \end{pmatrix}$$

in the Hilbert space $\mathcal{H}$. It is readily verified that the eigenvalues of the operator A coincide with those of the problem L.

Theorem 2.1. *The operator A is symmetric in $\mathcal{H}$.*

Proof. First, we prove that A is densely defined in $\mathcal{H}$. For this suppose $F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \in \mathcal{H}$ is orthogonal to all $G = \begin{pmatrix} g(x) \\ R'(g) \end{pmatrix} \in D(A)$, i.e.,

$$\langle F, G \rangle = \int_0^\pi f(x)\overline{g(x)}dx + \frac{1}{\rho} f_1 \overline{R'(g)} = 0. \tag{2.1}$$

Let $\widetilde{C}_0^\infty$ denote the set of functions

$$\phi(x) = \begin{cases} \phi_1(x), & x \in [0, \xi_1), \\ \phi_2(x), & x \in (\xi_1, \xi_2), \\ \phi_3(x), & x \in (\xi_2, \pi], \end{cases}$$

where $\phi_1(x) \in C_0^\infty[0, \xi_1)$, $\phi_2(x) \in C_0^\infty(\xi_1, \xi_2)$ and $\phi_3(x) \in C_0^\infty(\xi_2, \pi]$. Since $\widetilde{C}_0^\infty \oplus 0 \subseteq D(A)$ $(0 \in \mathbb{C})$, then any $G = \begin{pmatrix} g(x) \\ 0 \end{pmatrix} \in \widetilde{C}_0^\infty \oplus 0$ is orthogonal to F, namely

$$\langle F, G \rangle = \int_0^\pi f(x)\overline{g(x)}dx = 0.$$

Consequently, $f(x)$ vanishes, since $L_2(0,\pi)$ is complete with respect to its standard inner product. Then substituting $f(x)=0$ into (2.1) yields

$$\frac{1}{\rho}f_1\overline{R'(g)}=0$$

for all $G=\begin{pmatrix} g(x) \\ R'(g) \end{pmatrix}\in D(A)$. Since $R'(g)$ can be chosen arbitrary, hence $f_1=0$. Therefore, $F=0$, so $D(A)$ is dense in $\mathcal{H}$.
We prove that A is symmetric. Let

$$F=\begin{pmatrix} f(x) \\ R'(f) \end{pmatrix},\quad G=\begin{pmatrix} g(x) \\ R'(g) \end{pmatrix}$$

be arbitrary elements of $D(A)$. By twice integration by parts we get

$$\begin{aligned}\langle AF,G\rangle &= \langle F,AG\rangle - W(f,\overline{g};0)+W(f,\overline{g};\xi_1-0)-W(f,\overline{g};\xi_1+0)\\ &\quad +W(f,\overline{g};\xi_2-0)-W(f,\overline{g};\xi_2+0)\\ &\quad +W(f,\overline{g};\pi)-\frac{1}{\rho}(R(f)\overline{R'(g)}-R'(f)\overline{R(g)}),\end{aligned} \tag{2.2}$$

where as usual, $W(f,g;x)$ denotes the Wronskians $f(x)g'(x)-f'(x)g(x)$. Since $F,G\in D(A)$, it follows from (1.2) that

$$W(f,\overline{g};0)=0, \tag{2.3}$$

and from (1.4)–(1.7), we get

$$W(f,\overline{g};\xi_i-0)=W(f,\overline{g};\xi_i+0),\quad i=1,2. \tag{2.4}$$

Moreover, the direct calculations gives

$$\rho W(f,\overline{g};\pi)=R(f)\overline{R'(g)}-R'(f)\overline{R(g)}. \tag{2.5}$$

Now, inserting (2.3)–(2.5) into (2.2), yields the required equality

$$\langle AF,G\rangle=\langle F,AG\rangle,\quad F,G\in D(A).$$

So A is symmetric.◀

Corollary 2.2. *All eigenvalues of the problem L are real.*

We can now assume that all eigenfunctions of the problem L are real-valued.

Corollary 2.3. *If λ_1 and λ_2 are two different eigenvalues of the problem L, then corresponding eigenfunctions y_1 and y_2 of this problem are orthogonal in the following sense:*

$$\int_0^\pi y_1(x)y_2(x)dx+\frac{1}{\rho}R'(y_1)R'(y_2)=0.$$

3. Properties of the spectrum

In this section, properties of the spectrum of the discontinuous problem L will be investigated.

For what follows we need the following lemma, which can be proved similar to [30, Theorem 2].

Lemma 3.1. *Let $q(x) \in L_2(a,b)$, $a,b \in \mathbb{R}$, be a real-valued function and $f(\lambda)$, $g(\lambda)$ be given entire functions. Then for any $\lambda \in \mathbb{C}$ the equation*

$$-y'' + q(x)y = \lambda y, \quad x \in [a,b]$$

has a unique solution $y = y(x,\lambda)$ satisfying the initial conditions

$$y(a) = f(\lambda), \quad y'(a) = g(\lambda) \quad (or\, y(b) = f(\lambda), \quad y'(b) = g(\lambda)).$$

For each fixed $x \in [a,b]$, $y(x,\lambda)$ is an entire function of λ.

We shall define two solutions

$$\varphi(x,\lambda) = \begin{cases} \varphi_1(x,\lambda), & x \in [0,\xi_1), \\ \varphi_2(x,\lambda), & x \in (\xi_1,\xi_2), \\ \varphi_3(x,\lambda), & x \in (\xi_2,\pi] \end{cases}$$

and

$$\psi(x,\lambda) = \begin{cases} \psi_1(x,\lambda), & x \in [0,\xi_1), \\ \psi_2(x,\lambda), & x \in (\xi_1,\xi_2), \\ \psi_3(x,\lambda), & x \in (\xi_2,\pi] \end{cases}$$

of equation (1.1) as follows:
Let $\varphi_1(x,\lambda)$ be the solution of equation (1.1) on the interval $[0,\xi_1)$ satisfying the initial conditions

$$\varphi_1(0,\lambda) = 1, \quad \varphi_1'(0,\lambda) = h. \tag{3.1}$$

By virtue of Lemma 3.1, after defining this solution we can define the solution $\varphi_2(x,\lambda)$ of equation (1.1) on (ξ_1,ξ_2) by the nonstandard initial conditions

$$\varphi_2(\xi_1+0,\lambda) = \alpha_1\varphi_1(\xi_1-0,\lambda), \quad \varphi_2'(\xi_1+0,\lambda) = \alpha_1^{-1}\varphi_1'(\xi_1-0,\lambda) + \alpha_2\varphi_1(\xi_1-0,\lambda). \tag{3.2}$$

After defining this solution we can define the solution $\varphi_3(x,\lambda)$ of equation (1.1) on $(\xi_2,\pi]$ by the nonstandard initial conditions

$$\varphi_3(\xi_2+0,\lambda) = \beta_1\varphi_2(\xi_2-0,\lambda), \quad \varphi_3'(\xi_2+0,\lambda) = \beta_1^{-1}\varphi_2'(\xi_2-0,\lambda) + \beta_2\varphi_2(\xi_2-0,\lambda). \tag{3.3}$$

Obviously $\varphi(x,\lambda)$ satisfies equation (1.1) on J, the boundary condition (1.2) and the jump conditions (1.4)–(1.7).

Analogously first we define the solution $\psi_3(x,\lambda)$ on $(\xi_2,\pi]$ by the initial conditions

$$\psi_3(\pi,\lambda) = \lambda - H_1, \quad \psi_3'(\pi,\lambda) = -\lambda H + H_2. \tag{3.4}$$

Again, after defining this solution we define the solution $\psi_2(x,\lambda)$ of equation (1.1) on (ξ_1,ξ_2) by the nonstandard initial conditions

$$\psi_2(\xi_2-0,\lambda)=\beta_1^{-1}\psi_3(\xi_2+0,\lambda),\quad \psi_2'(\xi_2-0,\lambda)=\beta_1\psi_3'(\xi_2+0,\lambda)-\beta_2\psi_3(\xi_2+0,\lambda). \tag{3.5}$$

Using this solution, we define the solution $\psi_1(x,\lambda)$ of equation (1.1) on $[0,\xi_1)$ by the nonstandard initial conditions

$$\psi_1(\xi_1-0,\lambda)=\alpha_1^{-1}\psi_2(\xi_1+0,\lambda),\quad \psi_1'(\xi_1-0,\lambda)=\alpha_1\psi_2'(\xi_1+0,\lambda)-\alpha_2\psi_2(\xi_1+0,\lambda). \tag{3.6}$$

It is clear that $\psi(x,\lambda)$ satisfies equation (1.1), the boundary condition (1.3) and the jump conditions (1.4)–(1.7).

For any solution $y(x,\lambda)$ of equation (1.1) we shall use the notation

$$y_\lambda(x):=y(x,\lambda).$$

Let us consider the Wronskians

$$\chi_i(\lambda):=W(\psi_{i\lambda},\varphi_{i\lambda};x),\quad x\in\Omega_i,\quad i=\overline{1,3}, \tag{3.7}$$

where $\Omega_1=[0,\xi_1)$, $\Omega_2=(\xi_1,\xi_2)$ and $\Omega_3=(\xi_2,\pi]$. By virtue of Liouville's formula for the Wronakian (see [10, p. 83]), $\chi_i(\lambda)$ $(i=\overline{1,3})$ are independent of $x\in\Omega_i$ $(i=\overline{1,3})$. In view of (3.2), (3.3), (3.5) and (3.6), a short calculation gives

$$W(\psi_{1\lambda},\varphi_{1\lambda};\xi_1-0)=W(\psi_{2\lambda},\varphi_{2\lambda};\xi_1+0)=W(\psi_{2\lambda},\varphi_{2\lambda};\xi_2-0)=W(\psi_{3\lambda},\varphi_{3\lambda};\xi_2+0),$$

so, $\chi_1(\lambda)=\chi_2(\lambda)=\chi_3(\lambda)$ for each $\lambda\in\mathbb{C}$.

Now we may introduce the characteristic function

$$\Delta(\lambda):=\chi_3(\lambda). \tag{3.8}$$

Clearly

$$\Delta(\lambda)=V(\varphi_\lambda)=-U(\psi_\lambda). \tag{3.9}$$

It follows from Lemma 3.1 that $\Delta(\lambda)$ is an entire function of λ and it has an at most countable set of zeros $\{\lambda_n\}$.

Theorem 3.2. *The zeros $\{\lambda_n\}$ of the characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem L. The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions and there exists a sequence $\{k_n\}$ such that*

$$\psi(x,\lambda_n)=k_n\varphi(x,\lambda_n),\quad k_n\neq 0. \tag{3.10}$$

Proof. Let λ_0 be a zero of $\Delta(\lambda)$. Then from (3.7) and (3.8) we have $W(\psi_{1\lambda_0},\varphi_{1\lambda_0};x)=0$ for all $x\in\Omega_1$, and therefore, the functions $\varphi_1(x,\lambda_0)$ and $\psi_1(x,\lambda_0)$ are linearly dependent, i.e.,

$$\psi_1(x,\lambda_0)=k_0^{(1)}\varphi_1(x,\lambda_0),\quad x\in\Omega_1$$

for some $k_0^{(1)} \neq 0$. Consequently, $\psi(x,\lambda_0)$ satisfies also the boundary condition (1.2) and hence $\psi(x,\lambda_0)$ is an eigenfunction for the eigenvalue λ_0.

