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Inequalities for Convolutions of Functions on Commutative Hypergroupss

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Abstract. The generalized Young inequality on the Lorentz spaces for commutative hypergroups is introduced and its application to the theory of fractional integrals is given. The boundedness on the Lorentz space and the Hardy-Littlewood-Sobolev theorem for the fractional integrals on the commutative hypergroups are proved.

Key Words and Phrases: hypergroup, the Young inequality, fractional integral, the Hardy-Littlewood-Sobolev theorem.
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1. Introduction and preliminaries

It is known that a convolution of two functions on R^n is defined by

$$f *_{R^n} g(x) = \int_{R^n} f(x-y)g(y)dy.$$

Classical Young's inequality on the $L^p(R^n)$ spaces for the convolution of two functions on R^n states that if $f \in L^p(R^n)$ and $g \in L^q(R^n)$, then

$$\|f *_{R^n} g\|_r \le C\|f\|_p\|g\|_q,$$

where $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

The generalized Young inequality gives us the boundedness on the Lorentz spaces for the convolution of two functions on R^n.

Theorem 1. *([10] Theorem 2.10.1) If* $f \in L^{p_1,q_1}(R^n)$, $\varphi \in L^{p_2,q_2}(R^n)$ *and* $\dfrac{1}{p_1} + \dfrac{1}{p_2} > 1$, *then* $(f * \varphi) \in L^{p_0,q_0}(R^n)$, *where* $\dfrac{1}{p_1} + \dfrac{1}{p_2} - 1 = \dfrac{1}{p_0}$ *and* $q_0 \ge 1$ *is any number such that* $\dfrac{1}{q_1} + \dfrac{1}{q_2} \ge \dfrac{1}{q_0}$.
Moreover,

$$\|(f * \varphi)\|_{K,p_0,q_0} \le 3p_0\|f\|_{K,p_1,q_1}\|\varphi\|_{K,p_2,q_2}.$$

An extension of the Young inequality to the convolution

$$f *_G g(x) = \int_G f(xy)g(y^{-1})d\mu(y),$$

where μ is the Haar measure on local compact group G, was given in [6](see Theorem 20.18 in [6]).

In the theory of locally compact groups there arise certain spaces which, though not groups, have some of the structure of groups. Often, the structure can be expressed in terms of an abstract convolution of measures on the space.

A hypergroup $(K, *_K)$ consists of a locally compact Hausdorff space K together with a bilinear, associative, weakly continuous convolution on the Banach space of all bounded regular Borel measures on K with the following properties:

1. For all $x, y \in K$, the convolution of the point measures $\delta_x *_K \delta_y$ is a probability measure with compact support.
2. The mapping $K \times K \to \mathcal{C}(K)$, $(x, y) \mapsto \operatorname{supp}(\delta_x *_K \delta_y)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(K)$ of all nonvoid compact subsets of K, where this topology is generated by the sets
$$U_{V,W} = \{L \in \mathcal{C}(K) : L \cap V \neq \varnothing, L \subset W\}$$
with V, W open in K.
3. There is an identity $e \in K$ with $\delta_e *_K \delta_x = \delta_x *_K \delta_e = \delta_x$ for all $x \in K$.
4. There is a continuous involution $\sim$ on K such that
$$(\delta_x *_K \delta_y)^{\sim} = \delta_{y^\sim} *_K \delta_{x^\sim}$$
and $e \in supp(\delta_x *_K \delta_y) \Leftrightarrow x = y^\sim$ for $x, y \in K$ (see [7], [8], [2]).

A hypergroup K is called commutative if $\delta_x *_K \delta_y = \delta_y *_K \delta_x$ for all $x, y \in K$. It is well known that every commutative hypergroup K possesses a Haar measure which will be denoted by λ (see [8]). That is, for every Borel measurable function f on K,

$$\int\limits_K f(\delta_x *_K \delta_y)d\lambda(y) = \int\limits_K f(y)d\lambda(y) \ \ (x \in K).$$

Define the generalized translation operators T^x, $x \in K$, by

$$T^x f(y) = \int\limits_K f d(\delta_x *_K \delta_y)$$

for all $y \in K$. If K is a commutative hypergroup, then $T^x f(y) = T^y f(x)$ and the convolution of two functions is defined by

$$f *_K \varphi(x) = \int\limits_K T^x f(y)\varphi(y^\sim)d\lambda(y).$$

Note that $f *_K \varphi = \varphi *_K f$.
For $1 \le p \le \infty$, the Lebesgue space $L^p(K,\lambda)$ is defined as

$$L^p(K,\lambda) = \{f : f \text{ is } \lambda\text{-measurable on } K, \|f\|_{K,p} < \infty\}$$

where $\|f\|_{K,p}$ is defined by

$$\|f\|_{K,p} = \begin{cases} \left(\int\limits_K |f(x)|^p \, d\lambda(x)\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty \\ \operatorname*{ess\,sup}\limits_{x\in K} f(x), & \text{if } p = \infty. \end{cases}$$

Let $1 \le p \le \infty$. If f is in $L^p(K,\lambda)$ and φ is in $L^1(K,\lambda)$, then the function $f *_K \varphi$ belongs to $L^p(K,\lambda)$ and

$$\|f *_K \varphi\|_{K,p} \le \|f\|_{K,p}\|\varphi\|_{K,1}.$$

Let f be a λ-measurable function defined on the hypergroup K. The distribution function λ_f of the function f is given by

$$\lambda_f(s) = \lambda\{x : x \in K, |f(x)| > s\}, \text{ for } s \ge 0.$$

The distribution function λ_f is non-negative, non-increasing and continuous from the right. We associate to the distribution function the non-increasing rearrangement of f on $[0,\infty)$ defined by

$$f^{*_K}(t) = \inf\{s > 0 : \lambda_f(s) \le t\}.$$

Some elementary properties of λ_f and f^{*_K} are listed below. The proofs of them can be found in [1].

(1) If λ_f is continuous and strictly decreasing, then f^{*_K} is the inverse of λ_f, that is $f^{*_K} = (\lambda_f)^{-1}$.

(2) f^{*_K} is continuous from the right.

(3)

$$m_{f^{*_K}}(s) = \lambda_f(s), \quad \text{for all } s > 0,$$

where $m_{f^{*_K}}$ is a distribution function of the function f^{*_K} with respect to Lebesgue measure m on $(0,\infty)$.

(4)

$$\int\limits_0^t f^{*_K}(s)ds = tf^{*_K}(t) + \int\limits_{f^{*_K}(t)}^{\infty} \lambda_f(s)ds \tag{1}$$

(5) If $f \in L^p(K,\lambda)$, $1 \le p < \infty$, then

$$\left(\int_K |f(x)|^p \, d\lambda(x)\right)^{\frac{1}{p}} = \left(p\int_0^\infty s^{p-1}\lambda_f(s)ds\right)^{\frac{1}{p}} = \left(\int_0^\infty (f^{*_K}(t))^p \, dt\right)^{\frac{1}{p}}.$$

Furthermore, in the case $p = \infty$,

$$\operatorname*{ess\,sup}_{x\in K} f(x) = \inf\{s : \lambda_f(s) = 0\} = f^{*_K}(0).$$

f^{**_K} will denote the maximal function of f^{*_K} defined by

$$f^{**_K}(t) = \frac{1}{t}\int_0^t f^{*_K}(u)du, \text{ for } t > 0.$$

Note the following properties of f^{**_K}:

(1′) f^{**_K} is nonnegative, non-increasing and continuous on $(0,\infty)$ and $f^{*_K} \le f^{**_K}$.

(2′)

$$(f+g)^{**_K} \le f^{**_K} + g^{**_K}$$

(3′) If $|f_n| \uparrow |f|$ λ-a.e., then $f_n^{**_K} \uparrow f^{**_K}$.

For $1 \le p < \infty$ and $1 \le q \le \infty$, the Lorentz space $L^{p,q}(K,\lambda)$ is defined as

$$L^{p,q}(K,\lambda) = \{f : f \text{ is } \lambda\text{-measurable on } K, \|f\|_{K,p,q} < \infty\}$$

where $\|f\|_{K,p,q}$ is defined by

$$\|f\|_{K,p,q} = \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} f^{**_K}(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 1 \le p < \infty, 1 \le q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} f^{**_K}(t), & 1 \le p \le \infty, q = \infty. \end{cases}$$

Note that if $1 < p \le \infty$ then $L^{p,p}(K,\lambda) = L^p(K,\lambda)$. Moreover,

$$\|f\|_{K,p} \le \|f\|_{K,p,p} \le p'\|f\|_{K,p}, \tag{2}$$

where $p' = \begin{cases} \dfrac{p}{p-1}, & 1 < p < \infty, \\ 1, & p = \infty. \end{cases}$

For $p > 1$, the space $L^{p,\infty}(K,\lambda)$ is known as the Marcinkiewicz space or as Weak $L^p(K,\lambda)$.
Also note that $L^{1,\infty}(K,\lambda) = L^1(K,\lambda)$.
If $1 < p < \infty$ and $1 < q < r < \infty$, then

$$L^{p,q}(K,\lambda) \subset L^{p,r}(K,\lambda).$$

Moreover

$$\|f\|_{K,p,r} \le \left(\frac{q}{p}\right)^{\frac{1}{q}-\frac{1}{p}} \|f\|_{K,p,q}. \tag{3}$$

The Young inequality on Lebesgue spaces for compact commutative hypergroups was introduced in [9]. The generalized Young inequalities on the Lorentz spaces for Bessel and Dunkl convolution operators were introduced in [3] and [5], respectively.

In this paper we establish the generalized Young inequality on the Lorentz spaces for commutative hypergroups and give its application to the theory of fractional integrals. The boundedness on the Lorentz spaces of the fractional integrals on the commutative hypergroups is proved. We also prove the Hardy-Littlewood-Sobolev theorem for the fractional integrals on the commutative hypergroups.