Conversely, let λ_0 be an eigenvalue of L and let $y(x,\lambda_0)$ be a corresponding eigenfunction, but $\Delta(\lambda_0) \neq 0$. Then it follows from (3.7) and (3.8) that the pairs of functions $(\psi_{1\lambda_0}, \varphi_{1\lambda_0})$, $(\psi_{2\lambda_0}, \varphi_{2\lambda_0})$ and $(\psi_{3\lambda_0}, \varphi_{3\lambda_0})$ are linearly independent on $[0,\xi_1)$, (ξ_1,ξ_2) and $(\xi_2,\pi]$, respectively. Therefore, $y(x,\lambda_0)$ can be represented as follows:

$$y(x,\lambda_0) = \begin{cases} c_1\psi_{1\lambda_0}(x) + c_2\varphi_{1\lambda_0}(x), & x \in [0,\xi_1), \\ c_3\psi_{2\lambda_0}(x) + c_4\varphi_{2\lambda_0}(x), & x \in (\xi_1,\xi_2), \\ c_5\psi_{3\lambda_0}(x) + c_6\varphi_{3\lambda_0}(x), & x \in (\xi_2,\pi], \end{cases}$$

where at least one of the constants c_i $(i = \overline{1,6})$ is not zero. Since $y(x,\lambda_0)$ is an eigenfunction, then the equations

$$\begin{cases} U(y_{\lambda_0}) = 0, \\ V(y_{\lambda_0}) = 0, \\ l_j(y_{\lambda_0}) = 0, \quad j = \overline{1,4} \end{cases} \tag{3.11}$$

can be considered as a homogenous system of linear equations of the variables c_i $(i = \overline{1,6})$. It follows from (3.1)–(3.9) that the determinant of this system is

$$\begin{vmatrix} 0 & -\Delta(\lambda_0) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta(\lambda_0) & 0 \\ -\varphi_{2\lambda_0}(\xi_1+0) & -\psi_{2\lambda_0}(\xi_1+0) & \varphi_{2\lambda_0}(\xi_1+0) & \psi_{2\lambda_0}(\xi_1+0) & 0 & 0 \\ -\varphi'_{2\lambda_0}(\xi_1+0) & -\psi'_{2\lambda_0}(\xi_1+0) & \varphi'_{2\lambda_0}(\xi_1+0) & \psi'_{2\lambda_0}(\xi_1+0) & 0 & 0 \\ 0 & 0 & -\varphi_{3\lambda_0}(\xi_2+0) & -\psi_{3\lambda_0}(\xi_2+0) & \varphi_{3\lambda_0}(\xi_2+0) & \psi_{3\lambda_0}(\xi_2+0) \\ 0 & 0 & -\varphi'_{3\lambda_0}(\xi_2+0) & -\psi'_{3\lambda_0}(\xi_2+0) & \varphi'_{3\lambda_0}(\xi_2+0) & \psi'_{3\lambda_0}(\xi_2+0) \end{vmatrix}$$

$$= -\Delta^4(\lambda_0) \neq 0.$$

Therefore, the system (3.11) has only the trivial solution $c_i = 0$ $(i = \overline{1,6})$, which is a contradiction. Thus, $\Delta(\lambda_0) = 0$.

Now let λ_0 be an eigenvalue. It follows from (3.7) and (3.8) that

$$\chi_i(\lambda_0) = W(\psi_{i\lambda_0}, \varphi_{i\lambda_0}; x) = 0, \quad x \in \Omega_i, \quad i = \overline{1,3},$$

and therefore,

$$\psi_i(x,\lambda_0) = k_0^{(i)}\varphi_i(x,\lambda_0), \quad x \in \Omega_i, \quad i = \overline{1,3} \tag{3.12}$$

for some $k_0^{(i)} \neq 0$ $(i = \overline{1,3})$. From (3.12) we conclude that $\psi(x,\lambda_0)$ and $\varphi(x,\lambda_0)$ satisfies also the boundary conditions (1.2) and (1.3), respectively, and hence $\varphi(x,\lambda_0)$ and $\psi(x,\lambda_0)$ are eigenfunctions. We show that $k_0^{(1)} = k_0^{(2)} = k_0^{(3)}$. Suppose, if possible that $k_0^{(1)} \neq k_0^{(2)}$. Using (3.1)–(3.6) and (3.12), we have

$$\begin{aligned} (k_0^{(1)} - k_0^{(2)})\varphi_2(\xi_1+0,\lambda_0) &= k_0^{(1)}(\alpha_1\varphi_1(\xi_1-0,\lambda_0) + \alpha_1^{-1}\varphi_1'(\xi_1-0,\lambda_0)) \\ &\quad -k_0^{(2)}\varphi_2(\xi_1+0,\lambda_0) \\ &= \alpha_1\psi_1(\xi_1-0,\lambda_0) + \alpha_1^{-1}\psi_1'(\xi_1-0,\lambda_0) - \psi_2(\xi_1+0,\lambda_0) \end{aligned}$$

$$= 0.$$

Hence

$$\varphi_2(\xi_1 + 0, \lambda_0) = 0. \tag{3.13}$$

Analogously, starting from $(k_0^{(1)} - k_0^{(2)})\varphi_2'(\xi_1 + 0, \lambda_0)$ and following the same procedure, one can derive that

$$\varphi_2'(\xi_1 + 0, \lambda_0) = 0. \tag{3.14}$$

Since $\varphi_2(x, \lambda_0)$ is a solution of equation (1.1) on (ξ_1, ξ_2) and satisfies the initial conditions (3.13) and (3.14), it follows that $\varphi_2(x, \lambda_0) = 0$ identically on (ξ_1, ξ_2). taking this into account and using (1.4) and (1.5) we get

$$\varphi_1(\xi_1 - 0, \lambda_0) = \varphi_1'(\xi_1 - 0, \lambda_0) = 0. \tag{3.15}$$

Also making use of (3.3) we obtain

$$\varphi_3(\xi_2 + 0, \lambda_0) = \varphi_3'(\xi_2 + 0, \lambda_0) = 0. \tag{3.16}$$

From (3.15) and (3.16) by the same argument as for $\varphi_2(x, \lambda_0)$ it follows that $\varphi_1(x, \lambda_0) = 0$ identically on $[0, \xi_1)$ and $\varphi_3(x, \lambda_0) = 0$ identically on $(\xi_2, \pi]$. Hence $\varphi(x, \lambda_0) = 0$ identically on J. However, this contradicts (3.1). Thus, $k_0^{(1)} = k_0^{(2)}$. in the same way one can show that $k_0^{(2)} = k_0^{(3)}$. Consequently,

$$\psi(x, \lambda_0) = k_0\varphi(x, \lambda_0), \quad x \in J$$

for some $k_0 \neq 0$. This completes the proof of Theorem 3.2.◄

Recall that the set of eigenvalues $\{\lambda_n\}$ of the problem L coincide with the set of eigenvalues of the operators A. It is easy to show that

$$\Phi_n := \begin{pmatrix} \varphi(x, \lambda_n) \\ R'(\varphi_{\lambda_n}) \end{pmatrix}$$

are eigenelements of A. Here we define norming constants of the problem L by

$$\gamma_n := \|\Phi_n\|^2_{\mathcal{H}} = \int_0^\pi \varphi^2(x, \lambda_n)dx + \frac{1}{\rho}(R'(\varphi_{\lambda_n}))^2. \tag{3.17}$$

The numbers $\{\lambda_n, \gamma_n\}_{n\geq 0}$ are called the spectral data of the problem L.

Lemma 3.3. *The following relation holds:*

$$\dot{\Delta}(\lambda_n) = -k_n\gamma_n, \tag{3.18}$$

where the numbers k_n are defined by (3.10) *and* $\dot{\Delta}(\lambda) = d/d\lambda(\Delta(\lambda))$.

Proof. Using the Lagrange identity (see [29, Part II, p. 50]) for solutions $\varphi(x,\lambda)$ and $\varphi(x,\lambda_n)$, and taking into account (2.3) and (2.4) we get

$$\begin{aligned}\int_0^\pi \varphi(x,\lambda)\varphi(x,\lambda_n)dx &= \frac{W(\varphi_\lambda,\varphi_{\lambda_n};\pi)}{\lambda-\lambda_n}\\ &= \frac{W(\varphi_\lambda,\psi_{\lambda_n};\pi)}{k_n(\lambda-\lambda_n)}\\ &= \frac{R(\varphi_\lambda)-\lambda_n R'(\varphi_\lambda)}{k_n(\lambda-\lambda_n)}\\ &= \frac{(\lambda-\lambda_n)R'(\varphi_\lambda)-\Delta(\lambda)}{k_n(\lambda-\lambda_n)}.\end{aligned}$$

For $\lambda \to \lambda_n$, this yields

$$\begin{aligned}\dot{\Delta}(\lambda_n) &= -k_n\int_0^\pi \varphi^2(x,\lambda_n)dx - R'(\varphi_{\lambda_n})\\ &= -k_n\left(\gamma_n - \frac{1}{\rho}(R'(\varphi_{\lambda_n}))^2\right) - R'(\varphi_{\lambda_n}). \end{aligned}\tag{3.19}$$

Now putting $R'(\varphi_{\lambda_n}) = (1/k_n)R'(\psi_{\lambda_n}) = \rho/k_n$ in (3.19) we get (3.18).◀

Definition 3.4. *The algebraic multiplicity of an eigenvalue λ of the problem L is the order of it as a zero of the characteristic function $\Delta(\lambda)$. The geometric multiplicity of an eigenvalue λ is the dimension of its eigenspace, i.e., the number of its linearly independent eigenfunctions.*

Theorem 3.5. *The eigenvalues of the problem L are algebraically and geometrically simple.*

Proof. Let λ_0 be an eigenvalue of the problem L. By virtue of Lemma 3.3, we have $\dot{\Delta}(\lambda_0) \neq 0$, and hence λ_0 is algebraically simple.