2. Lemmas

Lemma 1. *Let f and φ be λ-measurable functions on the hypergroup K where $\sup_{x\in K}|f(x)| \le \beta$ and f vanishes outside a measurable set E with $\lambda(E) = r$. Then, for $t > 0$,*

$$(f *_K \varphi)^{**_K}(t) \le \beta r \varphi^{**_K}(r) \tag{4}$$

and

$$(f *_K \varphi)^{**_K}(t) \le \beta r \varphi^{**_K}(t). \tag{5}$$

Proof. Without loss of generality we can assume that the functions f and φ are nonnegative. Let $h = f *_K \varphi$. For $a > 0$, define

$$\varphi_a(x) = \begin{cases} \varphi(x), & \text{if } \varphi(x) \le a \\ a, & \text{if } \varphi(x) > a, \end{cases}$$

$$\varphi^a(x) = \varphi(x) - \varphi_a(x).$$

Also define functions h_1 and h_2 by

$$h = f *_K \varphi_a + f *_K \varphi^a = h_1 + h_2.$$

Then we have the following three estimate:

$$\sup_{x\in K} h_2(x) \le \sup_{x\in K} f(x)\|\varphi^a\|_{K,1} \le \beta \int_0^\infty \lambda_{\varphi^a}(s)ds = \beta \int_a^\infty \lambda_\varphi(s)ds, \tag{6}$$

$$\sup_{x\in K} h_1(x) \le \|f\|_{K,1} \sup_{x\in K} \varphi_a(x) \le \beta r a, \tag{7}$$

and

$$\sup_{x\in K} h_2(x) \le \|f\|_{K,1}\|\varphi^a\|_{K,1} \le \beta r \int_a^\infty \lambda_\varphi(s)ds. \tag{8}$$

Now set $a = \varphi_K^*(r)$ in (6) and (7). Then we obtain

$$h^{**_K}(t) = \frac{1}{t}\int_0^t h^{*_K}(s)ds \le \|h\|_{K,\infty} \le \|h_1\|_{K,\infty} + \|h_2\|_{K,\infty}$$

$$\le \beta r \varphi_K^*(r) + \beta \int_{\varphi_K^*(r)}^{\infty} \lambda_\varphi(s)ds,$$

and using (1) we have the inequality (4).

Let us prove the inequality (5). For this purpose set $a = \varphi_K^*(t)$ and use (7) and (8). Then

$$th^{**_K}(t) = \int_0^t h^{*_K}(s)ds \le \int_0^t h_1^{*_K}(s)ds + \int_0^t h_2^{*_K}(s)ds$$

$$\le t\|h_1\|_{K,\infty} + \int_0^t h_2^{*_K}(s)ds = t\|h_1\|_{K,\infty} + t\|h_2\|_{K,1}$$

$$\le t\beta r \varphi_K^*(t) + \beta r t \int_{\varphi_K^*(t)}^{\infty} \lambda_\varphi(s)ds$$

$$= \beta r t \left(\varphi_K^*(t) + \int_{\varphi^{*_K}(t)}^{\infty} \lambda_\varphi(s)ds \right) = \beta r t \varphi^{**_K}(t).$$

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Lemma 2. *Let f and φ be λ-measurable functions on hypergroup K Then for all $t > 0$ the following inequality holds:*

$$(f *_K \varphi)^{**_K}(t) \le t f^{**_K}(t)\varphi^{**_K}(t) + \int_t^{\infty} f_K^*(s)\varphi_K^*(s)ds. \tag{9}$$

Proof. Without loss of generality we can assume that the functions f and φ are nonnegative. Let $h = f *_K \varphi$ and fix $t > 0$. Select a nondecreasing sequence $\{s_n\}_{-\infty}^{+\infty}$ such that $s_0 = f_K^*(t)$, $\lim\limits_{n\to+\infty} s_n = +\infty$, $\lim\limits_{n\to-\infty} s_n = 0$.

Also let

$$f(x) = \sum_{n=-\infty}^{+\infty} f_n(x),$$

where

$$f_n(x) = \begin{cases} 0, & \text{if } f(x) \le s_{n-1} \\ f(x) - s_{n-1} & \text{if } s_{n-1} < f(x) \le s_n \\ s_n - s_{n-1} & \text{if } s_n < f(x). \end{cases}$$

Since the series $\{s_n\}_{-\infty}^{+\infty}$ converges absolutely we have

$$h = \int\limits_K T^x\varphi(y)\left(\sum_{n=-\infty}^{+\infty} f_n(y^\sim)\right) d\lambda(y)$$

$$= \sum_{n=-\infty}^{+\infty} \int\limits_K T^x\varphi(y) f_n(y^\sim) d\lambda(y) = \sum_{n=-\infty}^{+\infty} (f_n *_K \varphi)\,.$$

Define functions h_1 and h_2 by

$$h = \sum_{n=1}^{+\infty} (f_n *_K \varphi) + \sum_{n=-\infty}^{0} (f_n *_K \varphi) = h_1 + h_2.$$

Estimate $h_1^{**_K}(t)$. For this purpose use the inequality (5) with $E = \{x : f(x) > s_{n-1}\}$ and $\beta = s_n - s_{n-1}$. We have

$$h_1^{**_K}(t) \le \sum_{n=1}^{+\infty} ((f_n *_K \varphi)^{**_K})$$

$$\le \sum_{n=1}^{+\infty} (s_n - s_{n-1})\lambda_f(s_{n-1})\varphi^{**_K}(t)$$

$$= \varphi^{**_K}(t) \sum_{n=1}^{+\infty} \lambda_f(s_{n-1})(s_n - s_{n-1}).$$

Hence

$$h_1^{**_K}(t) \le \varphi^{**_K}(t) \int\limits_{f_K^*(t)}^{\infty} \lambda_f(s) ds. \tag{10}$$

To estimate $h_2^{**_K}(t)$ we use the inequality (4):

$$h_2^{**_K}(t) \le \sum_{n=-\infty}^{0} ((f_n *_K \varphi)^{**_K})$$

$$\le \sum_{n=1}^{+\infty} (s_n - s_{n-1})\lambda_f(s_{n-1})\varphi^{**_K}(\lambda_f(s_{n-1}))$$

$$= \sum_{n=1}^{+\infty} \lambda_f(s_{n-1})\varphi^{**_K}(\lambda_f(s_{n-1}))(s_n - s_{n-1}).$$

This implies that

$$h_2^{**K}(t) \le \int_0^{f_K^*(t)} \lambda_f(s)\varphi^{**K}(\lambda_f(s))ds. \tag{11}$$

We will estimate the integral on the right-hand side of (11) by making the substitution $s = f_K^*(\xi)$ and then integrating by parts. In order to justify the change of variable in the integral, consider a Riemann sum

$$\sum_{n=1}^{+\infty} \lambda_f(s_{n-1})\varphi^{**K}(\lambda_f(s_{n-1}))(s_n - s_{n-1}).$$

that provides a close approximation to

$$\int_0^{f_K^*(t)} \lambda_f(s)\varphi^{**K}(\lambda_f(s))ds.$$

By adding more points to the Riemann sum if necessary, we may assume that the left-hand end point of each interval on which λ_f is constant is included among the s_n. Then the Riemann sum is not changed if each s_n that is contained in the interior of an interval on which λ_f is constant, is deleted. It is now an easy matter to verify that for each of the remaining s_n there is precisely one element, ξ_n, such that $s_n = f^{*K}(\xi_n)$ and $\lambda\left(f^{*K}(\xi_n)\right) = \xi_n$. Therefore

$$\sum_{n=1}^{+\infty} \lambda_f(s_{n-1})\varphi^{**K}(\lambda_f(s_{n-1}))(s_n - s_{n-1}).$$

$$= \sum_{n=1}^{+\infty} \xi_{n-1}\varphi^{**K}(\xi_{n-1})(f^{*K}(\xi_n) - f^{*K}(\xi_{n-1}))$$

which, by adding more points if necessary, provides a close approximation to

$$-\int_t^{\infty} \xi\varphi^{**K}(\xi)df^{*K}(\xi).$$

If we recall (11) we get

$$h_2^{**K}(t) \le \int_0^{f^{*K}(t)} \lambda_f(s)\varphi^{**K}(\lambda_f(s))ds = -\int_t^{\infty} \xi\varphi^{**K}(\xi)df^{*K}(\xi). \tag{12}$$

Now let δ be an arbitrarily large number and choose ξ_j such that $t = \xi_1 \le \xi_2 \le \ldots \le \xi_{j+1} = \delta$. Then

$$\delta\varphi^{**K}(\delta)f^{*K}(\delta) - t\varphi^{**K}(t)f^{*K}(t)$$

$$= \sum_{n=1}^{j} \xi_{n+1}\varphi^{**K}(\xi_{n+1})\left(f^{*K}(\xi_{n+1}) - f^{*K}(\xi_n)\right)$$

$$+ \sum_{n=1}^{j} f^{*K}(\xi_n)\left(\varphi^{**K}(\xi_{n+1})\xi_{n+1} - \varphi^{**K}(\xi_n)\xi_n\right)$$

$$= \sum_{n=1}^{j} \xi_{n+1}\varphi^{**K}(\xi_{n+1})\left(f^{*K}(\xi_{n+1}) - f^{*K}(\xi_n)\right)$$

$$+ \sum_{n=1}^{j} f_K^{*}(\xi_n) \int\limits_{\xi_n}^{\xi_{n+1}} \varphi^{*K}(\tau)d\tau$$

$$\leq \sum_{n=1}^{j} \xi_{n+1}\varphi^{**K}(\xi_{n+1})\left(f^{*K}(\xi_{n+1}) - f^{*K}(\xi_n)\right)$$

$$+ \sum_{n=1}^{j} f^{*K}(\xi_n)\varphi^{*K}(\xi_n)\left(\xi_{n+1} - \xi_n\right).$$