Let us show that λ_0 is geometrically simple. Suppose on the contrary that there are two linearly independent eigenfunctions $y_1(x)$ and $y_2(x)$ corresponding to the same eigenvalue λ_0. Since $y_1(x)$ and $y_2(x)$ satisfy (1.2), we have $W(y_1,y_2;0)$=0. Therefore, $y_1(x)$ and $y_2(x)$ are linearly dependent which is a contradiction. This completes the proof of Theorem 3.5.◀

Lemma 3.6. *For $|s| \to \infty$, the following asymptotic formulae hold:*

$$\frac{d^k}{dx^k}\varphi_1(x,\lambda) = \frac{d^k}{dx^k}\cos sx + O\left(|s|^{k-1}e^{|\tau|x}\right), \quad k=0,1, \tag{3.20}$$

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_2(x,\lambda) &= \frac{d^k}{dx^k}\left(\alpha^+\cos sx + \alpha^-\cos s(2\xi_1 - x)\right)\\ &\quad +O\left(|s|^{k-1}e^{|\tau|x}\right), \quad k=0,1,\end{aligned}\tag{3.21}$$

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_3(x,\lambda) \;=\; & \frac{d^k}{dx^k}\Big(\alpha^+\beta^+\cos sx+\alpha^-\beta^+\cos s(2\xi_1-x)\\ & +\alpha^+\beta^-\cos s(2\xi_2-x)+\alpha^-\beta^-\cos s(2\xi_1-2\xi_2+x)\Big)\\ & +O\left(|s|^{k-1}e^{|\tau|x}\right),\quad k=0,1,\end{aligned}\tag{3.22}$$

uniformly with respect to $x\in\Omega_i$ *(*$i=\overline{1,3}$*). Here and in the sequel* $s=\sqrt{\lambda}$ *is the principle branch,* $\tau=\operatorname{Im}s$*, and*

$$\alpha^\pm=\frac{1}{2}\left(\alpha_1\pm\frac{1}{\alpha_1}\right),\quad \beta^\pm=\frac{1}{2}\left(\beta_1\pm\frac{1}{\beta_1}\right).\tag{3.23}$$

Proof. Let us show that

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_1(x,\lambda) \;=\; & \frac{d^k}{dx^k}\cos sx+\frac{h}{s}\frac{d^k}{dx^k}\sin sx\\ & +\frac{1}{s}\int_0^x\frac{d^k}{dx^k}\sin s(x-t)q(t)\varphi_1(t,\lambda)dt,\quad x\in\Omega_1,\ k=0,1,\end{aligned}\tag{3.24}$$

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_2(x,\lambda) \;=\; & \alpha_1\varphi_1(\xi_1-0,\lambda)\frac{d^k}{dx^k}\cos s(x-\xi_1)\\ & +\frac{1}{s}\Big(\alpha_1^{-1}\varphi_1'(\xi_1-0,\lambda)+\alpha_2\varphi_1(\xi_1-0,\lambda)\Big)\frac{d^k}{dx^k}\sin s(x-\xi_1)\\ & +\frac{1}{s}\int_{\xi_1}^x\frac{d^k}{dx^k}\sin s(x-t)q(t)\varphi_2(t,\lambda)dt,\quad x\in\Omega_2,\ k=0,1,\end{aligned}\tag{3.25}$$

$$\begin{aligned}\frac{d^k}{dx^k}\varphi_3(x,\lambda) \;=\; & \beta_1\varphi_1(\xi_1-0,\lambda)\frac{d^k}{dx^k}\cos s(x-\xi_2)\\ & +\frac{1}{s}\Big(\beta_1^{-1}\varphi_2'(\xi_2-0,\lambda)+\beta_2\varphi_2(\xi_2-0,\lambda)\Big)\frac{d^k}{dx^k}\sin s(x-\xi_2)\\ & +\frac{1}{s}\int_{\xi_2}^x\frac{d^k}{dx^k}\sin s(x-t)q(t)\varphi_3(t,\lambda)dt,\quad x\in\Omega_3,\ k=0,1.\end{aligned}\tag{3.26}$$

Since $\varphi_i(t,\lambda)(i=\overline{1,3})$ satisfy (1.1), we have

$$q(t)\varphi_i(t,\lambda)=\varphi_i''(t,\lambda)+s^2\varphi_i(t,\lambda),\quad t\in\Omega_i,\ i=\overline{1,3}.\tag{3.27}$$

Substituting right-hand side of these equalities in the integrals in (3.24)–(3.26) and twice integrating by parts the term involving $\varphi_i''(t,\lambda)$, we obtain (3.24)–(3.26).

Using (3.24), the asymptotic formulae for $\varphi_1(x,\lambda)$ can be found in the same way as in [13, Lemma 1.1.2]. Therefore, we shall formulate them without proof. Let us prove (3.21). Using (3.20) we have

$$\begin{aligned}\varphi_1(\xi_1-0,\lambda) \;=\; & \cos s\xi_1+O\left(|s|^{-1}e^{|\tau|\xi_1}\right),\\ \varphi_1'(\xi_1-0,\lambda) \;=\; & -s\sin s\xi_1+O\left(e^{|\tau|\xi_1}\right).\end{aligned}$$

Substituting these asymptotic expressions into (3.25) we obtain

$$\begin{aligned}\varphi_2(x,\lambda) \;=\; & \alpha^+ \cos sx + \alpha^- \cos s(2\xi_1 - x) \\ & + \frac{1}{s}\int_{\xi_1}^{x} \sin s(x-t) q(t)\varphi_2(t,\lambda)dt + O\left(|s|^{-1} e^{|\tau|x}\right). \end{aligned} \tag{3.28}$$

Multiplying through by $e^{-|\tau|x}$ and denoting $f(x,\lambda) := \varphi_2(x,\lambda)e^{-|\tau|x}$, we have

$$\begin{aligned}f(x,\lambda) \;=\; & \Big(\alpha^+ \cos sx + \alpha^- \cos s(2\xi_1 - x)\Big)e^{-|\tau|x} \\ & + \frac{1}{s}\int_{\xi_1}^{x} \sin s(x-t) e^{-|\tau|x} q(t) f(t,\lambda)dt + O\left(|s|^{-1}\right). \end{aligned}$$

Let $\mu(\lambda) = \sup\limits_{x\in\Omega_2} |f(x,\lambda)|$. Then using the inequalities

$$\begin{gathered} |\cos sx| \le e^{|\tau|x}, \quad |\cos s(2\xi_1 - x)| \le e^{|\tau|x}, \quad x \in \Omega_2, \\ |\sin s(x-t)| \le e^{|\tau|x}, \quad x\in\Omega_2,\ t \in (\xi_1, x] \end{gathered}$$

we obtain

$$\mu(\lambda) \le \alpha^+ + |\alpha^-| + \frac{1}{|s|}\mu(\lambda)\int_{\xi_1}^{\xi_2} |q(t)|dt + \frac{\mu_0}{|s|}$$

for some $\mu_0 > 0$. For sufficiently large values of $|s|$ this gives

$$\mu(\lambda) \le C\left(1 - \frac{\int_{\xi_1}^{\xi_2}|q(t)|dt}{|s|}\right)^{-1}.$$

Hence $|f(x,\lambda) \le \mu(\lambda)| = O(1)$, as $|s| \to \infty$, and therefore $\varphi_2(x,\lambda) = O\left(e^{|\tau|x}\right)$, uniformly with respect to $x \in \Omega_2$, as $|s| \to \infty$. Substituting this estimate into the right-hand side of (3.28), we get (3.21). The proof of (3.22) is similar to that of (3.21) and hence is omitted.◀

Similarly one can establish the following lemma for $\psi_i(x,\lambda)$ $(i = \overline{1,3})$:

Lemma 3.7. *For $|s| \to \infty$, the following asymptotic formulae hold:*

$$\begin{aligned}\frac{d^k}{dx^k}\psi_1(x,\lambda) \;=\; & s^2 \frac{d^k}{dx^k}\Big(\alpha^+\beta^+ \cos s(\pi - x) - \alpha^-\beta^+ \cos s(\pi - 2\xi_1 + x) \\ & -\alpha^+\beta^- \cos s(\pi - 2\xi_2 + x) + \alpha^-\beta^- \cos s(\pi - 2\xi_2 + 2\xi_1 - x)\Big) \\ & + O\left(|s|^{k+1} e^{|\tau|(\pi - x)}\right), \quad k = 0, 1, \end{aligned} \tag{3.29}$$

$$\begin{aligned}\frac{d^k}{dx^k}\psi_2(x,\lambda) \;=\; & s^2 \frac{d^k}{dx^k}\Big(\beta^+ \cos s(\pi - x) - \beta^- \cos s(\pi - 2\xi_2 + x)\Big) \\ & + O\left(|s|^{k+1} e^{|\tau|(\pi - x)}\right), \quad k = 0, 1, \end{aligned} \tag{3.30}$$

$$\frac{d^k}{dx^k}\psi_3(x,\lambda) \;=\; s^2 \frac{d^k}{dx^k}\cos s(\pi - x) + O\left(|s|^{k+1} e^{|\tau|(\pi - x)}\right) \quad k = 0, 1, \tag{3.31}$$

uniformly with respect to $x \in \Omega_i$ $(i = \overline{1,3})$.

For what follows we need to study the spectral properties of the discontinuous eigenvalue problem L_0 for the equation:

$$\ell_0 y := -y'' = \lambda y, \quad x \in J, \tag{3.32}$$

with the boundary conditions

$$U_0(y) := y'(0) = 0, \quad V_0(y) := y'(\pi) = 0, \tag{3.33}$$

and with the jump conditions

$$\begin{aligned} l_{01}(y) &:= y(\xi_1+0) - \alpha_1 y(\xi_1-0) = 0, && (3.34)\\ l_{02}(y) &:= y'(\xi_1+0) - \alpha_1^{-1} y'(\xi_1-0) = 0, && (3.35)\\ l_{03}(y) &:= y(\xi_2+0) - \beta_1 y(\xi_2-0) = 0, && (3.36)\\ l_{04}(y) &:= y'(\xi_2+0) - \beta_1^{-1} y'(\xi_2-0) = 0. && (3.37) \end{aligned}$$

Let $\varphi_0(x,\lambda)$ and $\psi_0(x,\lambda)$ be the solutions of (3.32) satisfying the initial conditions

$$\varphi_0(0,\lambda) = \psi_0(\pi,\lambda) = 1, \quad \varphi_0'(0,\lambda) = \psi_0'(\pi,\lambda) = 0. \tag{3.38}$$

Then we have

$$\varphi_0(x,\lambda) = \begin{cases} \cos sx, \quad x \in \Omega_1, \\ \alpha^+ \cos sx + \alpha^- \cos s(2\xi_1 - x), \quad x \in \Omega_2, \\ \alpha^+\beta^+ \cos sx + \alpha^-\beta^+ \cos s(2\xi_1 - x) \\ +\alpha^+\beta^- \cos s(2\xi_2 - x) + \alpha^-\beta^- \cos s(2\xi_1 - 2\xi_2 + x), \quad x \in \Omega_3, \end{cases} \tag{3.39}$$

$$\psi_0(x,\lambda) = \begin{cases} \alpha^+\beta^+ \cos s(\pi - x) - \alpha^-\beta^+ \cos s(\pi - 2\xi_1 + x) \\ -\alpha^+\beta^- \cos s(\pi - 2\xi_2 + x) - \alpha^-\beta^- \cos s(\pi - 2\xi_2 + 2\xi_1 - x), \quad x \in \Omega_1, \\ \beta^+ \cos s(\pi - x) - \beta^- \cos s(\pi - 2\xi_2 + x), \quad x \in \Omega_2, \\ \cos s(\pi - x), \quad x \in \Omega_3. \end{cases} \tag{3.40}$$

Let

$$\begin{aligned} \Delta_0(\lambda) := -s\Big(&\alpha^+\beta^+ \sin s\pi - \alpha^-\beta^+ \sin s(2\xi_1 - \pi) \\ &-\alpha^+\beta^- \sin s(2\xi_2 - \pi) + \alpha^-\beta^- \sin s(2\xi_1 - 2\xi_2 + \pi)\Big). \end{aligned} \tag{3.41}$$