This means that

$$\delta\varphi^{**K}(\delta)f^{*K}(\delta) - t\varphi^{**K}(t)f^{*K}(t) \leq \int\limits_{t}^{\delta} \xi\varphi^{**K}(\xi)df^{*K}(\xi) + \int\limits_{t}^{\delta} f^{*K}(\xi)\varphi^{*K}\xi)d\xi. \qquad (13)$$

Now we estimate the expression $\delta\varphi^{**K}(\delta)f^{*K}(\delta) - t\varphi^{**K}(t)f^{*K}(t)$ below.

$$\delta\varphi^{**K}(\delta)f^{*K}(\delta) - t\varphi^{**K}(t)f^{*K}(t)$$

$$= \sum_{n=1}^{j} \xi_n\varphi^{**K}(\xi_n)\left(f^{*K}(\xi_{n+1}) - f^{*K}(\xi_n)\right)$$

$$+ \sum_{n=1}^{j} f^{*K}(\xi_{n+1})\left(\varphi^{**K}(\xi_{n+1})\xi_{n+1} - \varphi^{**K}(\xi_n)\xi_n\right)$$

$$= \sum_{n=1}^{j} \xi_n\varphi^{**K}(\xi_n)\left(f^{*K}(\xi_{n+1}) - f^{*K}(\xi_n)\right)$$

$$+ \sum_{n=1}^{j} f^{*K}(\xi_{n+1}) \int\limits_{\xi_n}^{\xi_{n+1}} \varphi^{*K}(\tau)d\tau$$

$$\geq \sum_{n=1}^{j} \xi_n\varphi^{**K}(\xi_n)\Big(f^{*K(}\xi_{n+1}) - f^{*K}(\xi_n)\Big)$$

$$+\sum_{n=1}^{j} f_K^*(\xi_{n+1})\varphi^{*_K}(\xi_{n+1})\left(\xi_{n+1}-\xi_n\right).$$

In other words

$$\delta\varphi^{**_K}(\delta)f^{*_K}(\delta)-t\varphi^{**_K}(t)f^{*_K}(t)\geq\int_t^\delta \xi\varphi^{**_K}(\xi)df^{*_K}(\xi)+\int_t^\delta f^{*_K}(\xi)\varphi^{*_K}(\xi)d\xi. \qquad (14)$$

From (13) and (14) we obtain

$$-\int_t^\delta \xi\varphi^{**_K}(\xi)df^{*_K}(\xi)=t\varphi^{**_K}(t)f^{*_K}(t)-\delta\varphi^{**_K}(\delta)f^{*_K}(\delta)+\int_t^\delta f^{*_K}(\xi)\varphi^{*_K}(\xi)d\xi.$$

$$\leq t\varphi^{**_K}(t)f^{*_K}(t)+\int_t^\delta f^{*_K}(\xi)\varphi^{*_K}(\xi)d\xi.$$

Thus

$$-\int_t^\infty \xi\varphi^{**_K}(\xi)df^{*_K}(\xi)\leq t\varphi^{**_K}(t)f^{*_K}(t)+\int_t^\infty f^{*_K}(\xi)\varphi^{*_K}(\xi)d\xi.$$

By using this inequality and (12) we have

$$h_2^{**_K}(t)\leq\int_0^{f^{*_K}(t)}\lambda_f(s)\varphi^{**_K}(\lambda_f(s))ds\leq t\varphi^{**_K}(t)f^{*_K}(t)+\int_t^\infty f^{*_K}(\xi)\varphi^{*_K}(\xi)d\xi. \qquad (15)$$

Finally, from (10), (15) and (1) we get

$$h^{**_K}(t)\leq h_1^{**_K}(t)+h_2^{**_K}(t)$$

$$\leq\varphi^{**_K}(t)\int_{f^{*_K}(t)}^\infty\lambda_f(s)ds+t\varphi^{**_K}(t)f^{*_K}(t)+\int_t^\infty f^{*_K}(\xi)\varphi^{*_K}(\xi)d\xi$$

$$=f^{*_K}(t)\varphi^{**_K}(t)+\int_t^\infty f^{*_K}(\xi)\varphi^{*_K}(\xi)d\xi$$

$$=tf^{**_K}(t)\varphi^{**_K}(t)+\int_t^\infty f^{*_K}(\xi)\varphi^{*_K}(\xi)d\xi.$$

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Lemma 3. *Let f and φ be λ-measurable functions on hypergroup K. Then for all $t>0$ the following inequality holds:*

$$(f *_K \varphi)^{**_K}(t) \le \int_t^\infty f^{**_K}(s)\varphi^{**_K}(s)ds \tag{16}$$

Proof. Assume that the integral on the right of (16) is finite. Then it is easy to see that

$$sf^{**_K}(s)\varphi^{**_K}(s) \to 0, \text{ as } s \to \infty. \tag{17}$$

Let $h = f *_K \varphi$.
By Lemma 2 we have

$$h^{**_K}(t) \le tf^{**_K}(t)\varphi^{**_K}(t) + \int_t^\infty f^{*_K}(s)\varphi^{*_K}(s)ds$$

$$\le tf^{**_K}(t)\varphi^{**_K}(t) + \int_t^\infty f^{**_K}(s)\varphi^{*}_{K}(s)ds. \tag{18}$$

Since f^{**_K} and g^{**_K} are non-increasing,

$$\frac{df^{**_K}(s)}{ds} = -\frac{1}{s^2}\int_0^s f^{*_K}(\tau)d\tau + \frac{1}{s}f^{*_K}(s) = \frac{1}{s}\left(f^{*_K}(s) - f^{**_K}(s)\right), \tag{19}$$

$$\frac{d(s\varphi^{**_K}(s))}{ds} = \varphi^{**_K}(s) + s\left(\frac{1}{s}\left(\varphi^{*_K}(s) - \varphi^{**_K}(s)\right)\right) = \varphi^{*_K}(s) \tag{20}$$

for m-almost all s. Since f^{**_K} and g^{**_K} are absolutely continuous, we may use the integration by parts for $\int_t^\infty f^{**_K}(s)d\left(s\varphi^{**_K}(s)\right)$. Using (19), (20) and (17) we obtain

$$\int_t^\infty f^{**_K}(s)\varphi^{*_K}(s)ds = \int_t^\infty f^{**_K}(s)d\left(s\varphi^{**_K}(s)\right)$$

$$= f^{**_K}(s)s\varphi^{**_K}(s)|_t^\infty - \int_t^\infty s\varphi^{**_K}(s)df^{**_K}(s)$$

$$= -tf^{**_K}(t)\varphi^{**_K}(t) + \int_t^\infty \varphi^{**_K}(s)(f^{**_K}(s) - f^{*_K}(s))ds$$

$$\le -tf^{**_K}(t)\varphi^{**_K}(t) + \int_t^\infty \varphi^{**_K}(s)f^{**_K}(s)ds \tag{21}$$

By (18) and (21) we have

$$h^{**K}(t) \le \int\limits_t^\infty f^{**K}(s)\varphi^{**K}(s)ds.$$

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The next lemma is a classical estimate, known as Hardy's inequality.

Lemma 4. *If* $1 \le p < \infty$, $q > 0$ *and* f *is a nonnegative* m*-measurable function on* $(0,\infty)$, *then*

$$\int\limits_0^\infty \left(\frac{1}{s}\int\limits_0^s f(\tau)d\tau\right)^p s^{p-q-1}ds \le \left(\frac{p}{q}\right)^q \int\limits_0^\infty f(t)^p t^{p-q-1}dt. \tag{22}$$

3. Generalization of Young's inequality

Theorem 2. *If* $f \in L^{p_1,q_1}(K,\lambda)$, $\varphi \in L^{p_2,q_2}(K,\lambda)$ *and* $\frac{1}{p_1}+\frac{1}{p_2} > 1$, *then* $(f *_K \varphi) \in L^{p_0,q_0}(K,\lambda)$ *where* $\frac{1}{p_1}+\frac{1}{p_2}-1=\frac{1}{p_0}$ *and* $q_0 \ge 1$ *is any number such that* $\frac{1}{q_1}+\frac{1}{q_2} \ge \frac{1}{q_0}$. *Moreover,*

$$\|(f *_K \varphi)\|_{K,p_0,q_0} \le 3p_0\|f\|_{K,p_1,q_1}\|\varphi\|_{K,p_2,q_2}. \tag{23}$$

Proof. Let $h = f *_K \varphi$.
Suppose that q_1, q_2, q_0 are all different from ∞. Then, by (16), we have

$$(\|h\|_{K,p_0,q_0})^{q_0} = \int\limits_0^\infty \left(s^{\frac{1}{p_0}}h^{**K}(s)\right)^q \frac{ds}{s}$$

$$\le \int\limits_0^\infty \left(s^{\frac{1}{p_0}}\int\limits_s^\infty f^{**K}(\tau)\varphi^{**K}(\tau)d\tau\right)^q \frac{ds}{s}$$

$$= \int\limits_0^\infty \left(\frac{1}{t^{\frac{1}{p_0}}}\int\limits_0^t f^{**K}\left(\frac{1}{\eta}\right)\varphi^{**K}\left(\frac{1}{\eta}\right)\frac{d\eta}{\eta^2}\right)^q \frac{dt}{t}.$$