Clearly $\Delta_0(\lambda) = V_0(\varphi_{0\lambda})$. Analogous to the problem L, one can show that the zeros $\{\lambda_n^0 = (s_n^0)^2\}_{n\geq 0}$ of the entire function $\Delta_0(\lambda)$ coincide with the eigenvalues of the problem L_0; the functions $\varphi_0(x,\lambda_n^0)$ and $\psi_0(x,\lambda_n^0)$ are eigenfunctions and there exists a sequence $\{k_n^0\}_{n\geq 0}$ such that

$$\psi_0(x,\lambda_n^0) = k_n^0 \varphi_0(x,\lambda_n^0), \quad k_n^0 \neq 0. \tag{3.42}$$

Also, using the same techniques as in the problem L, we can prove that the zeros of $\Delta_0(\lambda)$ are real and eigenfunctions related to different eigenvalues are orthogonal in the Hilbert space $L_2(0,\pi)$. Denote norming constants of the problem L_0 by

$$\gamma_n^0 = \int_0^\pi \varphi_0^2(x,\lambda_n^0)dx. \tag{3.43}$$

Then using (3.39) we calculate

$$\gamma_n^0 = \hat{\gamma}_n^0 + \frac{\upsilon_n^0}{s_n^0}, \tag{3.44}$$

where

$$\begin{aligned}
\hat{\gamma}_n^0 &= \frac{\xi_1}{2} + \left(\frac{(\alpha^+)^2}{2} + \frac{(\alpha^-)^2}{2} + \alpha^+\alpha^- \cos 2s_n^0\xi_1\right)(\xi_2 - \xi_1) \\
&+\Bigg(\frac{(\alpha^+\beta^+)^2}{2} + \frac{(\alpha^-\beta^+)^2}{2} + \frac{(\alpha^+\beta^-)^2}{2} + \frac{(\alpha^-\beta^-)^2}{2} \\
&+\alpha^+\alpha^-\left((\beta^+)^2 + (\beta^-)^2\right)\cos 2s_n^0\xi_1 + (\alpha^+)^2\beta^+\beta^- \cos 2s_n^0\xi_2 \\
&+2\alpha^+\alpha^-\beta^+\beta^- \cos 2s_n^0(\xi_1 - \xi_2) + (\alpha^-)^2\beta^+\beta^- \cos 2s_n^0(2\xi_1 - \xi_2)\Bigg)(\pi - \xi_2)
\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}
\upsilon_n^0 &= \frac{(\alpha^+\beta^+)^2}{4}\sin 2s_n^0\pi - \frac{\alpha^+\alpha^-(\beta^+)^2}{2}\sin 2s_n^0(\xi_1 - \pi) \\
&-\frac{(\alpha^-\beta^+)^2}{4}\sin 2s_n^0(2\xi_1 - \pi) - \frac{1}{2}\beta^+\beta^-((\alpha^+)^2 + (\alpha^-)^2)\sin 2s_n^0(\xi_2 - \pi) \\
&-\frac{(\alpha^+\beta^-)^2}{4}\sin 2s_n^0(2\xi_2 - \pi) - \frac{1}{2}\alpha^+\alpha^-\beta^+\beta^- \sin 2s_n^0(\xi_1 + \xi_2 - \pi) \\
&+\frac{1}{2}\alpha^+\alpha^-\beta^+\beta^- \sin 2s_n^0(\xi_1 - \xi_2 + \pi) + \frac{1}{2}\alpha^+\alpha^-(\beta^-)^2 \sin 2s_n^0(\xi_1 - 2\xi_2 + \pi) \\
&+\frac{(\alpha^-\beta^-)^2}{4}\sin 2s_n^0(2\xi_1 - 2\xi_2 + \pi).
\end{aligned} \tag{3.46}$$

Similar to (3.18) one can get the following equality:

$$\dot{\Delta}_0(\lambda_n^0) = -k_n^0\gamma_n^0. \tag{3.47}$$

This shows that $\dot{\Delta}(\lambda_n^0) \neq 0$ for all $n \geq 0$, i.e., the zeros of $\Delta_0(\lambda)$ are simple. Using the study [18] (see also [22]), we obtain

$$s_n^0 = \sqrt{\lambda_n^0} = n + \eta_n, \quad \{\eta_n\}_{n\geq 0} \in l_\infty. \tag{3.48}$$

In the same way as [1, Lemma 1] we can prove the following lemma:

Lemma 3.8. *The sequence* $\{s_n^0\}_{n\ge 0}$ *is separated, i.e.,*

$$d := \inf_{n\neq m} |s_n^0 - s_m^0| > 0. \tag{3.49}$$

Theorem 3.9. *The discontinuous boundary value problem L has a countable set of eigenvalues* $\{\lambda_n\}_{n\ge 0}$. *Moreover, for* $n \ge 0$,

$$s_n := \sqrt{\lambda_n} = s_{n-1}^0 + \frac{\omega_n}{n} + \frac{\zeta_n}{n}, \qquad \{\zeta_n\}_{n\ge 0} \in l_2, \tag{3.50}$$

where

$$\begin{aligned} \omega_n \;&=\; -\Big(w_1 \cos s_{n-1}^0 \pi + w_2 \cos s_{n-1}^0 (2\xi_1 - \pi) + w_3 \cos s_{n-1}^0 (2\xi_2 - \pi) \\ &\quad + w_4 \cos s_{n-1}^0 (2\xi_1 - 2\xi_2 + \pi)\Big) / \left(2\dot{\Delta}_0(\lambda_{n-1}^0)\right), \qquad (3.51) \\ w_1 \;&=\; \alpha^+\beta^+ \left(H + h + \frac{1}{2}\int_0^\pi q(t)dt\right) + \frac{1}{2}\left(\alpha^+\beta_2 + \beta^+\alpha_2\right), \qquad (3.52) \\ w_2 \;&=\; \alpha^-\beta^+ \left(H - h + \frac{1}{2}\int_0^\pi q(t)dt - \int_0^{\xi_1} q(t)dt\right) + \frac{1}{2}\left(\alpha^-\beta_2 + \beta^+\alpha_2\right), \qquad (3.53) \\ w_3 \;&=\; \alpha^+\beta^- \left(H - h - \frac{1}{2}\int_0^\pi q(t)dt + \int_{\xi_2}^{\pi} q(t)dt\right) + \frac{1}{2}\left(\alpha^+\beta_2 - \beta^-\alpha_2\right), \qquad (3.54) \\ w_4 \;&=\; \alpha^-\beta^- \left(H + h + \frac{1}{2}\int_0^\pi q(t)dt - \int_{\xi_1}^{\xi_2} q(t)dt\right) + \frac{1}{2}\left(\alpha^-\beta_2 - \beta^-\alpha_2\right). \qquad (3.55) \end{aligned}$$

Proof. Substituting the asymptotics for $\varphi_1(x,\lambda)$ from (3.20) into the right-hand side of (3.24), we calculate

$$\varphi_1(x,\lambda) \;=\; \cos sx + f_{11}(x)\frac{\sin sx}{s} + \frac{1}{2s}\int_0^x q(t)\sin s(x-2t)dt + O\left(|s|^{-2}e^{|\tau|x}\right), \tag{3.56}$$

$$\varphi_1'(x,\lambda) \;=\; -s\sin sx + f_{11}(x)\cos sx + \frac{1}{2}\int_0^x q(t)\cos s(x-2t)dt + O\left(|s|^{-1}e^{|\tau|x}\right), \tag{3.57}$$

where

$$f_{11}(x) = h + \frac{1}{2}\int_0^x q(t)dt, \quad x \in \Omega_1. \tag{3.58}$$

Using (3.56), (3.57), and substituting the asymptotics for $\varphi_2(x,\lambda)$ from (3.21) into the right-hand side of (3.25) we obtain

$$\begin{aligned} \varphi_2(x,\lambda) \;&=\; \alpha^+\cos sx + \alpha^-\cos s(2\xi_1 - x) + \frac{1}{s}\Big(f_{21}(x)\sin sx + f_{22}(x)\sin s(2\xi_1 - x)\Big) \\ &\quad + \frac{1}{s}\left(\frac{\alpha^+}{2}\int_0^x q(t)\sin s(x-2t)dt + \frac{\alpha^-}{2}\int_0^{\xi_1} q(t)\sin s(2\xi_1 - x - 2t)dt\right. \end{aligned}$$

$$+\frac{\alpha^-}{2}\int_{\xi_1}^{x} q(t)\sin s(2\xi_1+x-2t)dt\Big)+O\left(|s|^{-2}e^{|\tau|x}\right), \tag{3.59}$$

$$\begin{aligned}\varphi_2'(x,\lambda) &= -s\Big(\alpha^+\sin sx-\alpha^-\sin s(2\xi_1-x)\Big)\\ &+\Big(f_{21}(x)\cos sx-f_{22}(x)\cos s(2\xi_1-x)\Big)+\frac{\alpha^+}{2}\int_0^x q(t)\cos s(x-2t)dt\\ &-\frac{\alpha^-}{2}\int_0^{\xi_1} q(t)\cos s(2\xi_1-x-2t)dt+\frac{\alpha^-}{2}\int_{\xi_1}^{x} q(t)\cos s(2\xi_1+x-2t)dt\\ &+O\left(|s|^{-1}e^{|\tau|x}\right),\end{aligned} \tag{3.60}$$

where

$$f_{21}(x) = \alpha^+\left(h+\frac{1}{2}\int_0^{\xi_1} q(t)dt+\frac{1}{2}\int_{\xi_1}^{x} q(t)dt\right)+\frac{\alpha_2}{2}, \tag{3.61}$$

$$f_{22}(x) = \alpha^-\left(h+\frac{1}{2}\int_0^{\xi_1} q(t)dt-\frac{1}{2}\int_{\xi_1}^{x} q(t)dt\right)-\frac{\alpha_2}{2}. \tag{3.62}$$