The last equality was obtained by the change of variables $s=\frac{1}{t}$ and $\tau=\frac{1}{\eta}$. Using (22) we get

$$\int\limits_0^\infty \left(\frac{1}{t^{\frac{1}{p_0}}}\int\limits_0^t f^{**K}\left(\frac{1}{\eta}\right)\varphi^{**K}\left(\frac{1}{\eta}\right)\frac{d\eta}{\eta^2}\right)^q \frac{dt}{t}$$

$$\le p_0^{q_0}\int\limits_0^\infty \left(t^{1-\frac{1}{p_0}}\frac{f^{**K}\left(\frac{1}{t}\right)\varphi^{**K}\left(\frac{1}{t}\right)}{t^2}\right)^{q_0}\frac{dt}{t}$$

$$= p_0^{q_0} \int_0^\infty \left(s^{1+\frac{1}{p_0}} f^{**K}(s) \varphi^{**K}(s) \right)^{q_0} \frac{ds}{s}.$$

The last equality was obtained by the change of the variable $t = \frac{1}{s}$. Since $\frac{q_0}{q_1} + \frac{q_0}{q_2} \geq 1$, one can find positive numbers n_1 and n_2 such that

$$\frac{1}{n_1} + \frac{1}{n_2} = 1 \text{ and } \frac{1}{n_1} \leq \frac{q_0}{q_1}, \ \frac{1}{n_2} \leq \frac{q_0}{q_2}.$$

By Hölder's inequality we obtain

$$\left(\|h\|_{K,p_0,q_0} \right)^{q_0} \leq p_0^{q_0} \int_0^\infty \frac{\left(s^{\frac{1}{p_1}} f^{**K}(s) \right)^{q_0}}{s^{\frac{1}{n_2}}} \frac{\left(s^{\frac{1}{p_2}} \varphi^{**K}(s) \right)^{q_0}}{s^{\frac{1}{n_1}}} ds$$

$$\leq p_0^{q_0} \left[\int_0^\infty \left(s^{\frac{1}{p_1}} f^{**K}(s) \right)^{q_0 n_1} \frac{ds}{s} \right]^{\frac{1}{n_1}} \left[\int_0^\infty \left(s^{\frac{1}{p_2}} \varphi^{**K}(s) \right)^{q_0 n_2} \frac{ds}{s} \right]^{\frac{1}{n_2}}$$

$$= p_0^{q_0} \left(\|f\|_{K,p_1,q_0 n_1} \right)^{q_0} \left(\|\varphi\|_{K,p_2,q_0 n_2} \right)^{q_0}.$$

Finally, by (3) we have

$$\|h\|_{K,p_0,q_0} \leq p_0 \|f\|_{K,p_1,q_0 n_1} \|\varphi\|_{K,p_2,q_0 n_2} \leq p_0 e^{\frac{1}{e}} e^{\frac{1}{e}} \|f\|_{K,p_1,q_1} \|\varphi\|_{K,p_2,q_2} \leq 3p_0 \|f\|_{K,p_1,q_1} \|\varphi\|_{K,p_2,q_2}.$$

Similar reasoning leads to the desired result in case one or more of q_1, q_2, q_0 are ∞.◄

4. Applications to the theory of fractional integrals

Consider the following particular case of Theorem 2. If we take $p_1 = \frac{N}{N-\alpha}$, with $o < \alpha < N$, $q_1 = \infty$ in Theorem 2, then the condition $\frac{1}{p_1} + \frac{1}{p_2} > 1$ is equivalent to $\alpha < \frac{N}{p_2}$, and the condition $\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p_0}$ is equivalent to $\frac{1}{p_0} = \frac{1}{p_2} - \frac{\alpha}{N}$. Thus we have the following result.

Theorem 3. *Let $(K, *_K)$ be a commutative hypergroup, with Haar measure λ. If $f \in L^{\frac{N}{N-\alpha},\infty}(K,\lambda)$, $\varphi \in L^{p,q}(K,\lambda)$, where $0 < \alpha < \frac{N}{p}$, $1 \leq q \leq \infty$, then $(f *_K \varphi) \in L^{r,q}(K,\lambda)$ and*

$$\|(f *_K \varphi)\|_{K,r,q} \leq 3r \|f\|_{K,\frac{N}{N-\alpha},\infty} \|\varphi\|_{K,p,q},$$

where $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{N}$.

Let K be a set. A function $\rho : K \times K \to [0,\infty)$ is called quasi-metric if:

1. $\rho(x,y) = 0 \Leftrightarrow x = y$;
2. $\rho(x,y) = \rho(y,x)$;
3. there is a constant $c \geq 1$ such that for every $x,y,z \in X$

$$\rho(x,y) \leq c\left(\rho(x,z) + \rho(z,y)\right).$$

Define the fractional integral (or Riesz potential)

$$I_\alpha f(x) = \int\limits_K T^x \rho(e,y)^{\alpha-N} f(y^\sim) d\lambda(y), \;\; 0 < \alpha < N$$

on commutative hypergroup $(K, *_K)$ equipped with the pseudo-metric ρ.
Also define a ball $B(e,r) = \{y \in K : \rho(e,y) < r\}$ centered at e with a radius r.

Theorem 4. *Let $(K, *_K)$ be a commutative hypergroup, with quasi-metric ρ and Haar measure λ satisfying $\lambda B(e,r) = Ar^N$, where A is a positive constant. Assume that $1 \leq q \leq \infty$, $1 \leq p < \infty$, $0 < \alpha < \frac{N}{p}$. If $f \in L^{p,q}(K,\lambda)$. Then $I_\alpha f \in L^{r,q}(K,\lambda)$ and*

$$\|I_\alpha f\|_{K,r,q} \leq C\|f\|_{K,p,q},$$

where $\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{N}$ and $C = \frac{3rN}{\alpha} A^{\frac{N-\alpha}{N}}$.

Proof. Let us show that $\rho(e,\cdot)^{\alpha-N} \in L^{\frac{N}{N-\alpha},\infty}(K,\lambda)$. For the distribution of $\rho(e,\cdot)^{\alpha-N}$ we can write

$$\lambda_{\rho(e,\cdot)^{\alpha-N}}(t) = \lambda\{x : x \in K, \rho(e,x)^{\alpha-N} > t\}$$
$$= \lambda\{x : x \in K, \rho(e,x) < t^{\frac{1}{\alpha-N}}\} = At^{\frac{N}{\alpha-N}}.$$

Since $\rho(e,\cdot)^{\alpha-N}$ is continuous and strictly decreasing, we have $\left(\rho(e,\cdot)^{\alpha-N}\right)^{*_K}$ is the inverse of the distribution function. That is $\left(\rho(e,\cdot)^{\alpha-N}\right)^{*_K}(t) = \left(\frac{t}{A}\right)^{\frac{\alpha-N}{N}}$. Then

$$\left(\rho(e,\cdot)^{\alpha-N}\right)^{**_K}(t) = \frac{1}{t}\int\limits_0^t \left(\frac{s}{A}\right)^{\frac{\alpha-N}{N}} ds = \frac{N}{\alpha}\left(\frac{t}{A}\right)^{\frac{\alpha-N}{N}}.$$

Therefore $\rho(e,\cdot)^{\alpha-N} \in L^{\frac{N}{N-\alpha},\infty}(K,\lambda)$ and

$$\|\rho(e,\cdot)^{\alpha-N}\|_{K,\frac{N}{N-\alpha},\infty} = \frac{N}{\alpha} A^{\frac{N-\alpha}{N}}.$$

Thus, from Theorem 3 we have the required result.◀

Theorem 5. *Let* $(K, *_K)$ *be a commutative hypergroup, with Haar measure* λ. *If* $f \in L^{\frac{N}{N-\alpha},\infty}(K,\lambda)$, $\varphi \in L^p(K,\lambda)$, *where* $1 < p < \infty$, $0 < \alpha < \dfrac{N}{p}$, *then* $(f *_K \varphi) \in L^r(K,\lambda)$ *and*

$$\|(f *_K \varphi)\|_{K,r} \le 3r\frac{p}{p-1}\left(\frac{p}{r}\right)^{\frac{1}{p}-\frac{1}{r}} \|f\|_{K,\frac{N}{N-\alpha},\infty}\|\varphi\|_{K,p}.$$

where $\dfrac{1}{r} = \dfrac{1}{p} - \dfrac{\alpha}{N}$.

Proof. From (2), (3) and (23) we have

$$\|(f *_K \varphi)\|_{K,r} \le \|(f *_K \varphi)\|_{K,r,r} \le \left(\frac{p}{r}\right)^{\frac{1}{p}-\frac{1}{r}} \|(f *_K \varphi)\|_{K,r,p}$$

$$\le 3r\left(\frac{p}{r}\right)^{\frac{1}{p}-\frac{1}{r}} \|f\|_{K,\frac{N}{N-\alpha},\infty}\|\varphi\|_{K,p,p} \le 3r\frac{p}{p-1}\left(\frac{p}{r}\right)^{\frac{1}{p}-\frac{1}{r}} \|f\|_{K,\frac{N}{N-\alpha},\infty}\|\varphi\|_{K,p}.$$

◄

The assertion following result give us the Hardy-Littlewood-Sobolev theorem for the fractional integrals on the commutative hypergroups.

Theorem 6. *Let* $(K, *_K)$ *be a commutative hypergroup, with quasi-metric* ρ *and Haar measure* λ *satisfying* $\lambda B(e,r) = Ar^N$, *where* A *is a positive constant. Assume that* $1 < p < \infty$, $0 < \alpha < \dfrac{N}{p}$. *If* $f \in L^p(K,\lambda)$, *then* $I_\alpha f \in L^r(K,\lambda)$ *and*

$$\|I_\alpha f\|_{K,r} \le C\|f\|_{K,p},$$

where $\dfrac{1}{r} = \dfrac{1}{p} - \dfrac{\alpha}{N}$ *and* $C = \dfrac{3pr}{p-1}\left(\dfrac{p}{r}\right)^{\frac{1}{p}-\frac{1}{r}}\dfrac{N}{\alpha}A^{\frac{N-\alpha}{N}}$.

Proof. This follows immediately from Theorem 5 and (4).◄

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