Using (3.59), (3.60), and substituting the asymptotics for $\varphi_3(x,\lambda)$ from (3.22) into the right-hand side of (3.26) we get

$$\begin{aligned}\varphi_3(x,\lambda) &= \alpha^+\beta^+\cos sx+\alpha^-\beta^+\cos s(2\xi_1-x)+\alpha^+\beta^-\cos s(2\xi_2-x)\\ &+\alpha^-\beta^-\cos s(2\xi_1-2\xi_2+x)+\frac{1}{s}\Big(f_{31}(x)\sin sx+f_{32}(x)\sin s(2\xi_1-x)\\ &+f_{33}(x)\sin s(2\xi_2-x)+f_{34}(x)\sin s(2\xi_1-2\xi_2+x)\Big)\\ &+\frac{1}{s}\int_0^x Q_1(x,t)\sin st\,dt+O\left(|s|^{-2}e^{|\tau|x}\right),\end{aligned} \tag{3.63}$$

$$\begin{aligned}\varphi_3'(x,\lambda) &= -s\Big(\alpha^+\beta^+\sin s\pi-\alpha^-\beta^+\sin s(2\xi_1-\pi)-\alpha^+\beta^-\sin s(2\xi_2-\pi)\\ &+\alpha^-\beta^-\sin s(2\xi_1-2\xi_2+\pi)\Big)+f_{31}(x)\cos sx-f_{32}(x)\cos s(2\xi_1-x)\\ &-f_{33}(x)\cos s(2\xi_2-x)+f_{34}(x)\cos s(2\xi_1-2\xi_2+x)\\ &+\int_0^x Q_2(x,t)\cos st\,dt+O\left(|s|^{-1}e^{|\tau|x}\right),\end{aligned} \tag{3.64}$$

where

$$f_{31}(x)=\alpha^+\beta^+\left(h+\frac{1}{2}\int_0^{\xi_1}q(t)dt+\frac{1}{2}\int_{\xi_1}^{\xi_2}q(t)dt+\frac{1}{2}\int_{\xi_2}^{x}q(t)dt\right)+\frac{1}{2}\left(\alpha^+\beta_2+\beta^+\alpha_2\right), \tag{3.65}$$

$$f_{32}(x)=\alpha^-\beta^+\left(h+\frac{1}{2}\int_0^{\xi_1}q(t)dt-\frac{1}{2}\int_{\xi_1}^{\xi_2}q(t)dt-\frac{1}{2}\int_{\xi_2}^{x}q(t)dt\right)-\frac{1}{2}\left(\alpha^-\beta_2+\beta^+\alpha_2\right), \tag{3.66}$$

$$f_{33}(x)=\alpha^+\beta^-\left(h+\frac{1}{2}\int_0^{\xi_1}q(t)dt+\frac{1}{2}\int_{\xi_1}^{\xi_2}q(t)dt-\frac{1}{2}\int_{\xi_2}^{x}q(t)dt\right)-\frac{1}{2}\left(\alpha^+\beta_2-\beta^-\alpha_2\right), \tag{3.67}$$

$$f_{34}(x)=\alpha^-\beta^-\left(h+\frac{1}{2}\int_0^{\xi_1}q(t)dt-\frac{1}{2}\int_{\xi_1}^{\xi_2}q(t)dt+\frac{1}{2}\int_{\xi_2}^{x}q(t)dt\right)+\frac{1}{2}\left(\alpha^-\beta_2-\beta^-\alpha_2\right), \tag{3.68}$$

and the terms

$$\int_0^x Q_1(x,t)\sin st\,dt, \quad Q_1(x,.)\in L_2(0,\pi), \quad x\in\Omega_3,$$
$$\int_0^x Q_2(x,t)\cos st\,dt, \quad Q_2(x,.)\in L_2(0,\pi), \quad x\in\Omega_3$$

are obtained by combining the integrals with integrands of the form $q(t)\sin sp(t)$ and $q(t)\cos sp(t)$, respectively.

According to (3.9),

$$\Delta(\lambda)=(\lambda-H_1)\varphi'(\pi,\lambda)+(\lambda H-H_2)\varphi(\pi,\lambda).$$

Hence by virtue of (3.63) and (3.64),

$$\begin{aligned}\Delta(\lambda) \;=\; & -s^3\Big(\alpha^+\beta^+\sin s\pi-\alpha^-\beta^+\sin s(2\xi_1-\pi)-\alpha^+\beta^-\sin s(2\xi_2-\pi)\\ & +\alpha^-\beta^-\sin s(2\xi_1-2\xi_2+\pi)\Big)+s^2\Big(w_1\cos s\pi+w_2\cos s(2\xi_1-\pi)\\ & +w_3\cos s(2\xi_2-\pi)+w_4\cos s(2\xi_1-2\xi_2+\pi)\Big)+s^2I(s),\end{aligned} \tag{3.69}$$

where w_1, w_2, w_3 and w_4 are given by (3.52)–(3.55), and

$$I(s)=\int_0^\pi Q_2(\pi,t)\cos st\,dt+O\left(|s|^{-1}e^{|\tau|\pi}\right). \tag{3.70}$$

Denote

$$\Gamma_n = \{\lambda\in\mathbb{C}:|\lambda|=\left(|s^0_{n-1}|+\tfrac{d}{2}\right)^2\}, \tag{3.71}$$
$$G_\delta = \{s:|s-s^0_k|\ge\delta,\ k=0,1,2,\ldots\}, \tag{3.72}$$

where d is defined by (3.49) and δ is sufficiently small positive number. Using known methods (see, e.g., [5, Theorem 12.4]) we get

$$|\Delta_0(\lambda)|\ge C_\delta|s|e^{|\tau|\pi}, \quad s\in G_\delta. \tag{3.73}$$

On the other hand, it follows from (3.69) that

$$\left|\Delta(\lambda)-s^2\Delta_0(\lambda)\right|<C'_\delta|s|^2e^{|\tau|\pi} \tag{3.74}$$

for sufficiently large values of $|s|$. Thus,

$$\left|s^2\Delta_0(\lambda)\right| > C_\delta|s|^3e^{|\tau|\pi} > C'_\delta|s|^2e^{|\tau|\pi} > \left|\Delta(\lambda) - s^2\Delta_0(\lambda)\right| \tag{3.75}$$

for sufficiently large values of $n \in \mathbb{N}$ and $s \in \Gamma_n$. Hence by Rouché's theorem [11, p. 125], we can establish that for sufficiently large values of $n \in \mathbb{N}$, the number of zeros of $s^2\Delta_0(\lambda) + \{\Delta(\lambda) - s^2\Delta_0(\lambda)\} = \Delta(\lambda)$ inside Γ_n coincides with the number of zeros of $s^2\Delta_0(\lambda)$, i.e., it equals $n+1$. Thus, in the circle $\{\lambda : |\lambda| < \left(|s^0_{n-1}| + \frac{d}{2}\right)^2\}$ there exists exactly $n+1$ eigenvalues of L: $\lambda_0, \ldots, \lambda_n$. Analogously, by using Rouché's theorem one can prove that for sufficiently large values of n, every circle $\sigma_n(\delta) = \{s : |s - s^0_{n-1}| \le \delta\}$ contains exactly one zero of $\Delta(\lambda)$, namely $s_n = \sqrt{\lambda_n}$. Since $\delta > 0$ is arbitrary, we must have

$$s_n = s^0_{n-1} + \varepsilon_n, \quad \varepsilon_n = o(1), \quad n \to \infty. \tag{3.76}$$

It is not difficult to see that

$$\varepsilon_n = O\left(\frac{1}{n}\right). \tag{3.77}$$

By virtue of (3.69) and the the relation $\Delta(\lambda_n) = 0$ we get

$$\Delta_0(\lambda_n) = O(1).$$

Taking into account that $\Delta_0(\lambda^0_n) = 0$ and using Taylor's expansion of $\Delta_0(s^2)$ at $s = s^0_{n-1}$, this yields

$$\varepsilon_n\dot{\Delta}_0(\lambda^0_{n-1}) = O\left(\frac{1}{s^0_{n-1}}\right) + O\left(\varepsilon_n^2\right). \tag{3.78}$$

It follows from (3.44) that

$$\left|\gamma^0_n\right| \asymp C. \tag{3.79}$$

It means that $\gamma^0_n = O(1)$ and $(\gamma^0_n)^{-1} = O(1)$. By virtue of (3.38) and (3.42) we have

$$k^0_n = \psi(0, \lambda^0_n) = \frac{1}{\varphi(\pi, \lambda^0_n)}.$$

Therefore, $k^0_n = O(1)$ and $(k^0_n)^{-1} = O(1)$, i.e., $k^0_n \asymp C$. Together with (3.47) and (3.79), this yields

$$\left|\dot{\Delta}_0(\lambda^0_n)\right| \asymp C. \tag{3.80}$$

Now (3.77) follows from (3.48), (3.78) and (3.80). By virtue of (3.48), (3.76), and using the method of [27, p. 66] (see also [27, Lemma 1.4.3]), we obtain

$$\left\{\int_0^\pi Q_2(\pi, t)\cos s_n t\, dt\right\}_{n\ge 0} \in l_2.$$

Taking this into account, it follows from (3.69), the relation $\Delta(\lambda_n) = 0$ and (3.77) that

$$\Delta_0(\lambda_n) + w_1\cos s_n\pi + w_2\cos s_n(2\xi_1 - \pi)$$

$$+w_3\cos s_n(2\xi_2-\pi)+w_4\cos s_n(2\xi_1-2\xi_2+\pi)+\kappa_{n1}=0,$$

where $\{\kappa_{n1}\}_{n\geq 0}\in l_2$. Using Taylor's expansions of $\Delta_0(s^2)$ and $w_1\cos s\pi+w_2\cos s(2\xi_1-\pi)+w_3\cos s(2\xi_2-\pi)+w_4\cos s(2\xi_1-2\xi_2+\pi)$ at $s=s^0_{n-1}$, this yields

$$2\varepsilon_n s^0_{n-1}\dot{\Delta}_0(\lambda^0_{n-1})+w_1\cos s^0_{n-1}\pi+w_2\cos s^0_{n-1}(2\xi_1-\pi)$$
$$+w_3\cos s^0_{n-1}(2\xi_2-\pi)+w_4\cos s^0_{n-1}(2\xi_1-2\xi_2+\pi)+\kappa_{n2}=0,$$

where, $\{\kappa_{n2}\}_{n\geq 0}\in l_2$. From this, (3.48), (3.77) and (3.80) we obtain (3.50). Theorem 3.9 is proved.◄

Theorem 3.10. *The eigenfunctions*

$$\varphi(x,\lambda_n)=\begin{cases}\varphi_1(x,\lambda_n), & x\in\Omega_1\\ \varphi_2(x,\lambda_n), & x\in\Omega_2\\ \varphi_3(x,\lambda_n), & x\in\Omega_3\end{cases}\tag{3.81}$$

of the discontinuous boundary value problem L satisfy the following asymptotic estimates:

$$\varphi_1(x,\lambda_n)=\cos s^0_{n-1}x+\frac{1}{n}\left(f_{11}(x)-x\omega_n\right)+\frac{\kappa_{n1}(x)}{n},\tag{3.82}$$

$$\begin{aligned}\varphi_2(x,\lambda_n)=&\alpha^+\cos s^0_{n-1}x+\alpha^-\cos s^0_{n-1}(2\xi_1-x)+\frac{1}{n}\Big(\big(f_{21}(x)-\alpha^+x\omega_n\big)\sin s^0_{n-1}x\\&+\big(f_{22}(x)-\alpha^-(2\xi_1-x)\omega_n\big)\sin s^0_{n-1}(2\xi_1-x)\Big)+\frac{\kappa_{n2}(x)}{n},\end{aligned}\tag{3.83}$$

$$\begin{aligned}\varphi_3(x,\lambda_n)=&\alpha^+\beta^+\cos s^0_{n-1}x+\alpha^-\beta^+\cos s^0_{n-1}(2\xi_1-x)+\alpha^+\beta^-\cos s^0_{n-1}(2\xi_2-x)\\&+\alpha^-\beta^-\cos s^0_{n-1}(2\xi_1-2\xi_2+x)+\frac{1}{n}\Big(\big(f_{31}(x)-\alpha^+\beta^+x\omega_n\big)\sin s^0_{n-1}x\\&+\big(f_{32}(x)-\alpha^-\beta^+(2\xi_1-x)\omega_n\big)\sin s^0_{n-1}(2\xi_1-x)\\&+\big(f_{33}(x)-\alpha^+\beta^-(2\xi_2-x)\omega_n\big)\sin s^0_{n-1}(2\xi_2-x)\\&+\big(f_{34}(x)-\alpha^-\beta^-(2\xi_1-2\xi_2+x)\omega_n\big)\sin s^0_{n-1}(2\xi_1-2\xi_2+x)\Big)+\frac{\kappa_{n3}(x)}{n},\end{aligned}\tag{3.84}$$

where $|\kappa_{ni}(x)|<C$ on Ω_i $(i=\overline{1,3})$ and $\{\kappa_{ni}(x)\}_{n\geq 0}\in l_2$ for $x\in\Omega_i$ $(i=\overline{1,3})$.

Proof. Let us consider only $\varphi_1(x,\lambda_n)$. Other cases can be considered in a similar way using (3.59) and (3.63). From the asymptotic formula (3.56) for $\lambda=\lambda_n$ we have

$$\varphi_1(x,\lambda_n)\ =\ \cos s_nx+f_{11}(x)\frac{\sin s_nx}{s_n}+\frac{1}{2s_n}\int_0^x q(t)\sin s_n(x-2t)dt+O\left(\frac{1}{s_n^2}\right).\tag{3.85}$$

Using (3.50) and Taylor's expansions of $\cos sx$, $\sin sx$ and $\sin s(x-2t)$ at $s=s^0_{n-1}$, this yields

$$\varphi_1(x,\lambda_n)\ =\ \cos s^0_{n-1}x+\frac{1}{n}\left(f_{11}(x)-x\omega_n-\zeta_nx\right)\sin s^0_{n-1}x$$

$$+\frac{1}{2n}\int_0^x q(t)\sin s_{n-1}^0(x-2t)\,dt+O\left(\frac{1}{n^2}\right). \tag{3.86}$$

Recall that $\{\zeta_n\}_{n\geq 0}\in l_2$ and $\{\int_0^x q(t)\sin s_{n-1}^0(x-2t)\,dt\}_{n\geq 0}\in l_2$. Also it is clear that the functions $\zeta_n\sin s_{n-1}^0 x$ and $\int_0^x q(t)\sin s_{n-1}^0(x-2t)\,dt$ are bounded on Ω_1. Consequently, we get (3.82) from (3.86).◄

Theorem 3.11. *The norming constants γ_n of the discontinuous boundary value problem L have the following asymptotic behavior:*

$$\gamma_n=\hat{\gamma}_{n-1}^0+\frac{\upsilon_n}{n}+\frac{\delta_n}{n},\quad \{\delta_n\}_{n\geq 0}\in l_2, \tag{3.87}$$

where $\hat{\gamma}_n^0$ is given by (3.45) *and*

$$\begin{aligned}
\upsilon_n\ =&\ b_1\sin 2s_{n-1}^0\xi_1+b_2\sin 2s_{n-1}^0\xi_2+b_3\sin 2s_{n-1}^0(\xi_1-\xi_2)\\
&+b_4\sin 2s_{n-1}^0(2\xi_1-\xi_2)+\frac{(\alpha^+\beta^+)^2}{4}\sin 2s_{n-1}^0\pi-\frac{\alpha^+\alpha^-(\beta^+)^2}{2}\sin 2s_{n-1}^0(\xi_1-\pi)\\
&-\frac{(\alpha^-\beta^+)^2}{4}\sin 2s_{n-1}^0(2\xi_1-\pi)-\frac{1}{2}\beta^+\beta^-((\alpha^+)^2+(\alpha^-)^2)\sin 2s_{n-1}^0(\xi_2-\pi)\\
&-\frac{(\alpha^+\beta^-)^2}{4}\sin 2s_{n-1}^0(2\xi_2-\pi)-\frac{1}{2}\alpha^+\alpha^-\beta^+\beta^-\sin 2s_{n-1}^0(\xi_1+\xi_2-\pi)\\
&+\frac{1}{2}\alpha^+\alpha^-\beta^+\beta^-\sin 2s_{n-1}^0(\xi_1-\xi_2+\pi)+\frac{1}{2}\alpha^+\alpha^-(\beta^-)^2\sin 2s_{n-1}^0(\xi_1-2\xi_2+\pi)\\
&+\frac{(\alpha^-\beta^-)^2}{4}\sin 2s_{n-1}^0(2\xi_1-2\xi_2+\pi),
\end{aligned} \tag{3.88}$$

$$\begin{aligned}
b_1\ =&\ \Big[-2\alpha^+\alpha^-\xi_1\omega_n+\alpha^+\alpha^-\Big(2h+\textstyle\int_0^{\xi_1}q(t)\,dt\Big)+\frac{\alpha_2}{2}(\alpha^--\alpha^+)\Big](\xi_2-\xi_1)\\
&+\Big[-2\alpha^+\alpha^-((\beta^+)^2+(\beta^-)^2)\xi_1\omega_n+\alpha^+\alpha^-((\beta^+)^2+(\beta^-)^2)\Big(2h+\textstyle\int_0^{\xi_1}q(t)\,dt\Big)\\
&-\frac{\alpha_2}{2}(\alpha^--\alpha^+)((\beta^+)^2+(\beta^-)^2)\Big](\pi-\xi_2),
\end{aligned} \tag{3.89}$$

$$\begin{aligned}
b_2\ =&\ \Big[-2(\alpha^+)^2\beta^+\beta^-\xi_2\omega_n-\alpha^+\alpha^-\beta^+\beta^-\Big(2h+\textstyle\int_0^{\xi_2}q(t)\,dt\Big)\\
&+\frac{1}{2}\alpha^+\beta^-(\alpha^+\beta_2+\beta^+\alpha_2)-\frac{1}{2}\alpha^+\beta^+(\alpha^+\beta_2-\beta^-\alpha_2)\Big](\pi-\xi_2),
\end{aligned} \tag{3.90}$$

$$b_3\ =\ \Big[-4\alpha^+\alpha^-\beta^+\beta^-(\xi_1-\xi_2)\omega_n-\alpha^+\alpha^-\beta^+\beta^-\textstyle\int_{\xi_1}^{\xi_2}q(t)\,dt$$

$$+\alpha^-\beta^+(\alpha^+\beta_2-\beta^-\alpha_2)-\alpha^+\beta^-(\alpha^-\beta_2+\beta^-\alpha_2)\Big](\pi-\xi_2), \tag{3.91}$$

$$b_4 = \Big[-2(\alpha^-)^2\beta^+\beta^-(2\xi_1-\xi_2)\omega_n+(\alpha^+)^2\beta^+\beta^-\left(2h+2\textstyle\int_0^{\xi_1}q(t)\,dt-\int_0^{\xi_2}q(t)\,dt\right)$$
$$+\frac{1}{2}\alpha^-\beta^+(\alpha^+\beta_2-\beta^-\alpha_2)-\frac{1}{2}\alpha^-\beta^-(\alpha^-\beta_2-\beta^-\alpha_2)\Big](\pi-\xi_2). \tag{3.92}$$

Proof. By virtue of (3.81), we can rewrite (3.17) as

$$\gamma_n=\int_0^{\xi_1}\varphi_1^2(x,\lambda_n)\,dx+\int_{\xi_1}^{\xi_2}\varphi_2^2(x,\lambda_n)\,dx+\int_{\xi_1}^{\pi}\varphi_3^2(x,\lambda_n)\,dx+\frac{1}{\rho}(R'(\varphi_{\lambda_n}))^2. \tag{3.93}$$

It follows from (1.3) and (3.22) that

$$\frac{1}{\rho}(R'(\varphi_{\lambda_n}))^2=\frac{1}{\rho\lambda_n^2}(R(\varphi_{\lambda_n}))^2=O\left(\frac{1}{n^2}\right). \tag{3.94}$$

Taking this into account and substituting (3.82)–(3.84) into (3.93) we obtain (3.87).◀

Theorem 3.12. *The characteristic function $\Delta(\lambda)$ can be represented as follows:*

$$\Delta(\lambda)=c_0(\lambda-\lambda_0)(\lambda_1-\lambda)\prod_{n=2}^{\infty}\frac{\lambda_n-\lambda}{\lambda_n^0}, \tag{3.95}$$

where

$$c_0=\alpha^+\beta^+\pi-\alpha^-\beta^+(2\xi_1-\pi)-\alpha^+\beta^-(2\xi_2-\pi)+\alpha^-\beta^-(2\xi_1-2\xi_2+\pi). \tag{3.96}$$

Proof. It follows from (3.9) and (3.22) that $\Delta(\lambda)$ is an entire function of λ of order 1/2 and hence by Hadamard's factorization theorem [11, p. 289], $\Delta(\lambda)$ is uniquely determined up to a multiplication constant by its zeros:

$$\Delta(\lambda)=C\prod_{n=0}^{\infty}\left(1-\frac{\lambda}{\lambda_n}\right). \tag{3.97}$$

The case $\Delta(0)=0$ requires minor modifications. We consider the function

$$\hat{\Delta}(\lambda):=s^2\Delta_0(\lambda)=-\lambda^2c_0\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_n^0}\right). \tag{3.98}$$

Then

$$\frac{\Delta(\lambda)}{\hat{\Delta}(\lambda)}=C\frac{(\lambda-\lambda_0)(\lambda_1-\lambda)}{c_0\lambda_0\lambda_1\lambda^2}\prod_{n=1}^{\infty}\frac{\lambda_n^0}{\lambda_{n+1}}\prod_{n=1}^{\infty}\left(1+\frac{\lambda_{n+1}-\lambda_n^0}{\lambda_n^0-\lambda}\right).$$

With the help of (3.41), (3.50) and (3.69), we calculate

$$\lim_{\lambda\to-\infty}\frac{\Delta(\lambda)}{\hat{\Delta}(\lambda)}=1,\qquad \lim_{\lambda\to-\infty}\prod_{n=1}^{\infty}\left(1+\frac{\lambda_{n+1}-\lambda_n^0}{\lambda_n^0-\lambda}\right)=1,$$

and hence

$$C=-c_0\pi\lambda_0\lambda_1\prod_{n=1}^{\infty}\frac{\lambda_{n+1}}{\lambda_n^0}.$$

Substituting this into (3.97), we get (3.95).◄

Remark 3.13. *Analogous results are valid for boundary value problems with other types of boundary conditions but the same jump conditions. Let us state some of these results for one of them which will be used below.*

Consider the discontinuous boundary value problem L_1 for equation (1.1) *with the boundary conditions $y(0)=V(y)=0$ and jump conditions* (1.4)–(1.7). *The eigenvalues $\{\mu_n\}_{n\geq 0}$ of L_1 are algebraically and geometrically simple and coincide with the zeros of characteristic function $\Delta_1(\lambda):=\psi(0,\lambda)$ and*

$$\Delta_1(\lambda)=c_1(\lambda-\mu_0)\prod_{n=1}^{\infty}\frac{\mu_n-\lambda}{\mu_{n-1}^0}, \tag{3.99}$$

$$t_n:=\sqrt{\mu_n}=t_{n-1}^0+\frac{\omega_{n1}}{\pi n}+\frac{\zeta_{n1}}{n},\quad \{\zeta_{n1}\}\in l_2, \tag{3.100}$$

where $\{\mu_n^0=(t_n^0)^2\}_{n\geq 0}$ is the set of zeros of the entire function

$$\begin{aligned}\Delta_{1,0}(\lambda)\ =\ &\alpha^+\beta^+\cos s\pi-\alpha^-\beta^+\cos s(\pi-2\xi_1)\\ &-\alpha^+\beta^-\cos s(\pi-2\xi_2)+\alpha^-\beta^-\cos s(\pi-2\xi_2+2\xi_1),\end{aligned}$$

and

$$c_1=\alpha^+\beta^+-\alpha^-\beta^+-\alpha^+\beta^-+\alpha^-\beta^-,$$

$$\begin{aligned}\omega_{n,1}\ =\ &-\Big(w_{1,1}\sin t_{n-1}^0\pi+w_{2,1}\sin t_{n-1}^0(\pi-2\xi_1)+w_{3,1}\sin t_{n-1}^0(\pi-2\xi_2)\\ &+w_{4,1}\sin t_{n-1}^0(\pi-2\xi_2+2\xi_1)\Big)/\left(2t_{n-1}^0\dot{\Delta}_{1,0}(\lambda_{n-1}^0)\right),\end{aligned}$$

$$\begin{aligned}w_{1,1}\ &=\ \alpha^+\beta^+\left(H+\frac{1}{2}\int_0^{\pi}q(t)dt\right)+\frac{1}{2}\left(\alpha^+\beta_2+\beta^+\alpha_2\right),\\ w_{2,1}\ &=\ \alpha^-\beta^+\left(-H-\frac{1}{2}\int_{\xi_1}^{\pi}q(t)dt+\int_0^{\xi_1}q(t)dt\right)-\frac{1}{2}\left(\alpha^-\beta_2+\beta^+\alpha_2\right),\\ w_{3,1}\ &=\ \alpha^+\beta^-\left(-H+\frac{1}{2}\int_{\xi_2}^{\pi}q(t)dt-\int_{\xi_2}^{\pi}q(t)dt\right)-\frac{1}{2}\left(\alpha^+\beta_2-\beta^-\alpha_2\right),\\ w_{4,1}\ &=\ \alpha^-\beta^-\left(H+\frac{1}{2}\int_0^{\pi}q(t)dt-\int_{\xi_1}^{\xi_2}q(t)dt\right)+\frac{1}{2}\left(\alpha^-\beta_2-\beta^-\alpha_2\right).\end{aligned}$$

4. Weyl solution and Weyl function

Let the function $\Phi(x,\lambda)$ be the solution of equation (1.1) which satisfy the boundary conditions $U(\Phi_\lambda)=1$ and $V(\Phi_\lambda)=0$ and jump conditions (1.4)–(1.7). The function $\Phi(x,\lambda)$ is called the Weyl solution of the discontinuous boundary value problem L.

Let $S(x,\lambda)$ be the solution of equation (1.1) which satisfy the initial conditions $S(0,\lambda)=0$, $S'(0,\lambda)=1$ and jump conditions (1.4)–(1.7). Then the function $\psi(x,\lambda)$ can be represented as follows:

$$\psi(x,\lambda)=\left(\psi'(0,\lambda)-h\psi(0,\lambda)\right)S(x,\lambda)+\psi(0,\lambda)\varphi(x,\lambda)$$

or

$$-\frac{\psi(x,\lambda)}{\Delta(\lambda)}=S(x,\lambda)-\frac{\psi(0,\lambda)}{\Delta(\lambda)}\varphi(x,\lambda).$$

Denote

$$M(\lambda)=-\frac{\psi(0,\lambda)}{\Delta(\lambda)}. \tag{4.1}$$

It is clear that

$$\Phi(x,\lambda)=S(x,\lambda)+M(\lambda)\varphi(x,\lambda), \tag{4.2}$$

$$M(\lambda)=-\frac{\Delta_1(\lambda)}{\Delta(\lambda)}, \tag{4.3}$$

$$W(\varphi_\lambda,\Phi_\lambda;x)\equiv 1. \tag{4.4}$$

The function $M(\lambda)=\Phi(0,\lambda)$ is called the Weyl function of the problem L. The notion of the Weyl function introduced here is a generalization of the Weyl function for the classical Sturm-Liouville operators (see [13, 24]). Since $\Delta(\lambda)$ and $\Delta_1(\lambda)$ have no common zeros, it follows from (4.3) that $M(\lambda)$ is a meromorphic function with poles $\{\lambda_n\}_{n\geq 0}$ and zeros $\{\mu_n\}_{n\geq 0}$.

Theorem 4.1. *The following representation holds:*

$$M(\lambda)=\sum_{n=0}^{\infty}\frac{1}{\gamma_n(\lambda-\lambda_n)}. \tag{4.5}$$

Proof. Consider the contour integral

$$J_N(\lambda)=\frac{1}{2\pi i}\int_{\Gamma_N}\frac{M(\mu)}{\lambda-\mu}\,d\mu,\quad \lambda\in int\,\Gamma_N,$$

where the contour Γ_N is defined by (3.71) and assumed to have the counterclockwise circuit. Since $\Delta_1(\lambda)=\psi(0,\lambda)$, it follows from (3.29) that

$$|\Delta_1(\lambda)|\leq C|s|^2e^{|\tau|\pi}. \tag{4.6}$$

Also, by virtue of (3.69) and (3.73) we get for sufficiently large values of $|s|$,

$$|\Delta(\lambda)| \geq C_\delta |s|^3 e^{|\tau|\pi}, \qquad s \in G_\delta. \tag{4.7}$$

Now using (4.3), (4.6) and (4.7), we conclude that for sufficiently large values of $|s|$,

$$|M(\lambda)| \leq \frac{C_\delta}{|s|}, \qquad s \in G_\delta. \tag{4.8}$$

Moreover, using (3.10), (3.18) and (4.3), we calculate

$$\operatorname*{Res}_{\lambda=\lambda_n} M(\lambda) = -\frac{\Delta_1(\lambda_n)}{\dot{\Delta}(\lambda_n)} = -\frac{k_n}{\dot{\Delta}(\lambda_n)} = \frac{1}{\gamma_n}. \tag{4.9}$$

In view of (4.8), $\lim_{N\to\infty} J_N(\lambda) = 0$. By virtue of (4.9) and residue theorem [11, p.112], we have

$$J_N(\lambda) = -M(\lambda) + \sum_{n=0}^{N} \frac{1}{\gamma_n(\lambda - \lambda_n)},$$

and consequently, (4.5) is proved.◀

5. Inverse problem

In this section, we investigate the inverse problem of reconstruction of the discontinuous boundary value problem L from its spectral characteristics. We consider three statements of the inverse problem of reconstruction of the problem L from the Weyl function, from the so-called spectral data $\{\lambda_n, \gamma_n\}_{n\geq 0}$, and from two spectra $\{\lambda_n, \mu_n\}_{n\geq 0}$.

Let us prove the uniqueness theorems for the solutions of the above mentioned inverse problems. For this purpose we agree that together with L we consider a discontinuous boundary value problem $\widetilde{L}$ of the same form but with different coefficients $\widetilde{q}(x)$, $\widetilde{h}$, $\widetilde{H}$, $\widetilde{H}_1$, $\widetilde{H}_2$, $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, and discontinuity points $\widetilde{\xi}_1$ and $\widetilde{\xi}_2$. Every where below if a certain symbol a denotes an object related to L, then the corresponding symbol $\widetilde{a}$ denotes the analogous object related to $\widetilde{L}$.

Theorem 5.1. *If $M(\lambda) = \widetilde{M}(\lambda)$, then $L = \widetilde{L}$. Thus, the specification of the Weyl function $M(\lambda)$ uniquely determines L.*

Proof. Denote $J_0 = J \cap \widetilde{J}$ where $J = [0, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, \pi]$. Let us define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$, $x \in J_0$ by the formula

$$P(x,\lambda) \begin{bmatrix} \widetilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda) \\ \widetilde{\varphi}'(x,\lambda) & \widetilde{\Phi}'(x,\lambda) \end{bmatrix} = \begin{bmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}. \tag{5.1}$$

Using (4.4) and (5.1) we calculate for $j = 1, 2$:

$$\begin{cases} P_{j1}(x,\lambda) = \varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}'(x,\lambda), \\ P_{j2}(x,\lambda) = \Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}(x,\lambda), \end{cases} \tag{5.2}$$

$$\begin{cases} \varphi(x,\lambda) = P_{11}(x,\lambda)\widetilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\widetilde{\varphi}'(x,\lambda), \\ \Phi(x,\lambda) = P_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda) + P_{12}(x,\lambda)\widetilde{\Phi}'(x,\lambda). \end{cases} \tag{5.3}$$

It follows from (4.2), (4.4) and (5.2) that

$$\begin{aligned} P_{11}(x,\lambda) &= 1 + \frac{\psi(x,\lambda)}{\Delta(\lambda)}(\widetilde{\varphi}'(x,\lambda) - \varphi'(x,\lambda)) + \frac{\varphi(x,\lambda)}{\Delta(\lambda)}(\psi'(x,\lambda) - \widetilde{\psi}'(x,\lambda)) \\ &\quad + \varphi(x,\lambda)\widetilde{\varphi}'(x,\lambda)\left(\frac{1}{\Delta(\lambda)} - \frac{1}{\widetilde{\Delta}(\lambda)}\right), \end{aligned}$$

$$\begin{aligned} P_{12}(x,\lambda) &= \frac{1}{\Delta(\lambda)}\Big(\varphi(x,\lambda)\widetilde{\psi}(x,\lambda) - \psi(x,\lambda)\widetilde{\varphi}(x,\lambda)\Big) \\ &\quad + \varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\left(\frac{1}{\Delta(\lambda)} - \frac{1}{\widetilde{\Delta}(\lambda)}\right). \end{aligned}$$

Denote $G_\delta^0 = G_\delta \cap \widetilde{G}_\delta$. By virtue of (3.20)–(3.22), (3.29)–(3.31) and (4.7), this yields

$$|P_{11}(x,\lambda) - 1| \le \frac{C_\delta}{|s|}, \qquad |P_{12}(x,\lambda)| \le \frac{C_\delta}{|s|}, \qquad s \in G_\delta^0. \tag{5.4}$$

for sufficiently large values of $|s|$. On the other hand according to (4.2) and (5.2),

$$\begin{aligned} P_{11}(x,\lambda) &= \varphi(x,\lambda)\widetilde{S}'(x,\lambda) - S(x,\lambda)\widetilde{\varphi}'(x,\lambda) + (\widetilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\widetilde{\varphi}'(x,\lambda), \\ P_{12}(x,\lambda) &= S(x,\lambda)\widetilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\widetilde{S}(x,\lambda) + (M(\lambda) - \widetilde{M}(\lambda))\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda). \end{aligned}$$

Since $M(\lambda) \equiv \widetilde{M}(\lambda)$, it follows that for each fixed $x \in J_0$, the functions $P_{11}(x,\lambda)$ and $P_{12}(x,\lambda)$ are entire in λ. With the help of (5.4) and well-known Liouville's theorem, this yields $P_{11}(x,\lambda) \equiv 1$, $P_{12}(x,\lambda) \equiv 0$. Substituting into (5.3), we get $\varphi(x,\lambda) \equiv \widetilde{\varphi}(x,\lambda)$, $\Phi(x,\lambda) \equiv \widetilde{\Phi}(x,\lambda)$ for all $x \in J_0$ and λ. Taking this into account, from (1.1) we get $q(x) = \widetilde{q}(x)$ a.e. on $(0,\pi)$, from (3.1) and (3.4) we obtain $h = \widetilde{h}$, $H = \widetilde{H}$, $H_1 = \widetilde{H}_1$, $H_2 = \widetilde{H}_2$, and from (1.4)–(1.7) we conclude that $\alpha_i = \widetilde{\alpha}_i$, $\beta_i = \widetilde{\beta}_i$, $\xi_i = \widetilde{\xi}_i$ $(i = 1,2)$. Consequently, $L = \widetilde{L}$.◀

Theorem 5.2. *If $\lambda_n = \widetilde{\lambda}_n$ and $\gamma_n = \widetilde{\gamma}_n$ for all $n \ge 0$, then $L = \widetilde{L}$. Thus the problem L uniquely defined by spectral data $\{\lambda_n, \gamma_n\}_{n\ge 0}$.*

Proof. If $\lambda_n = \widetilde{\lambda}_n$ and $\gamma_n = \widetilde{\gamma}_n$ for all $n \ge 0$, then from (4.5), we get that $M(\lambda) = \widetilde{M}(\lambda)$. Hence by virtue of Theorem 5.1, this implies $L = \widetilde{L}$.◀

Theorem 5.3. *If* $\lambda_n = \widetilde{\lambda}_n$ *and* $\mu_n = \widetilde{\mu}_n$ *for all* $n \geq 0$*, then* $L = \widetilde{L}$*. Thus the specification of two spectra* $\{\lambda_n, \mu_n\}_{n\geq 0}$ *uniquely determines* L*.*

Proof. According to Theorem 3.12 and Remark 3.13, the sets $\{\lambda_n\}_{n\geq 0}$ and $\{\mu_n\}_{n\geq 0}$ coincide with the set of zeros of the functions $\Delta(\lambda)$ and $\Delta_1(\lambda)$, respectively. If $\lambda_n = \widetilde{\lambda}_n$ and $\mu_n = \widetilde{\mu}_n$ for all $n \geq 0$, then from (3.95) and (3.99) we get

$$\frac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)} = \frac{\widetilde{c}_0}{c_0}, \quad \frac{\widetilde{\Delta}_1(\lambda)}{\Delta_1(\lambda)} = \frac{\widetilde{c}_1}{c_1}. \tag{5.5}$$

On the other hand using (3.29) and (3.69) we obtain

$$\lim_{\operatorname{Im} s\to\infty} \frac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)} = \frac{\widetilde{\alpha}^+\widetilde{\beta}^+}{\alpha^+\beta^+} \quad \text{for } \arg s = \frac{\pi}{2}, \tag{5.6}$$

$$\lim_{\operatorname{Im} s\to\infty} \frac{\widetilde{\Delta}_1(\lambda)}{\Delta_1(\lambda)} = \frac{\widetilde{\alpha}^+\widetilde{\beta}^+}{\alpha^+\beta^+} \quad \text{for } \arg s = \frac{\pi}{2}. \tag{5.7}$$

Comparing with (5.5), this yields

$$\frac{\widetilde{c}_0}{c_0} = \frac{\widetilde{c}_1}{c_1}.$$

Together with (4.3) and (5.5) this implies that $M(\lambda) = \widetilde{M}(\lambda)$. Therefore, by Theorem 5.1 we conclude that $L = \widetilde{L}$.◀

Remark 5.4. *By virtue of* (4.3)*, the specification of two spectra* $\{\lambda_n, \mu_n\}_{n\geq 0}$ *is equivalent to the specification of the Weyl function* $M(\lambda)$*. On the other hand, it follows from* (4.5) *that the specification of the Weyl function* $M(\lambda)$ *is equivalent to the specification of the spectral data* $\{\lambda_n, \gamma_n\}_{n\geq 0}$*. Consequently, three statements of the inverse problem of reconstruction of the problem* L *from the Weyl function, from the spectral data and from two spectra are equivalent.*

References

[1] E. N. Akhmedova and H. M. Huseynov, *On eigenvalues and eigenfunctions of one class of Sturm-Liouville operators with discontinuous coefficients*, Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **23**(4) (2003), Math. Mech., 7–18.

[2] R. Kh. Amirov, *On Sturm–liouville problems with discontinuity conditions inside an inteval*, J. Math. Anal. Appl. **317** (2006), 167–176.

[3] R. Kh. Amirov, A. S. Ozkan and B. Keskin, *Inverse problems for impulsive Sturm-Liouville operator with spectral parameter linearly contained in boundary conditions*, Integral Transforms Spec. Funct. **20**(8) (2009), 607–618.

[4] R.S. Anderssen, *The effect of discontinuities in density and shear velocity on the asymptotic overtone structure of tortional eigenfrequencies of the Earth*, Geophys. J. R. Astr. Soc. **50** (1997) 303–309.

[5] R. Bellman, K. Cooke, Differential–Difference Equations, Academic Press, New York, 1963.

[6] P. A. Binding and P. J. Browne, *Oscillation theory for indefinite Sturm–Liouville problems with eigenparameter dependent boundary conditions*, Proc. R. Soc. Edinburgh A, **127** (1997), 1123–36.

[7] P. A. Binding , P. J. Brown and K. Seddighi, *Sturm–Liouville equations with eigenparameter dependent boundary conditions*, Proc. Edinbourgh Math. Soc., **37** (1993), 57–72.

[8] P. A. Binding, P. J. Browne and B. A. Watson, *Inverse spectral problems for Sturm–Liouville equations with eigenparameter dependent boundary conditions*, J. London Math. Soc., **62** (2000), 161–182.

[9] P. J. Browne and D. B. Sleeman, *A uniqueness theorem for inverse eigenparameter dependent Sturm–Liouville problems*, Inverse Problems, **13** (1997), 1453–62.

[10] E. Coddington and N. Levinson, Theory of Ordinary Diffrential Equations, McGraw Hill, New York 1995.

[11] J. B. Conway, Functions of One Complex Variable, 2nd edition, Vol. I, Springer-Verlag, New York 1995.

[12] I. Dehghani Tazehkand and A. Jodayree Akbarfam, *On inverse Sturm-Liouville problems with spectral parameter linearly contained in the boundary conditions*, ISRN Mathematical Analysis, vol. 2011, Article ID 754718, 23 pages, 2011.

[13] G. Freiling and V. A. Yurko, Inverse Sturm-Liouville Problems and Their Applications, NY: Nova Sciences, Huntington 2001.

[14] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter in the boundary conditions*, Proc. R. Soc. Edinbourgh A, **87** (1977), 293–308.

[15] N. J. Guliyev, *Inverse eigenvalue problems for Sturm-Liouville equation with spectral parameter linearly contained in one of the boundary conditions*, Inverse Problems, **21** (2005), 1315–1330.

[16] O. H. Hald, *Discontinuous inverse eigenvalue problems*, Comm. Pure Appl. Math., **37** (1984), no. 5, 539–577.

[17] D. B. Hinton, *An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition*, Quart. J. Math. Oxford Ser. (2), **30** (1979), 33–42.

[18] B.F. Jdanovich, *Formulae for the zeros of Dirichlet polynomials and quasi-polynomials*, Dokl. Akad. Nauk SSSR **135**(8) (1960), 1046–1049(Russian).

[19] N. Yu. Kapustin, *Oscillation properties of solutions to a nonselfadjoint spectral problem with spectral parameter in the boundary condition*, Differ. Eqns. **35** (1999), 1031–4.

[20] N. Yu. Kapustin and E. I. Moiseev, 1997 *Spectral problems with the spectral parameter in the boundary conditions*, Differ. Equ., **33** (1997), 116–20.

[21] N. B. Kerimov and V. S. Mirzoev, *On the basis properties of one spectral problem with a spectral parameter in a boundary condition*, Siberian Math. J., **44** (2003), 13–6.

[22] M.G. Krein, B.Ya. Levin, *On entire almost periodic functions of exponential type*, Dokl. Akad. Nauk SSSR, **64**(3) (1949), 285–287(Russian).

[23] B. Keskin, A. S. Sinan and N. Yalçin, *Inverse spectral problems for discontinous Sturm–liouville operator with eigenparameter dependent boundary conditions*, Commun. Fac. Sci. Univ. Ank. Series A1, **60**(1) (2011) 15–25.

[24] B. M. Levitan and I. S. Sargsyan, Introduction to Spectral Theory, AMS Transl. Math. Monogr. vol. 39, Providence, RI, 1975.

[25] A. V. Likov and and Yu. A. Mikhailov, The Theory of Heat and Mass Transfer, Qosenergaizdat, 1963 (Russian).

[26] O.N. Litvinenko and V.I. Soshnikov, The Theory of Heterogenious Lines and Their Applications in Radio Engineering, Radio, Moscow, 1964(Russian).

[27] V.A. Marchenko, Sturm–Liouville Operators and Applications, Naukova Dumka, Kiev, 1977; English transl., Birkhäuser, Basel, 1986.

[28] O. Sh. Mukhtarov, Mahir Kadakal and F. S. Muhtarov, *On discontinuous Sturm-Liouville problems with transmission conditions*, J. Math. Kyoto Univ., **44**(4) (2004), 779–798.

[29] N. Naimark, Linear Differential Operators, Parts I and II, Frederick Ungar Publishing Co., New York, 1968.

[30] E. Tunç and O. Sh. Mukhtarov, Fundamental solutions and eigenvalues of one boundary value problem with transmission conditions, Applied Mathematics and Computation, **157** (2004), 347–355.

[31] J. Walter, *Regular eigenvalue problems with eigenvalue parameter in the boundary condition* , Math. Z. **133** (1973), 301–12.

[32] C. Willis, *Inverse Sturm–Liouville problems with two discontinuieties*, Inverse Problems **1** (1985), 263–289.

[33] S. D. Wray, *Absolutely convergent expansions associated with a boundary value problem with the eigenvalue parameter contained in one boundary condition* , Czeh. Math. J., **32** (1982), 608–22.

[34] V. A. Yurko, Integral transforms connected with discontinuous boundary value problems, Integral Transform. Spec. Funct., **10**(2) (2000), 141–164.

[35] V. A. Yurko, *Inverse nodal and inverse spectral problems for discontinuous boundary value problems*, J. Math. Anal. Appl. **347** (2008), 266–272.

Isa Dehghani
Department of Mathematics, Payame Noor University, I.R. of Iran
E-mail: isadehghani@gmail.com

Aliasghar Jodayree Akbarfam
Faculty of Mathematical Sciences,University of Tabriz, Tabriz, Iran
E-mail: akbarfam@yahoo.com

Received 04 June 2012
Accepted 02 November 2013